Proceedings of the Project Review, Geo-Mathematical Imaging Group (Purdue University, West Lafayette IN), Vol. 1 (2013) pp. 47-64.

LINEARIZATION AND MICROLOCAL ANALYSIS OF REFLECTION TOMOGRAPHY

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Abstract. We examine the linearization of the problem of reflection tomography with data taken to be either the scattering relation or travel time along reflected rays. We obtain coordinate invariant formulae for the Fréchet differentials with respect to perturbations of the wavespeed and reflector locator for the relevant nonlinear maps. The differential with respect to the wavespeed is a system of weighted reflected X-ray transforms and motivated by this we also begin a microlocal study of such transforms. Under a strong condition on the ray geometry we show that the normal operator associated to a weighted reflected X-ray is a pseudodifferential operator. As part of this we introduce a new approach to the microlocal analysis of X-ray transforms.

1. Introduction. We consider reflection tomography with multiple scattered waves. The data are represented by the (possibly multiply) reflected scattering relation. This relation can in principle be obtained from the Dirichlet-to-Neumann map associated with the boundary value problem for the wave equation at 'high' frequencies. This type of problem can arise in the study of Earth models with structure, that is, piecewise smooth with discontinuities (sometimes referred to in reflection seismology as 'blocky'). We assume that the presence (but not necessarily the location) of the interfaces, that is, conormal singularities is known. We express the linearization of this geometric inverse problem in terms of weighted geodesic X-ray transforms and begin to characterize such transforms microlocally. The key, here, is that our models contain only a finite number of interfaces. In case the data consist of single scattered waves and scatterers are present almost everywhere, a uniqueness result was obtained by Kurylev, Lassas & Uhlmann [9].

The scattering relation contains the surface source and receiver locations, the source and receiver 'horizontal' slownesses and travel times corresponding with broken geodesics. We note that the scattering relation is also used in stereotomography as the data (Billette & Lambaré [1] and Lambaré *et al.* [11]). In the case of transmission tomography, the inverse problem, in particular lens rigidity, using the scattering relation as the data was analyzed by Stefanov and Uhlmann [13].

Under slightly different formulations of the problem Nowack & Lyslo [12] calculated the linearization of inversion for slowness and interface location from travel times (and beam amplitudes) while Farra, Vireux, & Madariaga [4] examined how the paraxial ray propagator in Cartesian coordinates can be used to study the variation of rays reflected from a perturbed interface and with perturbations of the slowness. Jurado, Sinoquet & Ehinger [8] also considered reflection tomography for piecewise smooth models. They parametrized the smooth parts of the models by B-splines, and the interfaces by a level set function, that is, depth as a function of surface coordinates also using B-splines. The interfaces can intersect one another and thus form a pinch out by allowing a zero jump across portions of interfaces. The data were reflection times associated with single scattered waves. These could be multi-valued while they parameterized the data by source angle. They locally linearized the problem, incorporated inequality constraints, and applied an iterative Gauss-Newton method for reconstruction. In this context, we also mention the work of Lailly & Sinoquet [10] and Delprat-Jannaud & Lailly [3], who analyzed 'ill-posed and well-posed' formulations of the reflection tomography problem. More recently, Clapp, Biondi & Claerbout [2] incorporated geologic information into reflection tomography to constrain the presence of selected interfaces. The blocky representations also appear in the modelling and analysis of reflection seismograms by Sen & Frazer [6]. In applications, the interfaces which play a role in the reflection tomography will correspond with strong contrasts whereas any other interfaces will suppressed in the analysis.

In the next section we describe the exact mathematical framework in which we will work, introduce the precise problems we mean to study, and then give the outline for the remainder of the

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paper.

2. Reflection tomography. We assume that we have an isotropic medium given by a wavespeed function $c \in C^2(\Omega)$ where $\Omega = \{(y, y') \in \mathbb{R}^{n-1}_y \times \mathbb{R}_{y'} : \ y' \le h(y)\}$ for a given $h \in C^2(\mathbb{R}^{n-1})$; h gives the topography of the measurement surface. We will also write $\widetilde{h}(y) = y' - h(y)$ so that $\partial\Omega$ is defined by $\widetilde{h}(0) = 0$. The rays of this medium parametrized by travel time are precisely the geodesics of the Riemannian metric $\mathbf{g} = c(x)^{-2}\mathbf{e}$ where $\mathbf{e}$ is the Euclidean metric and the travel time along a segment of a ray is the length measured in this metric. Thus, we will refer to $\mathbf{g}$ as the travel time metric. The geodesics of $\mathbf{g}$ may be calculated as the projected Hamiltonian flow given by the Hamiltonian

$$H(\xi) = \tfrac{1}{2}(|\xi|^2_{\mathbf{g}} - 1) \in C^2(T^*\Omega). \tag{2.1}$$

Given any covector $\xi \in T^*\Omega$ we will use the notation γ_ξ for the maximally extended geodesic of $\mathbf{g}$ with $\dot{\gamma}_\xi(0) = \xi^{\sharp_{\mathbf{g}}}$ [1]. Also, we will write Ψ_s for the flow of the Hamiltonian vector field X_H given by H.

The rays will be reflected from an interface $\Gamma \subset \Omega$ which is represented as the level set of a function $f \in C^2(V)$ where V is an open subset of Ω. By this, we mean that

$$\Gamma = f^{-1}(\{0\})$$

and $\mathrm{d}f(x) \neq 0$ for any $x \in \Gamma$. Note that Γ could have more than one component representing multiple reflecting surfaces.

Now let $S^*\Omega$ denote the cosphere bundle over Ω, $\partial_- S^*\Omega$ the portion of the cosphere bundle over $\partial\Omega$ corresponding to those covectors oriented towards the interior of Ω, and $\partial_+ S^*\Omega$ those oriented towards the exterior. For us the *cosphere bundle* is the quotient of the cotangent bundle minus the zero section, $T^*\Omega \setminus \{0\}$, by $\mathbb{R}^+$ acting by multiplication in the fibers. There is a smooth map $P_{S^*\Omega} : T^*\Omega \setminus \{0\} \to S^*\Omega$ which gives the natural projection onto this quotient space. The Riemannian metric $\mathbf{g}$ on Ω allows us to identify $\nu_S \in S^*\Omega$ with the unique $\nu \in \nu_S \cap S^*_{\mathbf{g}}\Omega$ where $S^*_{\mathbf{g}}\Omega$ is the unit cosphere bundle with respect to $\mathbf{g}$. Thus the Riemannian metric $\mathbf{g}$ gives a right inverse for $P_{S^*\Omega}$ which is in fact a diffeomorphism when considered as a map into $S^*_{\mathbf{g}}\Omega$ and we will use this to identify $\nu \in S^*_{\mathbf{g}}\Omega$ with $[\nu] = P_{S^*\Omega}(\nu) \in S^*\Omega$.

Let U be an open subset compactly contained in $\partial_- S^*\Omega$. For $[\nu] \in U$ we suppose that there is a unit speed (with respect to $\mathbf{g}$) reflected geodesic beginning at ν, consisting of two legs, and ending on $\partial_+ S_{\mathbf{g}}\Omega$. More precisely we mean that γ_ν intersects Γ transversally at some time $t([\nu]) > 0$, $\gamma_\nu((0, t([\nu]))) \cap (\partial\Omega \cup \Gamma) = \emptyset$, and for

$$\nu'([\nu]) = \dot{\gamma}_\nu(t([\nu]))^{\flat_{\mathbf{g}}} - 2\left(\mathrm{d}f_{\gamma_\nu(t([\nu]))}\right) \frac{\mathrm{d}f_{\gamma_\nu(t([\nu]))}\left(\dot{\gamma}_\nu(t([\nu]))\right)}{|\mathrm{d}f_{\gamma_\nu(t([\nu]))}|^2_{\mathbf{g}}} =: \mathcal{R}_f\left(\dot{\gamma}_\nu(t([\nu]))^{\flat_{\mathbf{g}}}\right), \tag{2.2}$$

$\gamma_{\nu'}$ intersects $\partial\Omega$ transversally at $t'([\nu]) > 0$, and $\gamma_{\nu'}((0, t'([\nu]))) \cap (\partial\Omega \cup \Gamma) = \emptyset$. Here we are defining $\mathcal{R}_f$ to be, for every $y \in \Gamma$, a linear map from $T^*_y\Omega$ to itself which represents the map from the incident to the reflected covector.

We define a C^1 mapping $R_{c,f} : U \to \partial_+ S\Omega$ called the *reflected scattering relation* by

$$R_{c,f}([\nu]) = [\dot{\gamma}_{\nu'}(t'([\nu]))^{\flat_{\mathbf{g}}}].$$

Note also that the travel time along the reflected ray described by $\nu \in U$ is given by a function $T_{c,f} \in C^1(U)$

$$T_{c,f}([\nu]) = t([\nu]) + t'([\nu]).$$

[1] Here, $\sharp$, $\flat$ denote the musical isomorphism in Riemannian geometry.

We refer to T as the *reflected travel time function*. In the present work we are primarily interested in exploring the following two questions.

QUESTION 1. *If the reflection surface Γ (and thus the function f) is known to what extent can we recover the wave speed c from $R_{c,f}$ and $T_{c,f}$?*

QUESTION 2. *If Γ is also unknown can we hope to recover any information about c and Γ from $R_{c,f}$ and $T_{c,f}$?*

We begin studying these questions by analyzing the Fréchet differentiability of the maps $(c,f) \mapsto R_{c,f}$, $(c,f) \mapsto T_{c,f}$. The Fréchet derivatives of these maps are calculated in section 3. In section 4 we look at extending the derivative calculation to the case of multiple reflections. In section 5 we change the problem slightly and consider the linearization of travel time as a function of the two endpoints of the reflected geodesic. This linearization leads directly to the operator we call the *reflected geodesic X-ray transform*, and we finally perform a detailed microlocal analysis of this transform in section 6. It should be noted additionally that the linearization with respect to perturbation of the wavespeed c in any case leads to a system of weighted reflected X-ray transforms.

