Proceedings of the Project Review, Geo-Mathematical Imaging Group (Purdue University, West Lafayette IN), Vol. 1 (2013) pp. 1-16.

A VARIATIONAL FORMULATION FOR INTERPOLATION OF SEISMIC TRACES WITH DERIVATIVE INFORMATION

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Abstract. We construct a variational formulation for the problem of interpolating seismic data in the case of missing traces. We assume that we have derivative information available at the traces. The variational problem is essentially the minimization of the integral over the smallest eigenvector of the structure tensor associated with the interpolated data. This has the physical meaning of penalizing the local presence of more than one direction in the interpolation. We show that the solution to the variational problem also satisfies an elliptic partial differential equation. Moreover, this can be obtained by considering the steady state solution of a non-standard anisotropic diffusion problem. We show existence and uniqueness for this type of anisotropic diffusion. In particular, the uniqueness property is important as it guarantees that the unique solution to the variational problem that we constructed can be obtained by the numerical schemes we propose.

1. Introduction. Let $f = f(x, y)$ be a two-dimensional functon[1]. We assume that we know values of f along vertical lines (traces) $x = x_l$. In addition we assume that we have knowledge of the values of the derivative of f along the traces $x = x_l$. Since we know f along the traces, the only addition information are the values of $f_x(x, y)\Big|_{x=x_l}$. Assuming that f is continuous, this means that we know the values of f approximately in a vicinity of the traces. The problem then arises how to extrapolate this information. One simple way of extrapolation is to extend f using the derivative information and then use linear interpolation between different traces. This is illustrated in the middle upper panel of Figure 1. Another simple approach is to take use the extrapolation using the derivative information, and construct the interpolated image by picking the extension closest to the traces where there is information. This approach is depicted in the upper right panel of Figure 1.

Both the interpolation results obtained in the upper middle and upper right panels of Figure 1 are physically unrealistic. We would rather expect a smooth transition between the left trace and right trace. An apparent problem is the introduction of crossing events or discontinuities.

We are interested in phrasing the interpolation problem in terms of a variational problem. In this formulation, we would like to penalize the "artifacts" produced by the two naive approaches mentioned above. A good tool for measuring artifacts of this kind is by means of structure tensors. The structure tensor of an image f is defined by

$$T_f(x, y) = \begin{pmatrix} K * |f_x|^2 & K * (f_x f_y) \\ K * (f_x f_y) & K * |f_y|^2 \end{pmatrix}, \tag{1.1}$$

for some regularization function K. In this work we assume that K is a Gaussian.

Note that without the convolution with K, the tensor

$$\begin{pmatrix} |f_x|^2 & f_x f_y \\ f_x f_y & |f_y|^2 \end{pmatrix},$$

is of rank one for each fixed point (x, y). The eigenvectors are $f_x^2 + f_y^2$ and 0, respectively, and the eigenvectors are parallel to ∇f and perpendicular to ∇f, respectively. Now, if f locally describes a plane wave, then the convolution will describe a local averaging over areas where the eigenvector of the tensor above remain the same (parallel respectively perpendicular to ∇f). The (positive) eigenvectors of T_f will thus have the property that one will be large while the other one will be close to zero.

However, at regions where f depart from locally resembling a plane wave, the smallest eigenvalue of T_f will no longer be close to zero. Regions where the smallest eigenvalue of T_f not small thus

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[1]In seimic applications the variable y would usually be time and the letter t would be used instead of y. Since we will use time for other purposes we use y instead of t

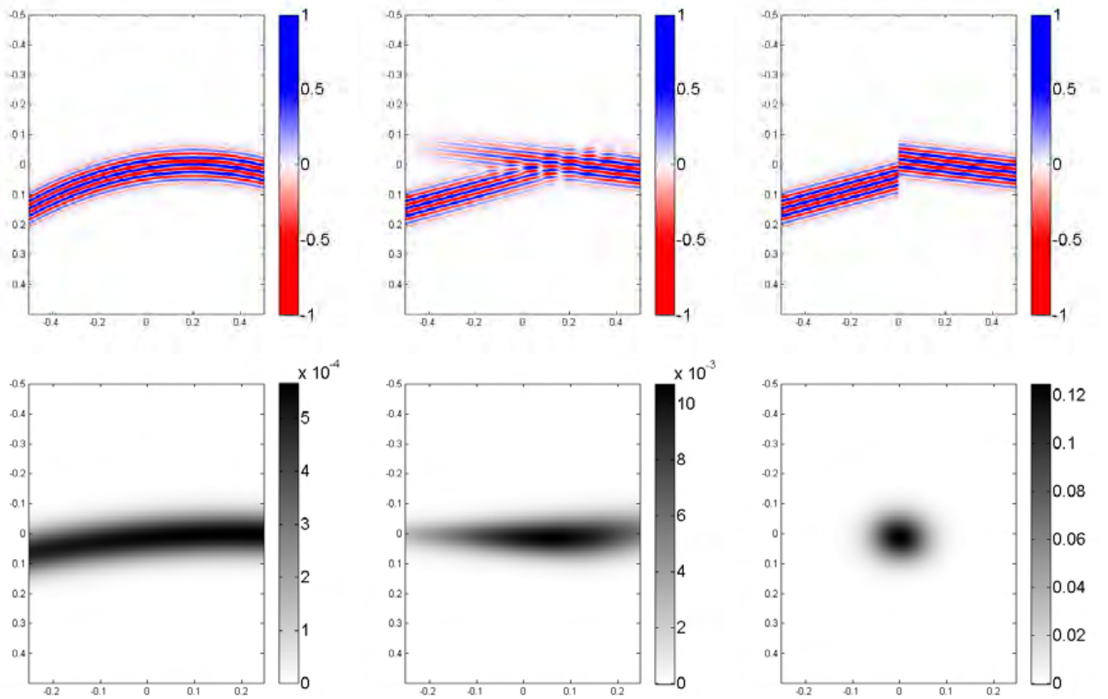

FIG. 1. *The left upper panel shows a simple synthetic data set. The middle and right upper panels shows two simple way of interpolation the interior of the image using derivative information f_x. In the lower panels the smallest eigenvalues of the structure tensor associated with the upper images are displayed. Note the large difference in magnitude for the three cases.*

