

## ALGORITHMS FOR UNEQUALLY SPACED FAST LAPLACE TRANSFORMS

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**Abstract.** We develop fast algorithms for unequally spaced discrete Laplace transforms with complex parameters, which are approximate up to prescribed choice of computational precision. The algorithms are based on modifications of algorithms for unequally spaced fast Fourier transforms using Gaussians. Various configurations of sums with equally and unequally spaced points can be dealt with. Numerical experiments show that the computational time is similar to that for computing ordinary discrete Fourier transforms by means of FFT. Results are given for the one-dimensional case, but it is straightforward to generalize them to arbitrary dimensions.

**1. Introduction.** The subject of this paper is to develop algorithms for the fast evaluation discrete Laplace sums (USFLT) of either of the form

$$(1.1) \quad \mathcal{L}_{\mathbb{Z} \rightarrow \mathbb{C}} : \hat{f}(\rho_j) = \sum_{l=-N/2}^{N/2-1} f(l)e^{\rho_j l},$$

$$(1.2) \quad \mathcal{L}_{\mathbb{C} \rightarrow \mathbb{Z}} : f(l) = \sum_{j=1}^J \hat{f}(\rho_j)e^{\rho_j l},$$

$$(1.3) \quad \mathcal{L}_{\mathbb{R} \rightarrow \mathbb{C}} : \hat{f}(\rho_j) = \sum_{l=1}^L f(\xi_l)e^{\rho_j \xi_l},$$

$$(1.4) \quad \mathcal{L}_{\mathbb{C} \rightarrow \mathbb{R}} : f(\xi_l) = \sum_{j=1}^J \hat{f}(\rho_j)e^{\rho_j \xi_l}.$$

where  $\rho_j = a_j - 2\pi i x_j$ , and where  $a_j$ ,  $x_j$ ,  $l$ , and  $\xi_l$  are restricted to certain intervals. Algorithms for unequally spaced fast Fourier transforms (USFFT) [1, 2] have proven to be a very useful complement to the standard discrete Fourier transform and the FFT algorithms for computing such. In this paper we show how to modify USFFT algorithms so that they can be used also to compute sums of the type above. The techniques presented here could be capable of dealing with the counterpart  $\mathcal{L}_{\mathbb{C} \rightarrow \mathbb{C}}$  to the sums above. The generalization is straightforward, but technical, and it is omitted here.

Fast methods for real Laplace sums of the type

$$(1.5) \quad \hat{f}(s_j) = \sum_l f(x_l)e^{-s_j x_l}, \quad s_j > 0, x_l > 0,$$

have been considered in [3, 4]. A fast method to compute (1.5) using an approximation-theoretic approach that employs Chebyshev expansions is presented in [3], while [4] makes use of Laguerre polynomials for the same purpose.

USFFT utilizes a combination of convolution style operations with FFT to achieve arbitrary, but fixed, approximations of sums (1.1), (1.2), (1.3) and (1.4), for purely imaginary  $\rho_j$  (for which (1.3) and (1.4) coincide). For a heuristic description of how this works, note that a sum of the form

$$\hat{f}(\xi) = \sum_j f(x_j)e^{-2\pi i x_j \xi}$$

can be considered as a Fourier transform of the distribution

$$F(x) = \sum_j f(x_j)\delta(x - x_j).$$

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Suppose now that  $\varphi$  is a “bump function”, e.g., a Gaussian or a B-spline. From the Fourier transform

$$\int_{-\infty}^{\infty} F * \varphi(x) e^{-2\pi i x \xi} d\xi = \widehat{F}(\xi) \varphi(\xi) = \widehat{f}(\xi) \varphi(\xi),$$

the relation

$$\widehat{f}(\xi) = \frac{1}{\varphi(\xi)} \int_{-\infty}^{\infty} \sum_j f(x_j) \varphi(x - x_j) e^{-2\pi i x \xi} d\xi$$

follows. What makes this formulation attractive is that if  $\varphi$  chosen sufficiently regular (with compact numerical support) then the integral above can be approximated by the trapezoidal rule, which allows a rapid approximation of the integral by means of a fast Fourier transform (FFT) for the case when the points  $\xi$  are chosen at equally spaced points. The approximation is thus done in three steps: 1) convolution in the space domain; 2) FFT; and 3) division in the frequency domain. There is an apparent loss of precision at the third step at points where  $\varphi$  is close or equal to zero. However, it turns out that approximation to prescribed precision for  $\xi$  in a certain range can be achieved by proper oversampling and choice of  $\varphi$ .

The algorithm sketched out above would be the Fourier transform correspondence of (1.2) and there are similar constructions using convolutions in one domain followed by division in the dual one for sums of the form (1.1), (1.3) and (1.4). Crucial for the construction of fast algorithms is that the function  $\varphi$  has small (numerical) support, in order to keep down the computational time for numerical evaluation of the convolutions. It turns out that the (numerical) width of  $\varphi$  is proportional to  $-\log \epsilon$ , where  $\epsilon$  denotes the computational precision, cf. Section 2.

In this paper, we will choose  $\varphi$  to be of Gaussian type, and make explicit use of this fact to generalize USFFT algorithms to deal with fast Laplace transforms, in a similar spirit to the explicit use of the structure of Gaussians for the USFFT style algorithms developed in [5, 6]. The extra cost for this generalization is that the (numerical) width of  $\varphi$  has to be increased slightly depending on the range of allowed decay parameters  $a$ .