3. Linearization. In order to proceed we first introduce some more notation and terminology. Define the two sets

$$\widetilde{U} = \{(P_{S^*\Omega} \circ \Psi_s)(\nu) \ : \ [\nu] \in U, \ s \in (0, t([\nu]))\} \subset S^*\Omega^{int}$$

and

$$\widetilde{U'} = \{(P_{S^*\Omega} \circ \Psi_s)(\nu'([\nu]) \ : \ [\nu] \in U, \ s \in (0, t'([\nu]))\} \subset S^*\Omega^{int}.$$

Note that $\widetilde{U}$ and $\widetilde{U'}$ are open subsets of $S^*\Omega$ and the maps $\Phi : \{([\nu], s) \ : \ [\nu] \in U, s \in (0, t([\nu]))\} \ni ([\nu], s) \mapsto [\Psi_s(\nu)] \in \widetilde{U}$ and $\Phi' : \{([\nu], s) \ : \ [\nu] \in U, s \in (0, t'([\nu]))\} \ni ([\nu], s) \mapsto [\Psi_s(\nu'([\nu]))] \in \widetilde{U'}$ are diffeomorphisms onto these sets. On these sets we introduce the functions $F_{c,f} : \widetilde{U} \to \partial_+ S^*\Omega$ and $F'_{c,f} : \widetilde{U'} \to \partial_+ S^*\Omega$ defined by

$$\begin{aligned} F_{c,f}(\omega) &= R_{c,f}(\pi_\nu \circ \Phi^{-1}(\omega)) \\ F'_{c,f}(\omega) &= R_{c,f}(\pi_\nu \circ (\Phi')^{-1}(\omega)) \end{aligned}$$

and

$$\begin{aligned} \tau_{c,f}(\omega) &= T_{c,f}(\pi_\nu \circ \Phi^{-1}(\omega)) - \pi_s \circ \Phi^{-1}(\omega) \\ \tau'_{c,f}(\omega) &= (T_{c,f} - t)(\pi_\nu \circ (\Phi')^{-1}(\omega)) - \pi_s \circ (\Phi')^{-1}(\omega) \end{aligned}$$

where π_s and π_ν are projections onto the s and $[\nu]$ components respectively. Intuitively $F_{c,f}$ and $F'_{c,f}$ map a given "codirection" ω to the place where the reflected ray beginning at this codirection reaches $\partial\Omega$ while $\tau_{c,f}$ and $\tau'_{c,f}$ map ω to the travel time from ω back to $\partial\Omega$ along the reflected ray.

3.1. Perturbation due to variation of the wave speed. We first consider only a variation of the wavespeed. For this we prove the following theorem.

THEOREM 3.1. *Let Ω, U, f, and c be as described in section (2). For fixed f the maps $C^2(\Omega) \ni c \mapsto R_{c,f} \in C(U, \partial_+ S\Omega)$ and $C^2(\Omega) \ni c \mapsto T_{c,f} \in C(U)$ are Fréchet differentiable at c and the differentials are given (for $[\nu] \in U$) by*

$$\begin{aligned} [D^c R_{c,f}(c, \delta c)]([\nu]) = &\int_0^{t([\nu])} D_{[\dot\gamma_\nu(s)^{\flat_{\mathbf{g}}}]} F_{c,f} \circ D_{\dot\gamma_\nu(s)^{\flat_{\mathbf{g}}}} P_{S^*\Omega}(X_{\delta H}(\dot\gamma_\nu(s)^{\flat_{\mathbf{g}}}))\, \mathrm{d}s \\ &+ \int_0^{t'([\nu])} D_{[\dot\gamma_{\nu'}(s)^{\flat_{\mathbf{g}}}]} F'_{c,f} \circ D_{\dot\gamma_{\nu'}(s)^{\flat_{\mathbf{g}}}} P_{S^*\Omega}(X_{\delta H}(\dot\gamma_{\nu'}(s)^{\flat_{\mathbf{g}}}))\, \mathrm{d}s \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
[D^c T_{c,f}(c,\delta c)]([\nu]) = &\int_0^{t([\nu])} D_{[\dot\gamma_\nu(s)^{\flat_{\mathbf{g}}}]}\tau_{c,f} \circ D_{\dot\gamma_\nu(s)^{\flat_{\mathbf{g}}}} P_{S^*\Omega}(X_{\delta H}(\dot\gamma_\nu(s)^{\flat_{\mathbf{g}}}))\ \mathrm{d}s \\
&+ \int_0^{t'([\nu])} D_{[\dot\gamma_{\nu'}(s)^{\flat_{\mathbf{g}}}]}\tau'_{c,f} \circ D_{\dot\gamma_{\nu'}(s)^{\flat_{\mathbf{g}}}} P_{S^*\Omega}(X_{\delta H}(\dot\gamma_{\nu'}(s)^{\flat_{\mathbf{g}}}))\ \mathrm{d}s
\end{aligned}
\tag{3.2}
$$

where $X_{\delta H}$ is the Hamiltonian vector field given by the Hamiltonian $\delta H(\xi) = c\ \delta c\ |\xi|^2_{\mathbf{e}} \in C^2(T^*\Omega)$.

Remark 1. *Actually $C(U, \partial_+ S\Omega)$ is a Banach manifold, and the Fréchet derivative of $R_{c,f}$ is the differential of $R_{c,f}$ considered as a map between Banach manifolds.*

Remark 2. *As part of the proof below we also find explicit formulae in coordinates for the differentials (3.1) and (3.2). See (3.10) and (3.11).*

Proof. We will begin by introducing some coordinate expressions. Firstly, on Ω we have the Cartesian coordinates, and these induce natural coordinate systems on $T\Omega$, $T^*\Omega$, and $T(T^*\Omega)$. As mentioned above we will identify $S^*\Omega$ with $S^*_{\mathbf{g}}\Omega$ and thus we can specify $[\nu] \in U$ using the natural coordinates on $T^*\Omega$ as

$$
\left((y_0)^j, \nu_j\right)
$$

where $|\nu_j| = c^{-1}(y_0)$. Here and in the following coordinate formulae the notation $|\cdot|$ refers to the regular Euclidean length of the coordinate expressions. In the same coordinates γ_ν is given by the solution of the Hamiltonian system

$$
\left\{ \begin{array}{l} \dot\gamma^j(s) = c(\gamma(s))^2\ \delta^{jl}\ \xi_l(s) \\ \dot\xi_l(s) = -c(\gamma(s))\ \partial_{x^l} c(\gamma(s))\ |\xi(s)|^2 \end{array} \right.
\tag{3.3}
$$

with initial conditions $\gamma^j_\nu(0) = (y_0)^j$ and $\xi^\nu_l(0) = \nu_l$. Let us write

$$
(x, \eta) = (x^j, \eta_l) = (\gamma^j_\nu(t([\nu])), \xi^\nu_l(t([\nu])))
$$

for the coordinates of the covector at which the ray γ_ν hits the interface Γ and (x, η') for the coordinates of ν' the reflected covector. The formula (2.2) in coordinates then becomes

$$
\eta'_j = \eta_j - 2\frac{\partial_{x^j} f(x)\ \partial_{x^l} f(x)}{|\partial_x f(x)|^2}\delta^{ll'}\eta_{l'}.
\tag{3.4}
$$

The second leg of the ray $\gamma_{\nu'}$ is then given in coordinates by solving (3.3) with initial conditions $\gamma^j_{\nu'}(0) = x^j$ and $\xi^{\nu'}_l(0) = \eta'_l$. Finally let us write in coordinates

$$
(y^j, \omega_l) = \left(\gamma^j_{\nu'}(t'([\nu])), \xi^{\nu'}_l(t'([\nu]))\right).
$$

Now we begin the calculation of the Gâteaux derivatives.

To start, let us consider a perturbation

$$
c_\epsilon = c + \epsilon\ \delta c
$$

of c in $C^2(\Omega)$ (i.e. $\delta c \in C^2(\Omega)$). When ϵ is sufficiently small all the constructions introduced above still work, and we can consider the corresponding mathematical objects which will be labeled with an ϵ (e.g. γ^ϵ_ν). We will also use the notation

$$
\delta\gamma_\nu = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \gamma^\epsilon_\nu
$$

and the same for the other objects depending on ϵ. Note that with this notation the Gâteaux derivatives of $R_{c,f}$ and $T_{c,f}$ in the direction $\delta c \in C^2(\Omega)$ are

$$
\delta R_{c,f}, \quad \text{and} \quad \delta T_{c,f}.
$$

Note that $\delta R_{c,f}([\nu])$ naturally lies in $T\partial_+ S^*\Omega$. We now begin calculating these quantities.

Differentiating (3.3) with respect to ϵ we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} (\delta\gamma)^j(s) \\ (\delta\xi)_l(s) \end{pmatrix} = \\
&\begin{pmatrix} 2\ c\ \partial_{x^{j'}} c\ \delta^{jk}\ \xi_k(s) & c^2\ \delta^{jl'} \\ -\left(c\ \partial^2_{x^l x^{j'}} c + \partial_{x^l} c\ \partial_{x^{j'}} c\right)|\xi(s)|^2 & -2\ c\ \partial_{x^l} c\ \delta^{l'k}\ \xi_k(s) \end{pmatrix}\begin{pmatrix} (\delta\gamma)^{j'}(s) \\ (\delta\xi)_{l'}(s) \end{pmatrix} \\
&+\begin{pmatrix} 2\delta^{jk}\ \xi_k(s)\ c\ \delta c \\ -\left(\partial_{x^l} c\ \delta c + c\ \partial_{x^l} \delta c\right)|\xi(s)|^2 \end{pmatrix}.
\end{aligned} \tag{3.5}
$$

We now point out that in the natural coordinates on $T(T^*\Omega)$ the Hamiltonian vector field $X_{\delta H} = X_{c\ \delta c\ |\xi|_{\mathbf{e}^2}}$ is given by

$$
\left(\begin{pmatrix} \gamma^j \\ \xi_l \end{pmatrix};\begin{pmatrix} 2\delta^{jk}\ \xi_k\ c\ \delta c \\ -\left(\partial_{x^l} c\ \delta c + c\ \partial_{x^l}\delta c\right)|\xi|^2 \end{pmatrix}\right). \tag{3.6}
$$

Let $P_\nu(t,s)$ be the fundamental matrix of the homogeneous equation corresponding to (3.5) with $\gamma = \gamma_\nu$, $\xi = \xi^\nu$, and $P_\nu(s,s) = \mathrm{Id}$. Note that in the natural coordinates P_ν corresponds with the differential of the Hamilton flow Ψ via the formula

$$
D_{\Psi_s(\nu)}\Psi_{t-s}\left(\begin{pmatrix} \gamma^j_\nu(s) \\ \xi^\nu_l(s) \end{pmatrix};\begin{pmatrix} a^j \\ b_l \end{pmatrix}\right) = \left(\begin{pmatrix} \gamma^j_\nu(t) \\ \xi^\nu_l(t) \end{pmatrix};\ P_\nu(t,s)\begin{pmatrix} a^j \\ b_l \end{pmatrix}\right).
$$

Using the fundamental matrix $P_\nu(t,s)$ gives

$$
\begin{pmatrix} (\delta\gamma_\nu)^j(t([\nu])) \\ (\delta\xi^\nu)_l(t([\nu])) \end{pmatrix} = \int_0^{t([\nu])} P_\nu(t([\nu]),s)\begin{pmatrix} 2\delta^{j'k}\ \xi^\nu_k(s)\ \delta c \\ -\left(\partial_{x^{l'}} c\ \delta c + c\ \partial_{x^{l'}}\delta c\right)|\xi^\nu(s)|^2 \end{pmatrix}\mathrm{d}s.
$$

Next we introduce the notation

$$
W_{ij}(x) = \partial_{x^i}\left(\frac{\partial_{x^j} f(x)}{|\partial_x f(x)|}\right)
$$

(recall that x is the coordinate expression for $\gamma_\nu(t([\nu]))$) and by differentiating (3.4) we find

$$
\begin{aligned}
(\delta\eta')_j &= \left(\delta_{jl} - 2\frac{\partial_{x^j} f(x)\ \partial_{x^l} f(x)}{|\partial_x f(x)|^2}\right)\delta^{ll'}\delta\eta_{l'} \\
&\quad - 2\frac{\partial_{x^j} f(x)}{|\partial_x f(x)|}W_{j'l}(x)\ \delta x^{j'}\delta^{ll'}\eta_{l'} - 2\frac{\partial_{x^l} f(x)}{|\partial_x f(x)|}W_{jj'}(x)\ \delta x^{j'}\delta^{ll'}\eta_{l'}.
\end{aligned} \tag{3.7}
$$

We mention that each of the terms on the right hand side of this equation can be expressed in terms of the Euclidean unit normal and shape or Weingarten operator for Γ. Next since $\gamma^\epsilon_\nu(t_\epsilon([\nu])) \in \Gamma$ we have

$$
f(\gamma^\epsilon_\nu(t_\epsilon([\nu]))) = 0
$$

and differentiating this gives

$$
\delta t([\nu]) = -c(x)^{-2}\frac{\partial_{x^j} f(x)\ \delta\gamma^j_\nu(t([\nu]))}{\partial_x f(x)\cdot\eta}. \tag{3.8}
$$

Thus we obtain

$$
\delta x^j = (\delta\gamma_\nu)^j(t([\nu])) - \frac{\partial_{x^{j'}} f(x)\ \delta\gamma^{j'}_\nu(t([\nu]))}{\partial_x f(x)\cdot\eta}\delta^{jl}\eta_l
$$

as well as

$$\delta\eta_l = (\delta\xi^\nu)_l(t([\nu])) + c(x)^{-1}\partial_{x^l}c(x)\ |\eta|^2\ \frac{\partial_{x^j}f(x)\ \delta\gamma_\nu^j(t([\nu]))}{\partial_x f(x)\cdot\eta}.$$

Putting the previous formulae together gives

$$\begin{pmatrix} \delta x^j \\ \delta\eta_l' \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \delta\gamma_\nu^{j'}(t([\nu])) \\ \delta\xi_{l'}^\nu(t([\nu])) \end{pmatrix}$$

where the entries in the matrix on the right hand side are given by

$$\begin{aligned} R_{11} =& \delta^j_{j'} - \frac{\partial_{x^{j'}}f(x)}{\partial_x f(x)\cdot\eta}\delta^{jl}\eta_l, \\ R_{21} =& \left(\delta_{lj} - 2\frac{\partial_{x^l}f(x)\ \partial_{x^j}f(x)}{|\partial_x f(x)|^2}\right)\delta^{jl'}c(x)^{-1}\partial_{x^{l'}}c(x)\ |\eta|^2\ \frac{\partial_{x^{j'}}f(x)}{\partial_x f(x)\cdot\eta} \\ &- 2\left(\frac{\partial_{x^l}f(x)}{|\partial_x f(x)|}W_{l'j}(x) + \frac{\partial_{x^{l'}}f(x)}{|\partial_x f(x)|}W_{lj}(x)\right)\delta^{l'k}\eta_k\left(\delta^j_{j'} - \frac{\partial_{x^{j'}}f(x)}{\partial_x f(x)\cdot\eta}\delta^{jk'}\eta_{k'}\right), \\ R_{22} =& \left(\delta_{lj} - 2\frac{\partial_{x^l}f(x)\ \partial_{x^j}f(x)}{|\partial_x f(x)|^2}\right)\delta^{jl'}. \end{aligned}$$

Next we have, using again (3.5)

$$\begin{aligned} \begin{pmatrix} (\delta\gamma_{\nu'})^j(t'([\nu])) \\ (\delta\xi^{\nu'})_l(t'([\nu])) \end{pmatrix} =& \int_0^{t'([\nu])} P_{\nu'}(t'([\nu]),s)\begin{pmatrix} 2\delta^{j'k}\ \xi_k^{\nu'}(s)\ \delta c \\ -(\partial_{x^{l'}}c\ \delta c + c\ \partial_{x^{l'}}\delta c)\,|\xi^{\nu'}(s)|^2 \end{pmatrix}\,\mathrm{d}s \\ &+ P_{\nu'}(t'([\nu]),0)\begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}\int_0^{t([\nu])} P_\nu(t([\nu]),s)\begin{pmatrix} 2\delta^{j'k}\ \xi_k^\nu(s)\ \delta c \\ -(\partial_{x^{l'}}c\ \delta c + c\ \partial_{x^{l'}}\delta c)\,|\xi^\nu(s)|^2 \end{pmatrix}\,\mathrm{d}s. \end{aligned}$$

Now, using the same technique that we applied when calculating δx and $\delta\eta$ we have

$$\delta y^j = \delta\gamma_{\nu'}^j(t'([\nu])) - \frac{\partial_{x^{j'}}\widetilde{h}\ \delta\gamma_{\nu'}^{j'}(t'([\nu]))}{\partial_x\widetilde{h}\cdot\omega}\delta^{jl}\omega_l$$

and

$$\delta\omega_l = \delta\xi_l^{\nu'}(t'([\nu])) + c^{-1}(y)\ \partial_{x^l}c(y)\ |\omega|^2\frac{\partial_{x^{j'}}\widetilde{h}\ \delta\gamma_{\nu'}^{j'}(t'([\nu]))}{\partial_x\widetilde{h}\cdot\omega}.$$

Thus

$$\begin{pmatrix} \delta y^j \\ \delta\omega_l \end{pmatrix} = \begin{pmatrix} R'_{11} & 0 \\ R'_{21} & R'_{22} \end{pmatrix}\begin{pmatrix} \delta\gamma_{\nu'}^{j'}(t'([\nu])) \\ \delta\xi_{l'}^{\nu'}(t'([\nu])) \end{pmatrix} \tag{3.9}$$

where R'_{11}, R'_{21}, and R'_{22} are given by

$$R'_{11} = \delta^j_{j'} - \frac{\partial_{x^{j'}}\widetilde{h}}{\partial_x\widetilde{h}\cdot\omega}\delta^{jl}\omega_l,$$

$$R'_{21} = c^{-1}(y)\ \partial_{\mathrm{x}^l}c(y)\ |\omega|^2\frac{\partial_{x^{j'}}\widetilde{h}}{\partial_x\widetilde{h}\cdot\omega},$$

and

$$R'_{22} = \delta_l^{l'}.$$

Combining this with the previous calculation we find that

$$\begin{aligned}\begin{pmatrix}\delta y^j\\ \delta\omega_l\end{pmatrix} &= \begin{pmatrix} R'_{11} & 0\\ R'_{21} & R'_{22}\end{pmatrix}\int_0^{t'([\nu])} P_{\nu'}(t'([\nu]),s)\begin{pmatrix} 2\delta^{j'k}\,\xi_k^{\nu'}(s)\,\delta c\\ -\left(\partial_{x^{l'}}c\,\delta c + c\,\partial_{x^{l'}}\delta c\right)|\xi^{\nu'}(s)|^2\end{pmatrix}\mathrm{d}s\\ &\quad+\begin{pmatrix} R'_{11} & 0\\ R'_{21} & R'_{22}\end{pmatrix}P_{\nu'}(t'([\nu]),0)\begin{pmatrix} R_{11} & 0\\ R_{21} & R_{22}\end{pmatrix}\\ &\quad\times\int_0^{t([\nu])} P_{\nu}(t([\nu]),s)\begin{pmatrix} 2\delta^{j'k}\,\xi_k^{\nu}(s)\,\delta c\\ -\left(\partial_{x^{l'}}c\,\delta c + c\,\partial_{x^{l'}}\delta c\right)|\xi^{\nu}(s)|^2\end{pmatrix}\mathrm{d}s.\end{aligned}\tag{3.10}$$

Since the Hamiltonian flow Ψ is homogeneous we can relate the above quantities to the differentials of the mappings $F_{c,f}\circ P_{S^*\Omega}$ and $F'_{c,f}\circ P_{S^*\Omega}$ in natural coordinates via

$$\begin{aligned}&D_{[\dot\gamma_\nu(s)^\flat\mathbf{g}]}F_{c,f}\circ D_{\dot\gamma_\nu(s)^\flat\mathbf{g}}P_{S^*\Omega}\left(\begin{pmatrix}\gamma_\nu^j(s)\\ \xi_l^\nu(s)\end{pmatrix};\begin{pmatrix}a^{j'}\\ b_{l'}\end{pmatrix}\right)\\ &\quad= D_{\dot\gamma_{\nu'}(t'([\nu]))^\flat\mathbf{g}}P_{S^*\Omega}\Bigg(\begin{pmatrix}\gamma_{\nu'}^j(t'([\nu]))\\ \xi_l^{\nu'}(t'([\nu]))\end{pmatrix};\\ &\qquad\qquad\begin{pmatrix} R'_{11} & 0\\ R'_{21} & R'_{22}\end{pmatrix}P_{\nu'}(t'([\nu]),0)\begin{pmatrix} R_{11} & 0\\ R_{21} & R_{22}\end{pmatrix}P_{\nu}(t([\nu]),s)\begin{pmatrix}a^{j'}\\ b_{l'}\end{pmatrix}\Bigg)\end{aligned}$$

and

$$\begin{aligned}&D_{[\dot\gamma_{\nu'}(s)^\flat\mathbf{g}]}F'_{c,f}\circ D_{\dot\gamma_{\nu'}(s)^\flat\mathbf{g}}P_{S^*\Omega}\left(\begin{pmatrix}\gamma_{\nu'}^j(s)\\ \xi_l^{\nu'}(s)\end{pmatrix};\begin{pmatrix}a^{j'}\\ b_{l'}\end{pmatrix}\right)\\ &\quad= D_{\dot\gamma_{\nu'}(t'([\nu]))^\flat\mathbf{g}}P_{S^*\Omega}\left(\begin{pmatrix}\gamma_{\nu'}^j(t'([\nu]))\\ \xi_l^{\nu'}(t'([\nu]))\end{pmatrix};\begin{pmatrix} R'_{11} & 0\\ R'_{21} & R'_{22}\end{pmatrix}P_{\nu'}(t'([\nu]),s)\begin{pmatrix}a^{j'}\\ b_{l'}\end{pmatrix}\right).\end{aligned}$$

From this, (3.10), and (3.6) we see that (3.1) gives the Gâteaux derivative of $R_{c,f}$ with respect to f. These coordinate expressions are linear in $\delta c\in C^2(\Omega)$ and continuous with respect to $c\in C^2(\Omega)$. Therefore $R_{c,f}$ is Fréchet differentiable as claimed with the differential given by the same formula.