The computation of the sums (1.1) and (1.2) corresponds to the application of complex Vandermonde matrices and its adjoints. Such are useful, for instance when trying to represent data using complex exponentials, cf. [7]. Another direct application of fast discrete Laplace transforms

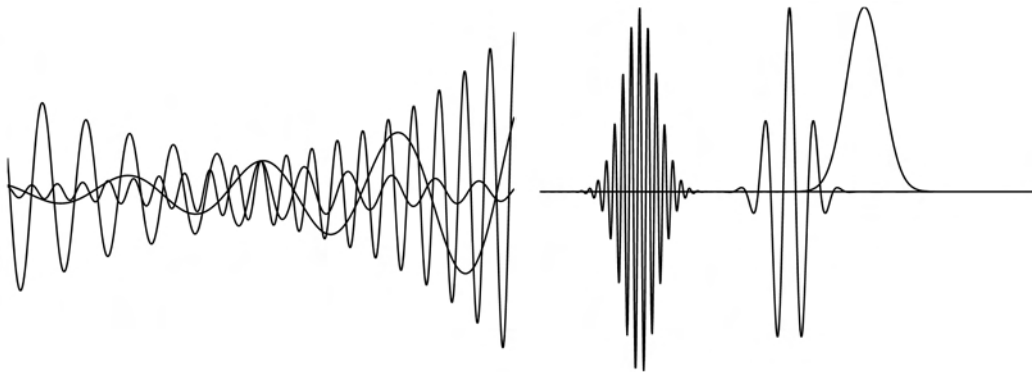


FIG. 1. Complex exponentials  $e^{(a_j - 2\pi i \xi_j)t}$  to the left and Gabor functions  $g_{s_j, \omega_j}$  to the right.

concerns Gabor transforms, using Gabor functions of the form  $g_{s_j, \omega_j}(t) = e^{-\lambda(t-s_j)^2} e^{2\pi i \omega_j t}$ . For instance, if  $f$  is sampled at points  $\xi_l$  (defining an inner product  $\langle \cdot, \cdot \rangle$ ), then

$$\langle f, g_{s_j, \omega_j} \rangle = e^{-\lambda s_j^2} \sum_l \left( f(\xi_l) e^{-\lambda \xi_l^2} \right) e^{2\lambda s_j \xi_l - 2\pi i \omega_j \xi_l}.$$

Once  $f$  has been pre-multiplied by  $e^{-\lambda\xi^2}$ , this corresponds to a sum of the form (1.3), with  $a_j = 2\lambda s_j$ , and  $x_j = \omega_j$ . Figure 1 illustrates the complex exponentials and the Gabor functions mentioned above.

Note that, just as for USFFT algorithms, the main objective is not to compute forward – inverse transform pairs, but simply to accurately evaluate sums of the types (1.1), (1.2), (1.3) and (1.4). The development of numerical methods for forward – inverse transforms is a different subject: however, one where the rapid evaluation of the type of sums considered here could prove useful.

**2. Approximations of exponentials.** THEOREM 2.1. *Given a precision parameter  $\epsilon > 0$ , an oversampling parameter  $\nu > 1$ , an interval parameter  $N > 1$ , and an exponential parameter  $a \in \mathbb{R}$ . Let*

$$\mu = \frac{a}{N(\nu - 1)} + \frac{-\ln(\epsilon)}{\nu N^2(\nu - 1)},$$

and let  $\rho = -2\pi i x + a$ , for some frequency parameter  $x$ , and define

$$(2.1) \quad \widehat{\varphi}_a(\omega) = e^{-\mu\omega^2 + a\omega}, \quad \varphi_a(t) = \sqrt{\frac{\pi}{\mu}} e^{-\frac{\pi^2}{\mu} \left(t - i\frac{a}{2\pi}\right)^2}$$

Then

$$(2.2) \quad \sup_{|\xi| < N/2} \left| e^{\rho\xi} - \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_{k \in \mathbb{Z}} \varphi_a\left(\frac{k}{\nu N} - x\right) e^{-2\pi i \frac{k}{\nu N} \xi} \right| < \epsilon + \mathcal{O}\left(\epsilon^{\min(4, \frac{\nu+1}{\nu-1})}\right).$$

*Proof.* Because of symmetry we can assume that  $a > 0$ . From the Poisson sum formula it follows that

$$\frac{1}{\nu N} \sum_{k \in \mathbb{Z}} \varphi_a\left(\frac{k}{\nu N} - x\right) e^{-2\pi i \frac{k}{\nu N} \xi} = \sum_{k \in \mathbb{Z}} \widehat{\varphi}_a(\xi + \nu N k) e^{-2\pi i x \xi}.$$

Division by  $\widehat{\varphi}_0(\xi)$  gives

$$\begin{aligned} \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_{k \in \mathbb{Z}} \varphi_a\left(\frac{k}{\nu N} - x\right) e^{-2\pi i \frac{k}{\nu N} \xi} &= \sum_{k \in \mathbb{Z}} \frac{\widehat{\varphi}_a(\xi + \nu N k)}{\widehat{\varphi}_0(\xi)} e^{-2\pi i x \xi} \\ &= \sum_{k \in \mathbb{Z}} e^{(a - 2\mu\nu N k)\xi - \mu\nu^2 N^2 k^2 + a\nu N k} e^{-2\pi i x \xi} \\ e^{\rho\xi} + \sum_{k \in \mathbb{Z} \setminus 0} e^{(a - 2\mu\nu N k)\xi - \mu\nu^2 N^2 k^2 + a\nu N k} e^{-2\pi i x \xi} &= e^{\rho\xi} + \sum_{k \in \mathbb{Z} \setminus 0} R_k(\xi). \end{aligned}$$

Note that each one of the terms  $R_k(\xi)$  are pure exponentials, and hence that they have their largest values at the endpoints  $\{-N/2, N/2\}$ . From the choice of  $\mu$ ,

$$|R_k(N/2)| = e^{-\frac{ak\nu N(\nu k + 2 - \nu)}{\nu - 1}} \epsilon^{\frac{k(\nu k + 1)}{\nu - 1}}, \quad |R_k(-N/2)| = e^{-\frac{ak\nu^2 N(k - 1)}{\nu - 1}} \epsilon^{\frac{k(\nu k - 1)}{\nu - 1}}.$$

From the exponential form of  $R_k$  it can now be easily shown that

$$\sup_{|\xi| < N/2} \left| \sum_{k \in \mathbb{Z} \setminus 0} R_k(\xi) \right| \leq \max \left( \sum_{k \in \mathbb{Z} \setminus 0} |R_k(N/2)|, \sum_{k \in \mathbb{Z} \setminus 0} |R_k(-N/2)| \right).$$

Now, for  $|k| > 1$ ,  $|R_k(\xi)|$  are smaller than  $\mathcal{O}\left(\epsilon^{\frac{4\nu - 1}{\nu - 1}}\right)$ . For  $|k| = 1$  we have that

$$\begin{aligned} R_1(N/2) &= \epsilon^{\frac{\nu+1}{\nu-1}} e^{-\frac{2a\nu N}{\nu-1}}, & R_1(-N/2) &= \epsilon \\ R_{-1}(N/2) &= e^{-2a\nu N} \epsilon, & R_{-1}(-N/2) &= e^{-\frac{2a\nu^2 N}{\nu-1}} \epsilon^{\frac{\nu+1}{\nu-1}} \end{aligned}$$

and

$$\sum_{k \in \mathbb{Z} \setminus 0} |R_k(-N/2)| \leq \sum_{k \geq 1} \epsilon^{\frac{k(\nu k - 1)}{\nu - 1}} + \sum_{k \geq 1} \epsilon^{\frac{k(\nu k + 1)}{\nu - 1}} \leq \epsilon + \sum_{k \geq 2} \epsilon^{2k} + \sum_{k \geq 0} \epsilon^{k \frac{\nu + 1}{\nu - 1}} = \epsilon + \frac{\epsilon^4}{1 - \epsilon^2} + \frac{\epsilon^{\frac{\nu + 1}{\nu - 1}}}{1 - \epsilon^{\frac{\nu + 1}{\nu - 1}}}.$$

The term  $\sum_{k \in \mathbb{Z} \setminus 0} |R_k(N/2)|$  can be estimated in a similar fashion, and it turns out that this is dominated by the estimate above, yielding (2.2).  $\square$

COROLLARY 2.2. *Given the assumption of Theorem 2.1, let*

$$\varphi_a^\epsilon(t) = \begin{cases} \varphi_a(t), & \text{if } |t| \leq \sqrt{-\mu \ln(\epsilon \widehat{\varphi}_0(N/2))}/\pi, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(2.3) \quad \sup_{|\xi| < N/2} \left| e^{\rho \xi} - \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_{k \in \mathbb{Z}} \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) e^{-2\pi i \frac{k}{\nu N} \xi} \right| = \mathcal{O}(\epsilon).$$

*Proof.* We use Poisson sum formula to rewrite the left hand side of (2.3) as

$$\begin{aligned} & \sup_{|\xi| < N/2} \left| e^{\rho \xi} - \frac{1}{\widehat{\varphi}_0(\xi)} \sum_{k \in \mathbb{Z}} \widehat{\varphi}_a^\epsilon(\xi + \nu N k) \right| \leq \sup_{|\xi| < N/2} \left| e^{\rho \xi} - \frac{1}{\widehat{\varphi}_0(\xi)} \sum_{k \in \mathbb{Z}} \widehat{\varphi}_a(\xi + \nu N k) \right| \\ & \quad + \sup_{|\xi| < N/2} \frac{1}{\widehat{\varphi}_0(\xi)} \left| \sum_{k \in \mathbb{Z}} \widehat{\varphi}_a^\epsilon(\xi + \nu N k) - \widehat{\varphi}_a(\xi + \nu N k) \right| \\ & \leq \mathcal{O}(\epsilon) + \sup_{|\xi| < N/2} \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \left| \sum_{k \in \mathbb{Z}} (\varphi_a^\epsilon - \varphi_a) \left( \frac{k}{\nu N} - x \right) e^{-2\pi i \frac{k}{\nu N} \xi} \right| \\ & \leq \mathcal{O}(\epsilon) + \frac{1}{\widehat{\varphi}_0(N/2)} \frac{1}{\nu N} \sum_{\{k: |\frac{k}{\nu N} - x| \geq \sqrt{-\mu \ln(\epsilon \widehat{\varphi}_0(N/2))}/\pi\}} \sqrt{\frac{\pi}{\mu}} e^{-\frac{\pi^2}{\mu} (\frac{k}{\nu N} - x)^2} \\ & \leq \mathcal{O}(\epsilon) + \frac{1}{\widehat{\varphi}_0(N/2)} 2 \int_{\sqrt{-\mu \ln(\epsilon \widehat{\varphi}_0(N/2))}/\pi}^{\infty} \sqrt{\frac{\pi}{\mu}} e^{-\frac{\pi^2}{\mu} t^2} dt \\ & \leq \mathcal{O}(\epsilon) + \frac{1}{\widehat{\varphi}_0(N/2)} \operatorname{erfc} \left( \sqrt{-\ln(\epsilon \widehat{\varphi}_0(N/2))} \right) = \mathcal{O}(\epsilon) \end{aligned}$$

where we have used that  $\operatorname{erfc}(t) \leq e^{-t^2}$ , which follows from [8, 7.1.13, p. 298] in the last step.  $\square$