To finish let us consider $T_{c,f}$. Using (3.8) we have

$$\begin{aligned}T_{c_\epsilon,f}([\nu]) = t_\epsilon + t'_\epsilon \Rightarrow \delta T_{c,f}([\nu]) &= \delta t([\nu]) + \delta t'([\nu])\\ &= -c(x)^{-2}\frac{\partial_{x^j}f(x)\,\delta\gamma_{\nu'}^j(t([\nu]))}{\partial_x f(x)\cdot\eta} - c(y)^{-2}\frac{\partial_{x^j}\widetilde{h}(y)\,\delta\gamma_{\nu}^j(t'([\nu]))}{\partial_x\widetilde{h}(y)\cdot\omega}.\end{aligned}$$

Therefore using the previous calculations

$$\begin{aligned}\delta T_{c,f}([\nu]) &= \begin{pmatrix} T'_{11} & 0\end{pmatrix}\int_0^{t'} P_{\nu'}(t',s)\begin{pmatrix} 2\delta^{j'k}\,\xi_k^{\nu'}(s)\,\delta c\\ -\left(\partial_{x^{l'}}c\,\delta c + c\,\partial_{x^{l'}}\delta c\right)|\xi^{\nu'}(s)|^2\end{pmatrix}\mathrm{d}s\\ &\quad+\left(\begin{pmatrix} T_{11} & 0\end{pmatrix}+\begin{pmatrix} T'_{11} & 0\end{pmatrix}P_{\nu'}(t',0)\begin{pmatrix} R_{11} & 0\\ R_{21} & R_{22}\end{pmatrix}\right)\\ &\quad\times\int_0^{t} P_{\nu}(t,s)\begin{pmatrix} 2\delta^{j'k}\,\xi_k^{\nu}(s)\,\delta c\\ -\left(\partial_{x^{l'}}c\,\delta c + c\,\partial_{x^{l'}}\delta c\right)|\xi^{\nu}(s)|^2\end{pmatrix}\mathrm{d}s\end{aligned}\tag{3.11}$$

where T_{11} and T'_{11} are given by

$$T_{11}(\delta\gamma_\nu(t)) = -c(x)^{-2}\frac{\partial_{x^j}f(x)}{\partial_x f(x)\cdot\eta}$$

and

$$T'_{11}(\delta\gamma_{\nu'}(t')) = -c(y)^{-2}\frac{\partial_{x^j}\widetilde{h}(y)}{\partial_x\widetilde{h}(y)\cdot\omega}.$$

Now note that (3.11) is a coordinate version of (3.2). Since the coordinate expression shows that the Gâteaux derivative is continuous in a $C^2(\Omega)$ neighborhood of c we find that $T_{c,f}$ is Fréchet differentiable with respect to c and the differential is given by (3.11) as claimed. □

3.2. Perturbation due to variation of the interface. Now we consider the case when c is held constant but f is perturbed. Many of the calculations from the proof of Theorem 3.1 will help here although we do need to introduce a few more concepts and notations before stating the analogous result. First, we note that if ξ and $\widetilde{\xi} \in T^*\Omega$ are covectors in the same fiber then ξ can be mapped to an element of $T^*_{\widetilde{\xi}}(T^*\Omega)$ by the pull-back $\pi^*_{T^*\Omega}|_{\widetilde{\xi}}$ of the natural bundle projection $\pi_{T^*\Omega} : T^*\Omega \to \Omega$ at $\widetilde{\xi}$. Further, the canonical two form $\mathfrak{w}$ on $T^*\Omega$ identifies ξ with a tangent vector $\mathfrak{i}_{\widetilde{\xi}}(\xi) \in T_{\widetilde{\xi}}(T^*\Omega)$ via the formula

$$\left[\pi^*_{T^*\Omega}|_{\widetilde{\xi}}(\xi)\right](\varphi) = \mathfrak{w}(\mathfrak{i}_{\widetilde{\xi}}(\xi), \varphi)$$

for all $\varphi \in T_{\widetilde{\xi}}(T^*\Omega)$. For $x \in \Gamma$ we will also write $P_{T^*_x\Gamma}$ for the Euclidean orthogonal projection of $T^*_x\Omega$ onto $T^*_x\Gamma$ considered as a subspace of $T^*_x\Omega$. With these we have the following result.

THEOREM 3.2. *Let Ω, U, f, and c be as described in section (2). For fixed c the maps $C_2(V) \ni f \mapsto R_{c,f} \in C(U, \partial_+S\Omega)$ and $C^2(V) \ni f \mapsto T_{c,f} \in C(U)$ are Fréchet differentiable at f and the differentials are given (for $[\nu] \in U$) by*

(3.12)

$$\begin{aligned}
[D^f R_{c,f}(f,\delta f)]([\nu]) = &- D_\omega P_{S^*\Omega} \circ \left(\mathrm{Id} - \frac{X_H(\omega)}{\langle \mathrm{d}\widetilde{h}(y), \omega\rangle}\,\pi^*_{T^*\Omega}|_\omega\,(\mathrm{d}\widetilde{h}(y))\right) \\
&\circ D_{\nu'}\Psi_{t'([\nu])}\left(\frac{\delta f(x)}{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}} D_\eta \mathcal{R}_f\left(X_H(\eta)\right)\right. \\
&\left.+ 2\,\mathfrak{i}_{\nu'}\left(\frac{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}}{|\mathrm{d}f(x)|^2_{\mathbf{g}}} P_{T^*_x\Gamma}\left(\mathrm{d}\delta f(x)\right) + \frac{\left\langle P_{T^*_x\Gamma}\left(\mathrm{d}\delta f(x)\right), \eta\right\rangle_{\mathbf{g}}}{|\mathrm{d}f(x)|^2_{\mathbf{g}}}\mathrm{d}f(x)\right)\right)
\end{aligned}$$

and

(3.13)

$$\begin{aligned}
[D^f T_{c,f}(f,\delta f)]([\nu]) = &- \frac{\delta f(x)}{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}} + \frac{\pi^*_{T^*\Omega}|_\omega\,(\mathrm{d}\widetilde{h}(y))}{\langle \mathrm{d}\widetilde{h}(y),\omega\rangle_{\mathbf{g}}}\left(D_{\nu'}\Psi_{t'([\nu])}\left(\frac{\delta f(x)}{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}} D_\eta \mathcal{R}_f\left(X_H(\eta)\right)\right.\right. \\
&\left.\left.+ 2\,\mathfrak{i}_{\nu'}\left(\frac{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}}{|\mathrm{d}f(x)|^2_{\mathbf{g}}} P_{T^*_x\Gamma}\left(\mathrm{d}\delta f(x)\right) + \frac{\left\langle P_{T^*_x\Gamma}\left(\mathrm{d}\delta f(x)\right), \eta\right\rangle_{\mathbf{g}}}{|\mathrm{d}f(x)|^2_{\mathbf{g}}}\mathrm{d}f(x)\right)\right)\right)
\end{aligned}$$

where $\omega = \dot{\gamma}_{\nu'}(t'([\nu]))^{\flat_{\mathbf{g}}}$, $y = \gamma_{\nu'}(t'([\nu]))$, $\eta = \dot{\gamma}_\nu(t([\nu]))^{\flat_{\mathbf{g}}}$, $x = \gamma_\nu(t([\nu]))$, and X_H is the Hamiltonian vector field corresponding to Hamiltonian H.

REMARK 3. *The coordinate invariant equations (3.12) and (3.13) are quite complicated, but we note that the dependence of the differentials on δf comes only through δf and $\mathrm{d}\delta f$ restricted to Γ. From the calculations in the proof coordinate formulae for both of the differentials can also be found.*

Proof. As in the proof of Theorem 3.1 we begin by introducing a perturbation

$$f_\epsilon = f + \epsilon\,\delta f.$$

We will use the same notation as before for the resulting perturbations of other objects. Additionally, we use the notations ω, y, η, and x for the invariant quantities and coordinate expressions interchangeably. Hopefully this will not cause confusion since the alternative of using different symbols for these two things causes the equations to become even longer and more unwieldy.

In this case the "source" leg γ_ν of the ray is not varied at all, but the position along that part of the ray where the reflection occurs (i.e. $t([\nu])$) does vary. Indeed, since

$$f_\epsilon(\gamma_\nu(t_\epsilon([\nu])) = 0$$

we have in the natural coordinates

$$\delta f(x) + \partial_{x^j} f(x) \dot{\gamma}_\nu^j(t([\nu]))\delta t([\nu]) = 0 \Rightarrow \delta t([\nu]) = -c(x)^{-2}\frac{\delta f(x)}{\partial_x f(x)\cdot\eta}.$$

We use this to get

$$\delta x^j = -\frac{\delta f(x)}{\partial_x f(x)\cdot\eta}\delta^{jl}\eta_l$$

and

$$\delta\eta_l = c(x)^{-1}\partial_{x^l} c(x)\ |\eta|^2\frac{\delta f(x)}{\partial_x f(x)\cdot\eta}.$$

Note that in coordinates

$$-\frac{\delta f(x)}{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}} X_H(\eta) = \left(\left(\begin{array}{c} x^j \\ \eta_l \end{array}\right);\left(\begin{array}{c} \delta x^j \\ \delta\eta_l \end{array}\right)\right). \tag{3.14}$$

Next we will use (3.4) to calculate $\delta\eta'$ in this case. To simplify the notation we introduce

$$\delta N_j(x) = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}\frac{\partial_{x^j} f_\epsilon(x)}{|\partial_x f_\epsilon(x)|} = \frac{\partial_{x^j}\delta f(x)}{|\partial_x f(x)|} - \frac{\partial_{x^j} f}{|\partial_x f(x)|^3}\langle\partial_x f, \partial_x \delta f\rangle$$

for the variation in the Euclidean unit conormal to Γ at x. With this notation we have

$$\begin{aligned}\delta\eta_l' = {} & \left(\delta_l^{l'} - 2\frac{\partial_{x^l} f(x)\partial_{x^j} f(x)}{|\partial_x f(x)|^2}\delta^{jl'}\right)\delta\eta_{l'} - 2\frac{\partial_{x^j} f(x)}{|\partial_x f(x)|}W_{j'l}(x)\ \delta x^{j'}\delta^{ll'}\eta_{l'} \\ & - 2\frac{\partial_{x^l} f(x)}{|\partial_x f(x)|}W_{jj'}(x)\ \delta x^{j'}\delta^{ll'}\eta_{l'} - 2\ \delta N_l(x)\ \frac{\langle\partial_x f(x),\eta\rangle}{|\partial_x f(x)|} - 2\frac{\partial_{x^l} f(x)}{|\partial_x f(x)|}\langle\delta N(x),\eta\rangle.\end{aligned}$$

We point out that the top line in this equation is a coordinate version of the "$\delta\eta'$ component" of the invariant expression

$$-\frac{\delta f(x)}{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}} D_\eta\mathcal{R}_f\Big(X_H(\eta)\Big)$$

while the second line corresponds with a coordinate version of

$$-2\ \mathrm{i}_{\nu'}\left(\frac{\langle \mathrm{d}f(x),\eta\rangle_{\mathbf{g}}}{|\mathrm{d}f(x)|_{\mathbf{g}}^2}P_{T_x^*\Gamma}\Big(\mathrm{d}\delta f(x)\Big) + \frac{\Big\langle P_{T_x^*\Gamma}\Big(\mathrm{d}\delta f(x)\Big),\eta\Big\rangle_{\mathbf{g}}}{|\mathrm{d}f(x)|_{\mathbf{g}}^2}\mathrm{d}f(x)\right).$$