COROLLARY 2.3. *Given the conditions of Theorem 2.1 and Corollary 2.2, and let*

$$(2.4) \quad M = \left\lceil \sqrt{-\mu \ln(\epsilon \widehat{\varphi}_0(N/2))} \nu N / \pi \right\rceil,$$

and suppose that unequally spaced Laplace nodes  $\{\rho_j = a_j - 2\pi i x_j\}_{j=1}^J$ , with  $|a_j| \leq a$  and  $|x_j| < 1/2 - M/(\nu N)$ , and coefficients  $f_j \in \mathbb{C}^J$  are given. Then the unequally spaced discrete (inverse) Laplace transform (1.2) can be rapidly computed by the following steps:

1.  $\widehat{G}(k) = \sum_{j=1}^J \widehat{f}(\rho_j) \varphi_{a_j}^\epsilon \left( \frac{k}{\nu N} - x_j \right)$ ,  $-\frac{\nu N}{2} \leq k < \frac{\nu N}{2}$ ,  $\mathcal{O}(MJ)$ ,
2.  $G(l) = \sum_{k=-\frac{\nu N}{2}}^{\frac{\nu N}{2}-1} \widehat{G}(l) e^{-2\pi i \frac{kl}{\nu N}}$ ,  $-\frac{\nu N}{2} \leq l < \frac{\nu N}{2}$ ,  $\mathcal{O}(\nu N \log(\nu N))$  by FFT,
3.  $f_{\text{approx}}(l) = G(l)/(\nu N \widehat{\varphi}_0(l))$ ,  $-N/2 \leq l < N/2$ ,  $\mathcal{O}(N)$ .

The approximation error satisfies

$$\sup_{-N/2 \leq l < N/2} |f(l) - f_{\text{approx}}(l)| = \mathcal{O}(\epsilon).$$

*Proof.* The result follows directly from Theorem 2.1 and Corollary 2.2 because of the requirements on  $a_j$ . Due to the small support of  $\varphi_{a_j}^\epsilon$ , only a few terms  $(2M+1)$  in  $k$  are affected for each  $j$ . Moreover, the limitation on  $x_j$  induces that  $\varphi_{a_j}^\epsilon\left(\frac{k}{\nu N} - x_j\right) = 0$  for  $|k| \geq \nu N/2$ . Thus the infinite sum in the Fourier series of (2.3) in Corollary 2.2 by a finite counterpart, which can be computed by FFT.  $\square$

The structural difference with the algorithm above and the USFFT counterpart is that in the first (convolution) step the smearing function ( $\varphi_{a_j}^\epsilon$ ) varies with  $j$ . Note that the impact of the exponential factor  $a_j$  only affects the phase of  $\varphi_{a_j}^\epsilon$ , and not the width, cf. (2.1).

**THEOREM 2.4.** *Let  $f \in \mathbb{C}^N$  indexed by  $l = -N/2, \dots, N/2-1$ , and make the same assumptions as in Theorem 2.1. Then*

$$(2.5) \quad \left| \sum_{l=-N/2}^{N/2-1} f(l)e^{\rho l} - \sum_{k \in \mathbb{Z}} \varphi_a \left( \frac{k}{\nu N} - x \right) \sum_{l=-N/2}^{N/2-1} f(l) \frac{1}{\nu N \widehat{\varphi}_0(l)} e^{-2\pi i \frac{k}{\nu N} l} \right| < \left( \frac{N(\nu-1)}{-\ln \epsilon} \epsilon + \mathcal{O}\left(\epsilon^{\frac{\nu+1}{\nu-1}}\right) \right) \|f\|_\infty.$$

*Proof.* After a rearrangements of terms, it follows that the the error in (2.5) can be written

$$\begin{aligned} & \left| \sum_l f(l) \left( e^{\rho l} - \frac{1}{\nu N \widehat{\varphi}_0(l)} \sum_{k \in \mathbb{Z}} \varphi_a \left( \frac{k}{\nu N} - x \right) e^{-2\pi i \frac{k}{\nu N} l} \right) \right| \\ & \leq \|f\|_\infty \int_{-N/2}^{N/2} \left| e^{\rho \xi} - \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_{k \in \mathbb{Z}} \varphi_a \left( \frac{k}{\nu N} - x \right) e^{-2\pi i \frac{k}{\nu N} \xi} \right| d\xi \\ & \leq \|f\|_\infty \sum_{k \in \mathbb{Z} \setminus 0} \int_{-N/2}^{N/2} |R_k(\xi)| d\xi, \end{aligned}$$

with  $R_k$  as in Theorem 2.1. Integration of the exponentials  $R_k$  gives

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus 0} \int_{-N/2}^{N/2} |R_k(\xi)| d\xi &= \sum_{k \in \mathbb{Z} \setminus 0} \frac{N(\nu-1)}{2k(a\nu N - \ln(\epsilon))} \left( \epsilon^{\frac{k(\nu k-1)}{\nu-1}} e^{-\frac{a k \nu^2 N(k-1)}{\nu-1}} - \epsilon^{\frac{k(\nu k+1)}{\nu-1}} e^{-\frac{a k \nu N(\nu k+2-\nu)}{\nu-1}} \right) \\ &\leq \frac{N(\nu-1)}{-\ln \epsilon} \epsilon + \mathcal{O}\left(\epsilon^{\frac{\nu+1}{\nu-1}}\right). \end{aligned}$$

$\square$

**COROLLARY 2.5.** *Given the conditions of Theorem 2.4 and Corollary 2.2, let  $M$  be defined by (2.4), and given unequally spaced Laplace nodes  $\{\rho_j = a_j - 2\pi i x_j\}_{j=1}^J$ , with  $|a_j| \leq a$  and  $|x_j| < 1/2 - M/(\nu N)$ , and  $f \in \mathbb{C}^N$ . Then the unequally spaced discrete Laplace transform (1.1) can be rapidly computed by the following steps:*

1.  $G(l) = f(l)/(\nu N \widehat{\varphi}_0(l))$ ,  $-N/2 \leq l < N/2$ ,  $\mathcal{O}(N)$
2.  $\widehat{G}(k) = \sum_{l=-N/2}^{N/2-1} G(l) e^{-2\pi i \frac{kl}{\nu N}}$ ,  $-\nu N/2 \leq k < \nu N/2$ ,  $\mathcal{O}(\nu N \log(\nu N))$  by FFT,
3.  $\widehat{f}_{\text{approx}}(\rho_j) = \sum_{k=[x_j \nu N]-M}^{[x_j \nu N]+M} \widehat{G}(k) \varphi_{a_j}^\epsilon \left( \frac{k}{\nu N} - x_j \right)$ ,  $1 \leq j \leq J$ ,  $\mathcal{O}(MJ)$ ,

The approximation error satisfies

$$\sup_{1 \leq j \leq J} \left| \widehat{f}_{\text{approx}}(\rho_j) - \widehat{f}(\rho_j) \right| = \mathcal{O} \left( \frac{N\epsilon}{-\ln \epsilon} \right).$$

*Proof.* A counterpart to Corollary 2.2 for Theorem 2.4 is easily constructed, and this provides the error estimate. Similarly as for Corollary 2.3, the small support of  $\varphi_{a_j}^\epsilon$  gives short sums over  $k$  in Step 3, and the conditions on the nodes assures that  $\varphi_{a_j}^\epsilon \left( \frac{k}{\nu N} - x_j \right) = 0$  for  $|k| \leq \nu N/2$ .  $\square$

**THEOREM 2.6.** *Given the assumptions of Theorem 2.1, and that that  $\rho = a - 2\pi i x$  with  $|x| < 1/2 - M_\mu/(\nu N)$  for  $M_\mu = M$  defined by (2.4). In addition, let*

$$(2.6) \quad \lambda = \frac{-\ln(\epsilon)}{\nu(\nu N)^2(\nu - 1)},$$

$$\widehat{\psi}(\omega) = e^{-\lambda\omega^2}, \quad \psi(t) = \sqrt{\frac{\pi}{\lambda}} e^{-\frac{\pi^2}{\lambda} t^2}, \quad \psi^\epsilon(t) = \begin{cases} \psi(t) & \text{if } t \text{ is such that } \psi(t)/\psi(0) \geq \epsilon \\ 0 & \text{otherwise,} \end{cases}$$

Then

$$(2.7) \quad \sup_{|\xi| \leq \frac{N}{2}} \left| e^{\rho\xi} - \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_m \psi^\epsilon \left( \frac{m}{\nu^2 N} - \frac{\xi}{\nu N} \right) \sum_{k=-\nu N/2}^{\nu N/2-1} \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) \frac{1}{\nu^2 N \widehat{\psi}(k)} e^{-2\pi i \frac{m}{\nu^2 N} k} \right|$$

$$(2.8) \quad = \sup_{|\xi| \leq \frac{N}{2}} \left| e^{\rho\xi} - \sum_k \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) \frac{1}{\nu^2 N \widehat{\psi}(k)} \sum_{m=-\nu N/2}^{\nu N/2-1} \psi^\epsilon \left( \frac{m}{\nu^2 N} - \frac{\xi}{\nu N} \right) e^{-2\pi i \frac{m}{\nu^2 N} k} \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \right| \\ = \mathcal{O} \left( \epsilon^{1 - \frac{1}{4\nu(\nu-1)}} \right)$$

*Proof.* Let

$$f_1(\xi) = \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_{k=-\nu N/2}^{\nu N/2-1} \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) e^{-2\pi i \frac{k}{\nu N} \xi}.$$

Then from Corollary 2.3 and the condition on  $x$  the estimate

$$(2.9) \quad \sup_{|\xi| < \frac{N}{2}} |e^{\rho\xi} - f_1(\xi)| = \mathcal{O}(\epsilon)$$

holds. Next, we would like to employ Theorem 2.4 to expand each of the terms  $e^{-\frac{2\pi i \xi}{\nu N} k}$ . In order to do this we have to take into account that the sum in the definition of  $f_1$  is over  $\nu N$  terms and not  $N$ , which motivates the introduction of  $\lambda$ ,  $\psi$  etc<sup>1</sup>.

We introduce

$$(2.10) \quad f_2(\xi) = \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \sum_{m \in \mathbb{Z}} \psi \left( \frac{m}{\nu^2 N} - \frac{\xi}{\nu N} \right) \sum_{k=-\nu N/2}^{\nu N/2-1} \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) \frac{1}{\nu N \widehat{\psi}(k)} e^{-2\pi i \frac{m}{\nu^2 N} k}$$

<sup>1</sup>Note that a scaled version  $\varphi_a$  could have been used, but that the impact of  $a$  decreases the performance slightly.