Next we have

$$\left(\begin{array}{c}\delta\gamma_{\nu'}(t'([\nu])) \\ \delta\xi^{\nu'}(t'([\nu]))\end{array}\right) = P_{\nu'}(t'([\nu]),0)\left(\begin{array}{c}\delta x \\ \delta\eta'\end{array}\right)$$

and using (3.9)

$$\begin{pmatrix} \delta y \\ \delta\omega \end{pmatrix} = \begin{pmatrix} R'_{11} & 0 \\ R'_{21} & R'_{22} \end{pmatrix} P_{\nu'}(t'([\nu]),0) \begin{pmatrix} \delta x \\ \delta\eta' \end{pmatrix}$$

where the entries in the matrix are the same as in (3.9). Finally, noting that in the natural coordinates on $T^*(T^*\Omega)$

$$\pi_{T^*\Omega}\big|_\omega (\mathrm{d}\widetilde{h}(y)) = \left(\begin{pmatrix} y \\ \omega \end{pmatrix} ; \begin{pmatrix} \partial_x h(y) \\ 0 \end{pmatrix} \right)$$

we see using coordinates

$$\frac{\pi_{T^*\Omega}\big|_\omega (\mathrm{d}\widetilde{h}(y))}{\langle \mathrm{d}\widetilde{h}(y), \omega\rangle_{\mathbf{g}}} \left(\begin{pmatrix} y \\ \omega \end{pmatrix} ; \begin{pmatrix} \delta\gamma_{\nu'}(t'([\nu])) \\ \delta\xi^{\nu'}(t'([\nu])) \end{pmatrix} \right) = c(y)^{-2} \frac{\partial_{x^j}\widetilde{h}(y)\delta\gamma^j_{\nu'}(t'([\nu]))}{\partial_x \widetilde{h}(y)\cdot\omega} = -\delta t'([\nu])$$

Combining this with the calculations in the proof of Theorem 3.1, and for $R_{c,f}$ projecting onto $S^*\Omega$, we find that the Gâteaux derivatives of $R_{c,f}$ and $T_{c,f}$ with respect to f are given by the invariant formulae (3.12) and (3.13). Since these Gâteaux derivatives are linear and continuous with respect to $f \in C^2(V)$ we conclude that $R_{c,f}$ and $T_{c,f}$ are Fréchet differentiable with differentials given by the formulae (3.12) and (3.13). □

4. Multiple reflections. The calculations of sections 3.1 and 3.2 can be extended to the case in which the rays are reflected multiple times before returning to the surface. Indeed let us suppose that the rays beginning at ν for $[\nu] \in U$ reflect m times at the points $\{x_j([\nu])\}_{j=1}^m \in \Gamma$, that the jth leg of each such ray begins transverse to either Γ or $\partial\Omega$ at covector $(\nu')^j([\nu])$ and intersects either Γ or $\partial\Omega$ transversally at covector $\nu^j([\nu]) \in T^*_{x_j}\Omega$, and that the travel time of the jth leg is $t^j([\nu])$. We note that $(\nu')^{j+1}$ and ν^j are related by (2.2). The multiply reflected scattering relation $R_{c,f}$ and $T_{c,f}$ are then defined in the same manner as in the singly scattered case by

$$R_{c,f}([\nu]) = [(\nu')^{m+1}] \quad , \quad \text{and} \quad T_{c,f}([\nu]) = \sum_{j=1}^{m+1} t^j([\nu]).$$

The sets $\left\{\widetilde{U^j}\right\}_{j=1}^{m+1}$ will be defined by

$$\widetilde{U^j} = \left\{(P_{S^*\Omega}\circ\Psi_s)(\nu^j([\nu])) \ : \ [\nu]\in U, \ s\in(0,t^j([\nu])\right\} \subset S^*\Omega$$

and as in section 3 the maps $\Phi^j([\nu],s) = (P_{S^*\Omega}\circ\Psi_s)(\nu^j([\nu]))$ are diffeomorphisms on their domains and we define mappings $F^j_{c,f} : \widetilde{U^j} \to \partial_+ S^*\Omega$ by

$$F^j_{c,f}(\omega) = R_{c,f}(\pi_\nu \circ (\Phi^j)^{-1}(\omega))$$

and $\tau^j_{c,f} : \widetilde{U^j} \to \mathbb{R}$ by

$$\tau^j_{c,f} : \left(T_{c,f} - \sum_{l=1}^{j-1} t^j \right)(\pi_\nu\circ(\Phi^j)^{-1}(\omega)) - \pi_s\circ(\Phi^j)^{-1}(\omega).$$

In order to study perturbations of the interface we also introduce sets $\{W^j\}_{j=1}^m$ and $\{(W')^j\}_{j=1}^m$

$$W^j = \{\lambda\nu^j([\nu]) \ : \ \nu\in U, \ \lambda\in\mathbb{R}^+\} \subset T^*\Omega,$$

$$(W')^j = \{\lambda(\nu')^j([\nu]) \ : \ \nu\in U, \ \lambda\in\mathbb{R}^+\} \subset T^*\Omega$$

which are embedded submanifolds of $T^*\Omega$ diffeomorphic to $\mathbb{R}^+ \times U$ via the maps $\widetilde{\Phi_{W^j}} : (\lambda, [\nu]) \mapsto \lambda\nu^j([\nu])$ and $\widetilde{\Phi_{W^j}} : (\lambda, [\nu]) \mapsto \lambda(\nu')^j([\nu])$. Now define mappings $\widetilde{\Psi}^j : (W')^j \to W^j$ given by

$$\widetilde{\Psi}^j(\xi) = \Psi_{t^j\left(\pi_\nu \circ \widetilde{\Phi_{W^j}}^{-1}(\xi)\right)}(\xi)$$

and $\widetilde{R^j} : (W')^j \to \partial_+ T^*\Omega$

$$\widetilde{R^j} = \widetilde{\Psi}^{m+1} \circ \prod_{j'=j}^{m} \mathcal{R}_f \circ \widetilde{\Psi}^{j'}.$$

Intuitively $\widetilde{\Psi}^j$ maps the initial covector for the jth leg of a reflected ray to the final covector of the jth leg of the same ray while $\widetilde{R^j}$ maps the initial covector of the jth leg of a ray to the covector at which the ray finally meets $\partial\Omega$.

Finally, given $\delta f \in C^2(V)$ we define vector fields φ^j on $(W')^j$ for $j = 2$ to $m+1$ by the formulae

$$\begin{aligned}\varphi^j((\nu')^j) = &\frac{\delta f(x_{j-1})}{\langle \mathrm{d}f(x_{j-1}), \nu^{j-1}\rangle_{\mathbf{g}}} D_{\nu^{j-1}}\mathcal{R}_f\Big(X_H(\nu^{j-1})\Big)\\ &+ 2\ \mathrm{i}_{(\nu')^j}\left(\frac{\langle \mathrm{d}f(x_{j-1}), \nu^{j-1}\rangle_{\mathbf{g}}}{|\mathrm{d}f(x_{j-1})|^2_{\mathbf{g}}} P_{T^*_{x_{j-1}}\Gamma}\Big(\mathrm{d}\delta f(x_{j-1})\Big)\right.\\ &\qquad\left.+ \frac{\Big\langle P_{T^*_{x_{j-1}}\Gamma}\Big(\mathrm{d}\delta f(x_{j-1})\Big), \nu^{j-1}\Big\rangle_{\mathbf{g}}}{|\mathrm{d}f(x_{j-1})|^2_{\mathbf{g}}} \mathrm{d}f(x_{j-1})\right).\end{aligned}$$

The field φ^j gives the perturbation in the ray created by the perturbation of the interface at the $(j-1)$st reflection given by δf.

Now we can give the generalization of Theorems 3.1 and 3.2 to the case of multiple reflections.

THEOREM 4.1. *Let Ω, U, f, and c be as described in section (2) and suppose the rays are multiply reflected m times as described above. For fixed f the maps $C^2(\Omega) \ni c \mapsto R_{c,f} \in C(U, \partial_+ S\Omega)$ and $C^2(\Omega) \ni c \mapsto T_{c,f} \in C(U)$ are Fréchet differentiable at c and the differentials are given (for $[\nu] \in U$) by*

$$[D^c R_{c,f}(c, \delta c)]([\nu]) = \sum_{j=1}^{m+1} \int_0^{t^j([\nu])} D_{[\dot\gamma_{(\nu')^j}(s)^{\flat_{\mathbf{g}}}]} F^j_{c,f} \circ D_{\dot\gamma_{(\nu')^j}(s)^{\flat_{\mathbf{g}}}} P_{S^*\Omega}(X_{\delta H}(\dot\gamma_{(\nu')^j}(s)^{\flat_{\mathbf{g}}}))\ \mathrm{d}s \tag{4.1}$$

and

$$[D^c T_{c,f}(c, \delta c)]([\nu]) = \sum_{j=1}^{m+1} \int_0^{t^j([\nu])} D_{[\dot\gamma_{(\nu')^j}(s)^{\flat_{\mathbf{g}}}]} \tau^j_{c,f} \circ D_{\dot\gamma_{(\nu')^j}(s)^{\flat_{\mathbf{g}}}} P_{S^*\Omega}(X_{\delta H}(\dot\gamma_{(\nu')^j}(s)^{\flat_{\mathbf{g}}}))\ \mathrm{d}s \tag{4.2}$$

where $X_{\delta H}$ is the Hamiltonian vector field given by the Hamiltonian $\delta H(\xi) = c\ \delta c\ |\xi|^2_{\mathbf{e}} \in C^2(T^*\Omega)$. For fixed c the maps $C_2(V) \ni f \mapsto R_{c,f} \in C(U, \partial_+ S\Omega)$ and $C^2(V) \ni f \mapsto T_{c,f} \in C(U)$ are Fréchet differentiable at f and the differentials are given (for $[\nu] \in U$) by

$$[D^f R_{c,f}(f, \delta f)]([\nu]) = -\sum_{j=2}^{m+1} D_{\widetilde{R}^j((\nu')^j)} P_{S^*\Omega} \circ D_{(\nu')^j} \widetilde{R}^j\left(\varphi^j\left((\nu')^j([\nu])\right)\right) \tag{4.3}$$

and

(4.4)
$$[D^f T_{c,f}(f,\delta f)]([\nu]) = \sum_{j=2}^{m+1}\Bigg(-\frac{\delta f(x_{j-1})}{\langle \mathrm{d}f(x_{j-1}),\nu^{j-1}\rangle_{\mathbf{g}}}$$
$$+\sum_{l=j}^{m}\frac{\pi^*_{T^*\Omega}|_{\nu^l}(\mathrm{d}f(x_l))}{\langle \mathrm{d}f(x_l),\nu^l\rangle_{\mathbf{g}}} D_{(\nu')^l}\Psi_{t^l([\nu])}\left(\prod_{l'=j}^{l-1} D_{\nu^{l'}}\mathcal{R}_f\circ D_{(\nu')^{l'}}\widetilde{\Psi}^{l'}\right)\Big(\varphi^j\big((\nu')^j([\nu])\big)\Big)$$
$$+\frac{\pi^*_{T^*\Omega}|_{\nu^l}(\mathrm{d}\widetilde{h}(x_l))}{\langle \mathrm{d}\widetilde{h}(x_l),\nu^l\rangle_{\mathbf{g}}} D_{(\nu')^l}\Psi_{t^l([\nu])}\left(\prod_{l'=j}^{m} D_{\nu^{l'}}\mathcal{R}_f\circ D_{(\nu')^{l'}}\widetilde{\Psi}^{l'}\right)\Big(\varphi^j\big((\nu')^j([\nu])\big)\Big)\Bigg).$$

In these formulae sums in which the bottom index is greater than the top are interpreted as zero while such products are interpreted as the identity mapping.