Then it follows from Theorem 2.4 that

$$\begin{aligned} \sup_{|\xi| < \frac{N}{2}} |f_1(\xi) - f_2(\xi)| &< \sup_{|\xi| < \frac{N}{2}} \left| \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \left( \sum_{k=-\nu N/2}^{\nu N/2-1} \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) e^{-2\pi i \frac{k}{\nu N} \xi} \right. \right. \\ &\times \left. \sum_{m \in \mathbb{Z}} \psi \left( \frac{m}{\nu^2 N} - \frac{\xi}{\nu N} \right) \sum_{k=-\nu N/2}^{\nu N/2-1} \varphi_a^\epsilon \left( \frac{k}{\nu N} - x \right) \frac{1}{\nu^2 N \widehat{\psi}(k)} e^{-2\pi i \frac{m}{\nu^2 N} k} \right) \left. \right| \\ &\times \sup_{|\xi| < \frac{N}{2}} \left| \frac{1}{\nu N \widehat{\varphi}_0(\xi)} \right| \left( \frac{\nu N (\nu - 1)}{-\ln \epsilon} \epsilon + \mathcal{O} \left( \epsilon^{\frac{\nu+1}{\nu-1}} \right) \right) \|\varphi_a\|_\infty = \mathcal{O} \left( \epsilon^{1 - \frac{1}{4\nu(\nu-1)}} \right). \end{aligned}$$

and in combination with (2.9) it follows that

$$\sup_{|\xi| < \frac{N}{2}} |e^{\rho \xi} - f_2(\xi)| = \mathcal{O} \left( \epsilon^{1 - \frac{1}{4\nu(\nu-1)}} \right).$$

To finalize the proof, note that the error introduced by substituting  $\psi$  by  $\psi^\epsilon$  in (2.10) is bounded by  $\mathcal{O} \left( \epsilon^{1 - \frac{1}{4\nu(\nu-1)}} \right)$ .  $\square$  Theorem 2.6 can now be used to construct fast algorithms for sums of the type (1.4) and (1.3), as illustrated in the following two Corollaries.

**COROLLARY 2.7.** *Given the conditions of Theorem 2.6 and Corollary 2.3, let  $M_\mu = M$  as defined by (2.4), and let*

$$M_\lambda = \left\lceil \sqrt{-\lambda \ln(\epsilon)} \nu^2 N / \pi \right\rceil.$$

*Then sums of the type (1.3) can be evaluated in  $\mathcal{O}(\nu N \log(\nu N) + N \sqrt{-\ln(\epsilon)}/\nu)$  time with an error of size  $\mathcal{O} \left( \epsilon^{1 - \frac{1}{4\nu(\nu-1)}} \right)$  by the following procedure*

1.  $G(\xi_l) = \frac{f(\xi_l)}{\nu N \widehat{\varphi}_0(\xi_l)}$ ,  $1 \leq l \leq L$ ,  $\mathcal{O}(L)$ ,
2.  $H(m) = \sum_l G(\xi_l) \psi^\epsilon \left( \frac{m}{\nu^2 N} - \frac{\xi_l}{\nu N} \right)$ ,  $-\frac{\nu N}{2} \leq m < \frac{\nu N}{2}$ ,  $\mathcal{O}(LM_\lambda)$ ,
3.  $\widehat{H}(k) = \sum_{m=-\frac{\nu N}{2}}^{\frac{\nu N}{2}-1} H(m) e^{-2\pi i \frac{km}{\nu^2 N}}$ ,  $-\frac{\nu^2 N}{2} \leq k < \frac{\nu^2 N}{2}$ ,  $\mathcal{O}(\nu^2 N \log(\nu^2 N))$  by FFT,
4.  $\widehat{G}(k) = \frac{\widehat{H}(k)}{\nu^2 N \widehat{\psi}(k)}$ ,  $-\frac{\nu N}{2} \leq k < \frac{\nu N}{2}$ ,  $\mathcal{O}(\nu N)$ ,
5.  $\widehat{f}_{\text{approx}}(\rho_j) = \sum_{k=[\nu N x] - M_\mu}^{[\nu N x] + M_\mu} \widehat{G}(k) \varphi_{a_j}^\epsilon \left( \frac{k}{\nu N} - x_j \right)$ ,  $1 \leq j \leq J$ ,  $\mathcal{O}(JM_\mu)$ .

*Proof.* When using Theorem 2.6 for the approximation of  $e^{\rho_j \xi_l}$  for each fixed pair  $(j, l)$ , only a few values of  $m$  in the sums over  $m$  and  $k$  are needed, with the sums over  $m$  being independent of  $j$  and the sums over  $k$  being independent of  $l$ , respectively. By a change of summation order, and by employing FFT for the discrete Fourier transform, fast algorithms are obtained.  $\square$

**COROLLARY 2.8.** *Given the conditions of Theorem 2.6 and Corollary 2.3, let  $M_\mu = M$  as defined by (2.4), and define in a similar fashion*

$$M_\lambda = \left\lceil \sqrt{-\lambda \ln(\epsilon \widehat{\psi}(\nu N/2))} \nu^2 N / \pi \right\rceil.$$

*Then sums of the type (1.4) can be evaluated in  $\mathcal{O}(\nu N \log(\nu N) + N \sqrt{-\ln(\epsilon)}/\nu)$  time with an error of size  $\mathcal{O} \left( \epsilon^{1 - \frac{1}{4\nu(\nu-1)}} \right)$  by the following procedure*