REMARK 4. *We comment on the meaning of each of the terms in the formulae of Theorem 4.1. In (4.1) and (4.2) the jth term in the sum gives the contribution of the perturbation of the wave speed c along the jth leg of the ray. In (4.3) the jth term of the sum gives the contribution due to the perturbation of the interface at the $(j-1)$st reflection point. The same is true of the outer sum in (4.4). The first term within the sum gives the perturbation of the travel time along the $(j-1)$st leg of the ray from the perturbation at the $(j-1)$st reflection, while the lth term in the second sum gives the perturbation in the travel time of the lth leg due to the perturbation of the $(j-1)$st reflection point. The final term gives the pertubation of the final leg due to the perturbation of the $(j-1)$st reflection point.*

REMARK 5. *In fact Theorems 3.1 and 3.2 are special cases of Theorem 4.1.*

We do not include a proof of Theorem 4.1 but comment that the same essential method works as for the previous theorems. Looking in coordinates shows that the given formulas are the Gâteaux derivatives for a perturbation of either c or f and the continuity in C^2 then shows the Fréchet differentiability.

5. Parametrization by the source and receiver locations. When we parametrize reflected rays between points on $\partial\Omega$ by the starting and ending points, which we refer to as the source and receiver locations and label here as y_s and y_r respectively, the linearization of travel time becomes much nicer. Indeed, let us suppose that we have have two open set U_s and U_r in $\partial\Omega$, and that every pair $(y_s,y_r)\in U_s\times U_r$ corresponds continuously with a ray traveling through $\partial\Omega$, reflecting transversally at m places on Γ, and intersecting $\partial\Omega$ transversally at y_s and y_r. We will label this ray as γ_{y_s,y_r}. The travel time along the entire ray gives a function $T_{c,f}\in C(U_s\times U_r)$ and with this we have the following result.

THEOREM 5.1. *For fixed f the map $C^2(\Omega)\ni c\mapsto T_{c,f}\in C(U_s\times U_r)$ is Fréchet differentiable with Fréchet differential given by*

$$[D^c T_{c,f}(c,\delta c)](y_s,y_r) = -\int_0^{T_{c,f}(y_s,y_r)} c(\gamma_{y_s,y_r}(s))^{-1}\delta c(\gamma_{y_s,y_r}(s))\,\mathrm{d}s. \tag{5.1}$$

Proof. Suppose that we introduce a perturbation

$$c_\epsilon = c+\epsilon\delta c$$

of the wave speed and label the resulting piecewise smooth reflected ray with respect to the perturbed metric as $\gamma^\epsilon_{y_s,y_r}$. Note that $T_{c,f}(y_s,y_r)$ is the arc length of $\gamma_{y_s,y_r}([0,T_{c,f}(y_s,y_r)])$ with respect to the metric $\mathbf{g}$, and so $T_{c_\epsilon,f}(y_s,y_r)$ is the arc length of $\gamma^\epsilon_{y_s,y_r}([0,T_{c,f}(y_s,y_r)])$ with respect to $\mathbf{g}^\epsilon = c_\epsilon^{-2}\mathbf{e}$. Since arc length is independent of parametrization we can introduce a parametrization $\widetilde{\gamma}^\epsilon_{y_s,y_r}$ of $\gamma^\epsilon_{y_s,y_r}([0,T_{c,f}(y_s,y_r)])$ so that for any ϵ the domain is $t\in[0,T_{c,f}(y_s,y_r)]$ and $\widetilde{\gamma}^0_{y_s,y_r}$ is parametrized by arc length with respect to $\mathbf{g}$. Then we have

$$T_{c_\epsilon,f}(y_s,y_r) = \int_0^{T_{c,f}(y_s,y_r)} |\dot{\widetilde{\gamma}}^\epsilon(s)|_{\mathbf{e}}\ c_\epsilon(\widetilde{\gamma}^\epsilon(s))^{-1}\ \mathrm{d}s$$

and so, using that $|\dot\gamma_{y_s,y_r}(s)|\ c(\gamma_{y_s,y_r}(s)^{-1} = 1$,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} T_{c_\epsilon,f}(y_s,y_r) = -\int_0^{T_{c,f}(y_s,y_r)} c(\gamma_{y_s,y_r}(s))^{-1}\delta c(\gamma_{y_s,y_r}(s))\ \mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} A_{\mathbf{g}}[\gamma^\epsilon_{y_s,y_r}]$$

where $A_{\mathbf{g}}[\gamma^\epsilon_{y_s,y_r}]$ is the arc length of $\gamma^\epsilon_{y_s,y_r}([0,T_{c,f}(y_s,y_r)])$ with respect to $\mathbf{g}$. The first variation formula for the arc length functional of piecewise smooth curves can now be applied to show that the last term in the previous equation is zero, and so the Gâteaux derivative is given by the formula in the statement of the theorem. The Fréchet differentiability follows as usual.□

6. Microlocal analysis of reflected X-ray transforms. Having computed in the previous sections the linearization of the problem of recovering the wavespeed c and level set function f for Γ from both the reflected scattering relation and travel time we now focus on the inversion of the linear problem for recovery of δc. For the case considered in section 5 we saw in Theorem 5.1 that the linearized problem is exactly inversion of the "reflected geodesic X-ray transform" of δc given by (5.1). Also, we point out that considered in coordinates as a transform acting on $X_{\delta H}$ the formulas (4.1) and (4.2) may be considered as systems of weighted reflected geodesic X-ray transforms and so in this section we will begin to study the microlocal analysis of such transforms.

First we will state more precisely what we mean by the reflected geodesic X-ray transform. Here we will consider only the case of single reflections. We warn the reader that some of the notations in this section are different from those given in the previous sections, and in order to apply microlocal analysis we must assume that the relevant functions are C^∞. The set Ω and reflecting surface Γ will be as before although we assume now that Γ and $\partial\Omega$ are C^∞. Let $\mathbf{g}$ be a smooth (C^∞) Riemannian metric on Ω (we do not assume that $\mathbf{g}$ is conformal to $\mathbf{e}$). Let U and U' be open subsets compactly contained in $\partial_- S_{\mathbf{g}}\Omega$ (the set of inward pointing unit vectors at the boundary of Ω) such that $U' \Subset U$. Suppose that $\phi \in C_c^\infty(\partial_- S_{\mathbf{g}}\Omega)$ is a cutoff function equal to 1 on U' and equal to 0 on the complement of U.

We will now define the weighted reflected geodesic X-ray transform restricted to U. Given $\nu \in T\Omega$, let γ_ν be the maximally extended geodesic for $\mathbf{g}$ with initial condition $\dot\gamma_\nu(0) = \nu$. For every $\nu \in U$ we suppose that there is a reflected geodesic consisting of two legs ending at $\partial_+ S_{\mathbf{g}}\Omega$. In particular this means that γ_ν intersects the reflecting interface $\Gamma \subset \Omega$ transversally at some time $t > 0$, $\gamma_\nu([0,t]) \notin \partial\Omega$, and $\gamma_{\nu'}$ intersects $\partial\Omega$ transversally at a time $t' > 0$ where

$$\nu' = \dot\gamma_\nu(t) - 2\ \mathrm{d}f^{\sharp_g}_{\gamma_\nu(r)}\mathrm{d}f_{\gamma_\nu(t)}(\dot\gamma_\nu(t)) := \mathcal{R}_f(\dot\gamma_\nu(t))$$

(this is the tangent vector version of (2.2)) and $\gamma_{\nu'}([0,t']) \notin \partial\Omega$. Let $t(\nu)$ and $t'(\nu)$ be the travel times for the two legs as functions of the initial vector $\nu \in U$.

Definition 6.1. *Let $w \in C^\infty(S_{\mathbf{g}}\Omega^{int})$. For $\varphi \in C_c^\infty(\Omega^{int})$ we define the weighted reflected geodesic X-ray transform of φ to be a function on U defined by*

$$\begin{aligned}\mathcal{I}_{g,w,\Gamma}[\varphi](\nu) &= \phi(\nu)\int_0^{t(\nu)} w(\dot\gamma_\nu(s))\ \varphi(\gamma_\nu(s))\ \mathrm{d}s + \phi(\nu)\int_0^{t'(\nu')} w(\dot\gamma_{\nu'}(s))\ \varphi(\gamma_{\nu'}(s))\ \mathrm{d}s\\ &= \mathcal{I}^1_{g,w,\Gamma}[\varphi](\nu) + \mathcal{I}^2_{g,w,\Gamma}[\varphi](\nu).\end{aligned}$$

We are defining $\mathcal{I}^1_{g,w,\Gamma}[\varphi](\nu)$ and $\mathcal{I}^2_{g,w,\Gamma}[\varphi](\nu)$ to be the first and second integrals respectively.

Since we have required that all intersections are transversal in fact $\mathcal{I}_{g,w,\Gamma}[\varphi] \in C^\infty(U)$. Our object now is to apply methods of microlocal analysis to understand the associated normal operator

$N_{g,w,\Gamma} = (\mathcal{I}_{g,w,\Gamma})^t \circ \mathcal{I}_{g,w,\Gamma}$ where $(\mathcal{I}_{g,w,\Gamma})^t$ is the transpose with respect to some measure on $\partial_- S_{\mathbf{g}}\Omega$. Of course we have

$$N_{g,w,\Gamma} = \left(\mathcal{I}^1_{g,w,\Gamma}\right)^t \circ \mathcal{I}^1_{g,w,\Gamma} + \left(\mathcal{I}^2_{g,w,\Gamma}\right)^t \circ \mathcal{I}^1_{g,w,\Gamma} + \left(\mathcal{I}^1_{g,w,\Gamma}\right)^t \circ \mathcal{I}^2_{g,w,\Gamma} + \left(\mathcal{I}^2_{g,w,\Gamma}\right)^t \circ \mathcal{I}^2_{g,w,\Gamma} \tag{6.1}$$

and we will analyze each of these operators using the calculus of Fourier Integral Operators (FIOs). We will make extensive use of the clean composition calculus for FIOs and use as a reference for this [7, Section 25.2].

We begin by writing $\mathcal{I}^1_{g,w,\Gamma}$ and $\mathcal{I}^2_{g,w,\Gamma}$ each as compositions of two operators which are simpler to understand. First, let us introduce the natural projection map $\pi_\Omega : S_{\mathbf{g}}\Omega^{int} \to \Omega^{int}$. Since π_Ω has surjective differential and the fibers over every point in Ω^{int} are all compact the pullback $\pi^*_\Omega : C^\infty_c(\Omega^{int}) \to C^\infty_c(S_{\mathbf{g}}\Omega^{int})$ is a properly supported FIO in the space $\mathcal{I}^{(1-n)/4}_1(S_{\mathbf{g}}\Omega^{int} \times \Omega^{int}, \Lambda'_{\pi^*_\Omega})$ where

$$\Lambda_{\pi^*_\Omega} = \left\{((D\pi_\Omega)^t_\omega\ \xi, \xi)\ :\ \omega \in S_{\mathbf{g}}\Omega^{int},\ \xi \in T^*_{\pi_\Omega(\omega)}\Omega\right\}$$

is a canonical relation. By the calculus of FIOs the transpose $(\pi^*_\omega)^t$ is also an FIO in the space $\mathcal{I}^{(1-n)/4}_1(\Omega^{int} \times S_{\mathbf{g}}\Omega^{int}, (\Lambda^t_{\pi^*_\Omega})')$ where

$$\Lambda^t_{\pi^*_\Omega} = \left\{(\xi, (D\pi_\Omega)^t_\omega\ \xi)\ :\ \omega \in S_{\mathbf{g}}\Omega^{int},\ \xi \in T^*_{\pi_\Omega(\omega)}\Omega\right\}.$$

We view $\mathcal{I}^1_{g,w,\Gamma}$ as a composition of π^*_Ω with the operator $\mathcal{I}^{1,S}_{g,w,\Gamma} : C^\infty_c(S_{\mathbf{g}}\Omega^{int}) \to C^\infty_c(U)$ defined for $\varphi \in C^\infty_c(S_{\mathbf{g}}\Omega^{int})$ by

$$\mathcal{I}^{1,S}_{g,w,\Gamma}[\varphi](\nu) = \phi(\nu) \int_0^{t(\nu)} w(\dot\gamma_\nu(s))\ \varphi(\dot\gamma_\nu(s))\ \mathrm{d}s$$

and $\mathcal{I}^2_{g,w,\Gamma}$ as a composition of π^*_Ω with the operator $\mathcal{I}^{2,S}_{g,w,\Gamma} : C^\infty_c(S_{\mathbf{g}}\Omega^{int}) \to C^\infty_c(U)$ defined by

$$\mathcal{I}^{2,S}_{g,w,\Gamma}[\varphi](\nu) = \phi(\nu) \int_0^{t'(\nu)} w(\dot\gamma_{\nu'}(s))\ \varphi(\dot\gamma_{\nu'}(s))\ \mathrm{d}s$$

It is then straightforward to see that

$$\mathcal{I}^j_{g,w,\Gamma} = \mathcal{I}^{j,S}_{g,w,\Gamma} \circ \pi^*_\Omega$$

for $j = 1$ or 2. We now proceed to analyze the two operators $\{\mathcal{I}^{j,S}_{g,w,\Gamma}\}^2_{j=1}$. To this end, define the sets

$$\widetilde{U^1}\,\{\dot\gamma_\nu(t)\ :\ \nu \in U, \quad 0 < t < t(\nu)\}$$

and

$$\widetilde{U^2}\,\left\{\dot\gamma_{\nu'(\nu)}(t)\ :\ \nu \in U, \quad 0 < t < t'(\nu)\right\}$$

which are both open subsets of $S_{\mathbf{g}}\Omega^{int}$. The mappings

$$\Phi^1 : \{(\nu, t) \in U \times \mathbb{R}\ : 0 < t < t(\nu)\} \to \widetilde{U^1} \quad \text{and} \quad \Phi^1 : \{(\nu, t) \in U \times \mathbb{R}\ : 0 < t < t'(\nu)\} \to \widetilde{U^2}$$

defined by

$$\Phi^1(\nu, t) = \dot\gamma_\nu(t) \quad \text{and} \quad \Phi^2(\nu, t) = \dot\gamma_{\nu'}(t)$$

are diffeomorphisms and we define two smooth mappings using their inverses by

$$F_j(\omega) = \pi_\nu \circ (\Phi^j)^{-1}(\omega)$$

for $j = 1$ and 2. These maps take ω in $\widetilde{U^j}$ to the initial vector ν for the reflected ray passing through ω. We now note that another expression for $\mathcal{I}^{j,S}_{g,w,\Gamma}$ is the following

$$\mathcal{I}^{j,S}_{g,w,\Gamma}[\varphi](\nu) = \phi(\nu) \int_{F_j^{-1}(\{\nu\})} w(\omega)\ \varphi(\omega)\ \mathrm{d}H^1_g(\omega)$$

in which $\mathrm{d}H^1_g$ is the one-dimensional Hausdorff measure on $S_{\mathbf{g}}\Omega^{int}$ induced by the metric g. Applying the coarea formula (see [5, Theorem 3.1]) we have for any $\widetilde{\varphi} \in C^\infty_c(U)$

$$\begin{aligned}\int_U \widetilde{\varphi}(\nu)\ \phi(\nu) \left(\int_{F_j^{-1}(\{\nu\})} w(\omega)\ \varphi(\omega)\ \mathrm{d}H^1_g \right)\ \mathrm{d}(\partial_- S_{\mathbf{g}}\Omega)(\nu) \\ = \int_{S_{\mathbf{g}}\Omega^{int}} \phi(F_j(\omega))\ w(\omega)\ \varphi(\omega)\ \widetilde{\varphi}(F_j(\omega))\ \mathcal{J}F_j(\omega)\ \mathrm{d}S_{\mathbf{g}}\Omega^{int}(\omega)\end{aligned}$$

where the integrand on the right is interpreted as 0 outside of $\widetilde{U^j}$. The functions $\mathcal{J}F_j$ are defined in [5] and are in $C^\infty(S_{\mathbf{g}}\Omega^{int})$ since F is a submersion. Thus $\mathcal{I}^{j,S}_{g,w,\Gamma} \in \mathcal{I}^{-1/4}_1(\partial_- S_{\mathbf{g}}\Omega \times S(\Omega^{int} \setminus \Gamma), \Lambda'_{\mathcal{I}^{j,S}_{g,w,\Gamma}})$ where

$$\Lambda_{\mathcal{I}^{j,S}_{g,w,\Gamma}} = \left\{ (\xi, (DF_j)^t_\omega\ \xi)\ :\ \omega \in \widetilde{U^j}\ ,\ \xi \in T^*_{F_j(\omega)} \partial_- S_{\mathbf{g}}\Omega \right\}.$$

We can see immediately from the previous formula that the formal transpose of $\mathcal{I}^{j,S}_{g,w,\Gamma}$ is given by

$$\left(\mathcal{I}^{j,S}_{g,w,\Gamma} \right)^t [\widetilde{\varphi}]\,(\omega) = \phi(F_j(\omega))\ \mathcal{J}F_j(\omega)\ \widetilde{\varphi}(F_j(\omega)).$$

Since $\Lambda_{\mathcal{I}^{j,S}_{g,w,\Gamma}}$ is a canonical relation by the calculus of FIOs we may conclude that $\left(\mathcal{I}^{j,S}_{g,w,\Gamma} \right)^t \in \mathcal{I}^{-1/4}_1(S_{\mathbf{g}}(\Omega^{int} \setminus \Gamma) \times \partial_- S_{\mathbf{g}}\Omega, (\Lambda^t_{\mathcal{I}^{j,S}_{g,w,\Gamma}})')$ where

$$\Lambda^t_{\mathcal{I}^{j,S}_{g,w,\Gamma}} = \left\{ ((DF_j)^t_\omega\ \xi, \xi)\ :\ \omega \in \widetilde{U^j}, \xi \in T^*_{F_j(\omega)} \partial_- S_{\mathbf{g}}\Omega \right\}.$$

Now we are prepared to apply these results to analyze the operator $\mathcal{N}_{g,w,\Gamma}$ defined by (6.1). We will prove a theorem below (see Theorem 6.2) concerning one case in which it can be shown that $\mathcal{N}_{g,w,\Gamma}$ is a pseudodifferential operator on $\Omega^{int} \setminus \Gamma$, but we first we continue to show how far the clean composition calculus alone can carry us in the general case.

Let us examine $(\mathcal{I}^1_{g,w,\Gamma})^t \circ \mathcal{I}^1_{g,w,\Gamma}$. By the clean composition calculus for FIOs and the calculations above $\left(\mathcal{I}^{1,S}_{g,w,\Gamma} \right)^t \circ \mathcal{I}^{1,S}_{g,w,\Gamma} \in \mathcal{I}^{-1/2}_1 \left(S_{\mathbf{g}}(\Omega^{int} \setminus \Gamma) \times S_{\mathbf{g}}(\Omega^{int} \setminus \Gamma), \left(\Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \right)' \right)$ where we can calculate

$$\Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} = \left\{ \left((DF_1)^t_{\widetilde{\omega}}\ \xi, (DF_1)^t_\omega\ \xi \right)\ :\ \omega,\ \widetilde{\omega} \in \widetilde{U^1},\ F_1(\omega) = F_1(\widetilde{\omega}),\ \xi \in T^*_{F_1(\omega)} \partial_- S_{\mathbf{g}}\Omega \right\}.$$

Then, precomposing with π^*_Ω and again using the calculus we find that

$$\left(\mathcal{I}^{1,S}_{g,w,\Gamma} \right)^t \circ \mathcal{I}^{1,S}_{g,w,\Gamma} \circ \pi^*_\Omega \in \mathcal{I}^{-(n+1)/4}_1 \left(S_{\mathbf{g}}(\Omega^{int} \setminus \Gamma) \times \Omega^{int} \setminus \Gamma, \left(\Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega} \right)' \right)$$

where

$$\Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega} = \Big\{ \big((DF_1)^t_{\widetilde{\omega}}\ \xi, \eta\big)\ :\ \widetilde{\omega} \in \widetilde{U^1},\ \xi \in T^*_{F_1(\widetilde{\omega})}\partial_- S_{\mathbf{g}}\Omega,\ \exists\ \omega \in \widetilde{U^1} \text{ such that } F_1(\omega) = F_1(\widetilde{\omega}),\ \eta \in T^*_{\pi_\Omega(\omega)}\Omega^{int},\ (D\pi_\Omega)^t_{\pi_\Omega(\omega)}\eta = (DF_1)^t_\omega \xi \Big\}.$$

Finally, we need to compose this with $(\pi^*_\Omega)^t$ to obtain the first term in (6.1), but the FIO calculus fails us here in the general case. Indeed, this composition is not an FIO in general, although we expect that it can be decomposed into a sum of terms which are each FIOs including one pseudodifferential operator (ΨDO) of order -1. If we make a fairly strong simplifying assumption about the geometry of the reflected rays then we can show that $\big(\mathcal{I}^1_{g,w,\Gamma}\big)^t \circ \mathcal{I}^1_{g,w,\Gamma}$ (and in fact $\mathcal{N}_{g,w,\Gamma}$) is a pseudodifferential operator as in the following theorem. For the theorem we introduce the following definition. First let Ψ_s be the geodesic flow in $S_{\mathbf{g}}\Omega$ (i.e. $\Psi_s(\omega) = \dot{\gamma}_\omega(s)$) Two points $\gamma_\nu(s)$ and $\gamma_{\nu'}(s')$ will be called conjugate along the reflected geodesic corresponding to $\nu \in U$ if

$$D\pi_\Omega \circ D\Psi_{s'} \circ D\mathcal{R} \circ D\Psi_{t(\nu)-s}$$

restricted to the tangent space of the fiber over $\gamma_\nu(s)$ in $S_{\mathbf{g}}\Omega$ is injective at $\dot{\gamma}_\nu(s)$. Intuitively this is the requirement that the "reflected exponential map" is a local diffeomorphism. Now we have the following.