1.  $\widehat{G}(k) = \sum_{j=1}^J \widehat{f}(\rho_j) \varphi_{a_j}^\epsilon \left( \frac{k}{\nu N} - x_j \right), \quad -\frac{\nu N}{2} \leq k < \frac{\nu N}{2}, \quad \mathcal{O}(M_\mu J),$
2.  $\widehat{H}(k) = \frac{\widehat{G}(k)}{\nu N \widehat{\psi}(k)}, \quad -\frac{\nu N}{2} \leq k < \frac{\nu N}{2}, \quad \mathcal{O}(\nu N).$
3.  $H(m) = \sum_{k=-\frac{\nu N}{2}}^{\frac{\nu N}{2}-1} \widehat{H}(k) e^{-2\pi i \frac{km}{\nu^2 N}}, \quad -\frac{\nu^2 N}{2} \leq m < \frac{\nu^2 N}{2}, \quad \mathcal{O}(\nu^2 N \log(\nu^2 N)) \text{ by FFT},$
4.  $G(l) = \sum_{k=[\nu \xi_l]-M_\lambda}^{[\nu \xi_l]+M_\lambda} H(m) \psi^\epsilon \left( \frac{m}{\nu^2 N} - \frac{\xi_l}{\nu N} \right), \quad 1 \leq l \leq L, \quad \mathcal{O}(M_\lambda L)$
5.  $f_{\text{approx}}(\xi_l) = \frac{G(l)}{\nu N \widehat{\varphi}_0(\xi_l)}, \quad 1 \leq l \leq L, \quad \mathcal{O}(L).$

REMARK 1. *It is possible to construct fast algorithms for the evaluation of sums of the form*

$$\widehat{f}(\zeta_l) = \sum_j f(\rho_j) e^{\rho_j \sigma_l}, \quad \rho_j, \zeta_l \in \mathbb{C}.$$

in a similar fashion as above. If  $\rho_j = -2\pi i x_j + a_l$  and  $\zeta_l = \xi_l - 2\pi i \sigma_l$ , then error estimate in the counterpart of Theorem 2.1 will have an additional factor  $e^{4\pi^2 x \sigma}$ , where  $x = \max_j |x_j|$  and  $\sigma = \max_j |\sigma_l|$ . These details are excluded in order to not make the results more technical than they already are.

**3. Numerics.** In this section, results from numerical simulations for the four discrete Laplace transforms (1.1), (1.2), (1.3) and (1.4) are shown. We satisfy by showing that the time complexity of the algorithms scale as expected, without aiming at hardware optimized performance.

One optimization method that we do make use of is a recursive scheme to reduce the number of intrinsic function calls. To this end, note that

$$\frac{\varphi_{a_j} \left( \frac{k+1}{\nu N} - x_j \right)}{\varphi_{a_j} \left( \frac{k}{\nu N} - x_j \right)} = e^{\frac{\pi (-2\pi k - \pi + 2\xi_j \nu N \pi + i a_j \nu N)}{\mu \nu^2 N^2}},$$

is of exponential type with respect to  $k$ . A recursive technique can therefore be used to compute the elements as a function of (integers)  $k$ . This procedure consists of an initialization step

$$\kappa(k_{\min}) = e^{-\frac{\pi^2}{\mu} \left( \left( \frac{k_{\min}}{\nu N} - x_j \right) - i \frac{a_j}{2\pi} \right)^2} \sqrt{\frac{\pi}{\mu}}, \quad \alpha = e^{\frac{-\pi^2 + 2\pi^2 x_j \nu N + i a_j \pi \nu N - 2\pi^2 k_{\min}}{\mu \nu^2 N^2}}, \quad \beta = e^{\frac{-2\pi^2}{\mu \nu^2 N^2}},$$

followed by recursive steps

$$\kappa(k+1) = \kappa(k)\alpha, \quad \alpha := \alpha\beta.$$

Then  $\kappa(k) = \varphi_{a_j} \left( \frac{k}{\nu N} - x_j \right)$ ,  $k = k_{\min}, k_{\min} + 1, \dots$ . In this way, only three intrinsic function calls (exponentials) are needed for each  $j$ . A similar technique for USFFT algorithms is used in [6].

The convolutions in step 1 of Corollary 2.3, step 3 of Corollary 2.5 and steps 2 and 4 for Corollary 2.7, and Corollary 2.8, respectively, will typically be the most time consuming part of the algorithms. In Table 3.1 we present the time spent on these steps in relation to the time spent on FFT. The simulations were done using  $J = L = N$  for  $\epsilon = 10^{-10}$ . The parameters  $x_j$  and  $\xi_l$  were chosen randomly with a uniform distribution on the intervals of approximation validity, and the exponential decay parameters  $a_j$  were chosen randomly with a uniform distribution on the interval  $[-\ln(1000)/N, \ln(1000)/N]$ , i.e., in such a way that maximum ratio between the largest and the smallest absolute value for each of the exponentials is less than 1000.

In Figure 2 we show the approximation results for (1.2) for one single complex exponential ( $J = 1$ ), for  $\epsilon = 10^{-10}$  and  $N = 1024$ , and with  $a = \log(1000)/N$ . The left panel of Figure 2 shows the complex exponential, and the right panel shows the approximation error for  $-N/2 \leq l < N/2$ .