THEOREM 6.2. *Assume that there are no conjugate points along any of the reflected geodesics. Then $\mathcal{N}_{g,w,\Gamma}$ acting on $C^\infty_c(\Omega^{int} \setminus \Gamma)$ is a pseudodifferential operator of order -1.*

Proof. We examine each of the terms in (6.1) separately starting with the first which we have already begun analyzing prior to the statement of the theorem. To prove that the first term is a ΨDO it is sufficient to show that the composition of the canonical relations $\Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega}$ and $\Lambda^t_{(\pi^*_\Omega)}$ is clean and contained in the diagonal

$$\Delta_\Omega = \big\{(\xi,\xi)\ :\ \xi \in T^*\Omega^{int}\big\}.$$

To begin we simply calculate

$$\Lambda^t_{(\pi^*_\Omega)} \circ \Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega} = \Big\{ (\widetilde{\eta}, \eta) \in T^*\Omega^{int} \times T^*\Omega^{int}\ :\ \exists\ \widetilde{\omega},\ \omega \in \widetilde{U^1}, \text{ and } \xi \in T^*\partial_- S\Omega \text{ such that } F_1(\omega) = F_1(\widetilde{\omega}),\ \xi \in T^*_{F_1(\widetilde{\omega})}\partial_- S_{\mathbf{g}}\Omega,\ (D\pi_\Omega)^t_{\pi_\Omega(\omega)}\eta = (DF_1)^t_\omega \xi,\ (D\pi_\Omega)^t_{\pi_\Omega(\widetilde{\omega})}\widetilde{\eta} = (DF_1)^t_{\widetilde{\omega}}\xi \Big\}.$$

Let us look closely at this set. Suppose that η, $\widetilde{\eta}$, ω, $\widetilde{\omega}$, and ξ are as specified. First, the requirement that $F_1(\omega) = F_1(\widetilde{\omega})$ means that ω and $\widetilde{\omega}$ must be along the same geodesic. Using natural coordinates on $T^*\Omega$ given by the Cartesian coordinates x^j on Ω the requirement that $(D\pi_\Omega)^t_{\pi_\Omega(\widetilde{\omega})}\widetilde{\eta} = (DF_1)^t_{\widetilde{\omega}}\xi$ means that $(DF_1)^t_{\widetilde{\omega}}\xi$ has a coordinate expression of the form

$$(DF_1)^t_{\widetilde{\omega}}\xi = \widetilde{\eta}_l \mathrm{d}x^l$$

and similarly

$$(DF_1)^t_\omega \xi = \eta_l \mathrm{d}x^l.$$

Now, from the definition of F_1 we have locally

$$F_1(\omega) = F_1 \circ \Psi_s(\omega)$$

and so since ω and $\widetilde{\omega}$ lie along the same geodesic there is an s so that

$$(DF_1)_{\widetilde{\omega}} = (DF_1)_{\widetilde{\omega}}(D\Psi_s)_\omega \Rightarrow (DF_1)^t_{\widetilde{\omega}} = (D\Psi_s)^t_\omega (DF_1)^t_{\widetilde{\omega}}.$$

Therefore

$$(D\Psi_s)^t_\omega \widetilde{\eta}_l \mathrm{d}x^l = \eta_l \mathrm{d}x^l.$$

This last equation implies that either $\pi_\Omega(\omega) = \pi_\Omega(\widetilde{\omega})$ or $\pi_\Omega(\omega)$ and $\pi_\Omega(\widetilde{\omega})$ are conjugate along the geodesic. Since we have eliminated the latter possibility by hypothesis this implies that $\pi_\Omega(\omega) = \pi_\Omega(\widetilde{\omega})$. Since $(D\pi_\Omega)^t_{\pi_\Omega(\widetilde{\omega})}$ is injective we can also conclude that $\eta = \widetilde{\eta}$ and so

$$\Lambda_{(\pi^*_\Omega)^t} \circ \Lambda^t_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega} \subset \Delta_\Omega.$$

We must also show that the composition is clean, and to do this we analyze the set

$$C_\eta = \Big\{ \big(\eta,\ (D_{\pi^*_\Omega})^t\eta,\ (DF_1)^t_\omega \xi,\ \eta\big)\ :\ \omega \in \widetilde{U^j},\ \pi_\Omega(\omega) = \pi_\Omega(\eta), \\ \xi \in T^*_{F_1(\omega)}\partial_- S_{\mathbf{g}}\Omega,\ (D_{\pi^*_\Omega})^t \eta = (DF_1)^t_\omega \xi \Big\}.$$

Since F_1 is a submersion $(DF_1)^t_\omega$ is injective and so C_η can be parametrized entirely by $\omega \in \widetilde{U^j} \cap \pi_\Omega^{-1}(\pi_\Omega(\eta))$. Also $(D_{\pi^*_\Omega})^t\eta = (DF_1)^t_\omega\xi$ implies that $\eta(\omega) = 0$ (this is the duality pairing). Thus C_η is diffeomorphic to $\widetilde{U^j} \cap \eta^\perp$ and the composition is clean. Finally we apply the calculus to show that $\left(\mathcal{I}^1_{g,w,\Gamma}\right)^t \circ \mathcal{I}^1_{g,w,\Gamma}$ is a pseudodifferential operator. Since the excess of the composition is the dimension of C_η, which by the argument of this paragraph is $n-2$, we see that the order of $\left(\mathcal{I}^1_{g,w,\Gamma}\right)^t \circ \mathcal{I}^1_{g,w,\Gamma}$ is $-(n+1)/4 + (1-n)/4 + (n-2)/2 = -1$.

For the term $\left(\mathcal{I}^2_{g,w,\Gamma}\right)^t \circ \mathcal{I}^2_{g,w,\Gamma}$ the same analysis applies if we replace F_1 and $\widetilde{U^1}$ by F_2 and $\widetilde{U^2}$. The two cross terms in (6.1) are actually smoothing operators under the hypothesis that there are no conjugate points. Indeed, we have for example

$$\left(\mathcal{I}^{2,S}_{g,w,\Gamma}\right)^t \circ \mathcal{I}^{1,S}_{g,w,\Gamma} \circ \pi^*_\Omega \in \mathcal{I}_1^{-(n+1)/4}\left(S_{\mathbf{g}}(\Omega^{int} \setminus \Gamma) \times \Omega^{int} \setminus \Gamma, \left(\Lambda^t_{\mathcal{I}^{2,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega}\right)'\right)$$

where

$$\Lambda^t_{\mathcal{I}^{2,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega} = \Big\{ \big((DF_2)^t_{\widetilde{\omega}}\ \xi, \eta\big)\ :\ \widetilde{\omega} \in \widetilde{U^2},\ \xi \in T^*_{F_1(\widetilde{\omega})}\partial_- S_{\mathbf{g}}\Omega,\ \exists\ \omega \in \widetilde{U^1} \\ \text{such that } F_1(\omega) = F_1(\widetilde{\omega}),\ \eta \in T^*_{\pi_\Omega(\omega)}\Omega^{int},\ (D\pi_\Omega)^t_{\pi_\Omega(\omega)}\eta = (DF_1)^t_\omega \xi \Big\}$$

and so we would have

$$\begin{aligned} &\Lambda^t_{(\pi^*_\Omega)} \circ \Lambda^t_{\mathcal{I}^{2,S}_{g,w,\Gamma}} \circ \Lambda_{\mathcal{I}^{1,S}_{g,w,\Gamma}} \circ \Lambda_{\pi^*_\Omega} \\ &\quad = \Big\{ (\widetilde{\eta}, \eta) \in T^*\Omega^{int} \times T^*\Omega^{int}\ :\ \exists\ \widetilde{\omega} \in \widetilde{U^2},\ \omega \in \widetilde{U^1},\ \text{and } \xi \in T^*\partial_- S\Omega \\ &\qquad \text{such that } F_1(\omega) = F_2(\widetilde{\omega}),\ \xi \in T^*_{F_2(\widetilde{\omega})}\partial_- S_{\mathbf{g}}\Omega,\ (D\pi_\Omega)^t_{\pi_\Omega(\omega)}\eta = (DF_1)^t_\omega \xi, \\ &\qquad (D\pi_\Omega)^t_{\pi_\Omega(\widetilde{\omega})}\widetilde{\eta} = (DF_2)^t_{\widetilde{\omega}}\xi \Big\}. \end{aligned}$$

Arguing as before we find that under the hypothesis on conjugate points along the reflected geodesic this set must be empty and so $\left(\mathcal{I}^2_{g,w,\Gamma}\right)^t \circ \mathcal{I}^1_{g,w,\Gamma}$ is a smoothing operator, and the same argument applies for the last term. □

7. Conclusion. We have studied the linearization of problems related to the recovery of a wavespeed and reflector location from the reflected scattering relation and the travel time along reflected rays. These linearizations are systems of weighted reflected X-ray transforms and we have found coordinate invariant formulae for the relevant Fréchet differentials.

Motivated by this we have also begun the microlocal study of a scalar weighted reflected X-ray transform, and showed that, as in the non-reflected case under sufficiently strong assumptions on the geometry of the reflected rays the associated normal operator is a pseudodifferential operator. In so doing we have also introduced a new technique for performing the microlocal analysis by decomposing the X-ray transform into a pull-back and integration over a family of foliating curves in the tangent bundle defined as the level surfaces of a function, and then applying the clean composition calculus for Fourier integral operators. As part of our ongoing research program we plan to extend this technique to the case of more complicated geometries in order to study microlocally the case of the X-ray, or reflected X-ray, transform in the case when there are conjugate points. Currently the normal operator associated to a X-ray transform has been characterized microlocally only in the presence of fold caustics [14], but we believe the method initiated in the current paper will extend to other types of caustics. Also, we would like to use this method to explicitly calculate the principal symbol of the normal operator in the reflected case. Once we have done this we will be able to study the microlocal invertibility of the reflected X-ray transform which can lead to stability results for the inversion, and also lead to results for the nonlinear problems of recovery from the scattering relation and travel time.

Acknowledgements. Thanks to the members of the GMIG group for supporting this research.

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