TABLE 3.1  
Computation times

$N$	Time				Ratio to FFT			
	$\mathbb{C} \rightarrow \mathbb{Z}$	$\mathbb{Z} \rightarrow \mathbb{C}$	$\mathbb{R} \rightarrow \mathbb{C}$	$\mathbb{C} \rightarrow \mathbb{R}$	$\mathbb{C} \rightarrow \mathbb{Z}$	$\mathbb{Z} \rightarrow \mathbb{C}$	$\mathbb{R} \rightarrow \mathbb{C}$	$\mathbb{C} \rightarrow \mathbb{R}$
$2^8$	$8.9e-5$	$9.2e-5$	$1.7e-4$	$1.7e-4$	2.1	2.6	2.7	3.0
$2^9$	$2.3e-4$	$2.3e-4$	$4.5e-4$	$4.6e-4$	2.5	2.7	2.6	2.7
$2^{10}$	$5.1e-4$	$5.1e-4$	$1.0e-3$	$1.0e-3$	2.8	2.8	2.6	2.5
$2^{11}$	$1.0e-3$	$1.1e-3$	$2.1e-3$	$2.1e-3$	2.8	2.7	2.8	2.8
$2^{12}$	$2.2e-3$	$2.4e-3$	$5.1e-3$	$4.7e-3$	3.4	3.4	3.6	3.1
$2^{13}$	$4.1e-3$	$4.4e-3$	$9.3e-3$	$8.7e-3$	3.5	3.6	3.6	3.2
$2^{14}$	$8.4e-3$	$9.2e-3$	$1.8e-2$	$1.8e-2$	3.5	3.7	5.0	4.5
$2^{15}$	$1.6e-2$	$1.7e-2$	$3.6e-2$	$3.6e-2$	4.6	5.0	5.8	5.5
$2^{16}$	$3.2e-2$	$3.4e-2$	$7.4e-2$	$7.2e-2$	5.2	5.8	5.8	5.6
$2^{17}$	$6.4e-2$	$6.7e-2$	$1.5e-1$	$1.6e-1$	5.0	5.8	5.7	5.9
$2^{18}$	$1.3e-1$	$1.4e-1$	$3.3e-1$	$3.4e-1$	4.7	5.5	5.9	6.2
$2^{19}$	$2.5e-1$	$2.8e-1$	$7.0e-1$	$7.3e-1$	4.1	5.2	5.8	6.3
$2^{20}$	$5.0e-1$	$5.6e-1$	$1.5e0$	$1.6e0$	3.8	4.7	5.3	5.7
$2^{21}$	$1.0e0$	$1.1e0$	$3.2e0$	$3.3e0$	3.4	4.0	5.3	5.5
$2^{22}$	$2.1e0$	$2.2e0$	$6.8e0$	$7.0e0$	3.1	3.6	5.3	5.5
$2^{23}$	$4.2e0$	$4.4e0$	$1.5e1$	$1.5e1$	2.9	3.4	5.2	5.3
$2^{24}$	$8.8e0$	$9.2e0$	$3.2e1$	$3.2e1$	2.5	2.7	4.5	4.7

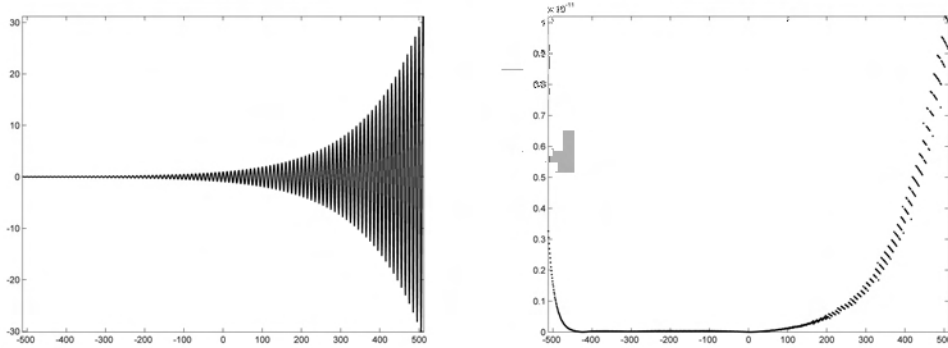


FIG. 2. A single complex exponential to the left and the approximation error to the right for (1.2) with  $\epsilon = 10^{-10}$ .

Figure 3 shows the same setup as for Figure 2, but for  $J = 1024$  exponentials, with (complex) normally distributed coefficients  $\hat{f}(\rho_j)$ , and where the exponential parameters  $a_j$  were chosen randomly (uniformly) in the interval  $[-\ln(1000)/N, \ln(1000)/N]$ . The left panel of Figure 3 shows the exponential, and the right panel shows the error for  $-N/2 \leq l < N/2$ .

**4. Conclusions.** By modifying algorithms for unequally spaced FFT using Gaussians, we can use evaluate unequally spaced discrete Laplace transforms with complex parameters (USFLT). As for USFFT, the unequal positioning are dealt with by short convolutions in time (or space) which are then corrected for in the frequency (time) domain. A similar speedup as for USFFT is obtained for USFLT, and the proposed method display similar accuracy behavior.

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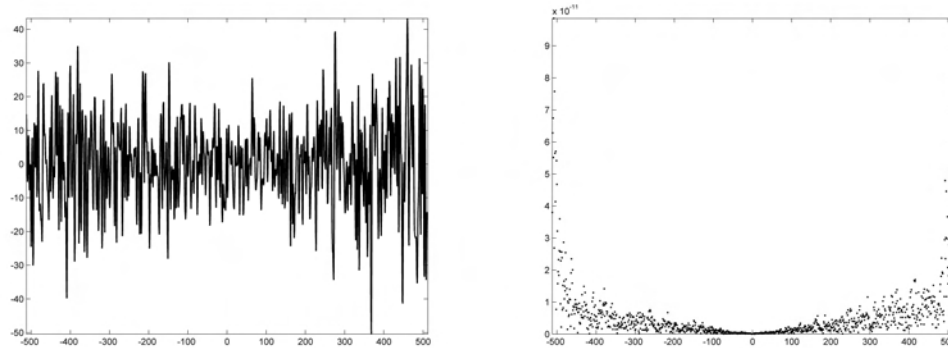


FIG. 3. A sum of 100 complex exponentials to the left and the approximation to the right for (1 2) with  $\epsilon = 10^{-10}$ .

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