indicate that f does not locally look like a plane wave. The bottom panel of Figure 1 show the smallest eigenvalues (as function of x and y), for the corresponding three images in the upper panels. Note the variation in magnitude for the three different cases. The eigenvalues have been normalized in relation to the maximum values of the largest eigenvalues of the structure tensor[2].

This motivates the construction of the following variational problem

$$\min_g \int s_2(T_g)(x,y)\,dx\,dy \quad \text{such that } g(x_l,y) = f(x_l,y),$$

where $s_2(T_g)(x,y)$ denotes the smallest eigenvalue of the structure tensor at the point (x,y).

In the next section we formalize this choice of variational problem, and show that solutions to it can be found by means of solutions to certain partial differential equations.

2. Problem description. In Weickert [12, Chapter 2], a result by Catté et al [3] concerning isotropic diffusion filters and its generalization to the anisotropic case is discussed. The result of Catté et al [3] gives existence and uniqueness of solutions to the problem

$$\partial u/\partial t - \operatorname{div}[g(|\nabla G_\sigma * u|)\nabla u] = 0 \quad \text{on } (0,T]\times\Omega,$$
$$u(0) = u_0,$$

[2]To avoid aliasing effects originating from the periodic convolutions, the eigenvalue maps are depicted for $|x| \leqslant \frac{1}{4}$ instead of $|x| \leqslant \frac{1}{2}$.

where g is a function exemplified by $g(t) = (1 + t^2)^{-1}$. Here Ω is the open set $(0, 1) \times (0, 1)$ in $\mathbb{R}^2$, and G_σ is a Gaussian filter given by

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(- \frac{|x|^2}{4\sigma} \right).$$

One would also need to impose some boundary conditions on the solution $u(t, x)$ on the boundary Γ of Ω, for example

$$\partial u/\partial\eta = 0 \quad \text{on } (0, T] \times \Gamma,$$

where η is the outward normal direction. (Note that the type of boundary conditions affects the definition of the convolution $G_\sigma * u$, since one needs to extend the domain of definition of u in an appropriate way.) The generalization of this problem to the anisotropic case involves replacing g with a smooth, symmetric positive definite diffusion tensor acting on the structure tensor $K_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma)$; the convolution with the Gaussian K_ρ is understood to be componentwise, and u_σ denotes the regularization $G_\sigma * u$.

We are interested in a closely related problem, derived from the following interpolation problem. We shall throughout this note assume that all functions are real valued unless stated otherwise. For a 2×2 symmetric matrix M, let $M = U\Sigma U^T$ be the singular value decomposition of M where $s_1 \geqslant s_2$ are the elements of the diagonal matrix Σ and U is unitary. Let $g : \mathbb{R} \to \mathbb{R}$ be a function to be specified later, and for real parameters a, b and c, consider the map $L : \mathbb{R}^3 \to \mathbb{R}$ given by

$$L(a, b, c) = g(s_1) + s_2, \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} = U \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} U^T. \tag{2.1}$$

For $f \in C^1(\mathbb{R}^2)$ we let f_x and f_y denote the partial derivatives, and similarly we denote by L_a, L_b and L_c the partial derivatives of L. Let K be a Gaussian function, and define the structure tensor T_f by (1.1). We will identify $T_f(x, y)$ with a vector in $\mathbb{R}^3$, also denoted by $T_f(x, y)$, and permit us to write $L \circ T_f$ to represent the map from $\mathbb{R}^2$ to $\mathbb{R}$ given by

$$L \circ T_f(x, y) = L(T_f(x, y)) = L(K * |f_x|^2(x, y), K * (f_x f_y)(x, y), K * |f_y|^2(x, y)).$$

We also permit us to evaluate the partial derivatives of L at $T_f(x, y)$ in this way.

Let $\Omega \subset \mathbb{R}^2$ be the open set $\Omega = (0, 1) \times (0, 1)$, and let $\lambda \in \mathscr{C}(\bar{\Omega})$ satisfy $\mu \leqslant \lambda(x, y) \leqslant 1$ for all $(x, y) \in \bar{\Omega}$, where $\mu > 0$ is a positive constant. For a given function $f_0 \in \mathscr{C}(\bar{\Omega})$ consider now the functional

$$\psi(f) = \int_\Omega (1 - \lambda(x, y))(f(x, y) - f_0(x, y))^2 dxdy + \int_\Omega \lambda(x, y) L(T_f(x, y)) dxdy. \tag{2.2}$$

We remark that we throughout this note always will assume that if given a function u defined on Ω, the convolution $K * u$ of u and a Gaussian K is defined as $K * \tilde{u}$, where $\tilde{u}$ is a linear and continuous extension of u to $\mathbb{R}^2$. Since we will be considering Neumann boundary data below, the extension will be given by

$$\begin{aligned} \tilde{u}(x, y) &= u(-x, y), \quad -1 \leqslant x \leqslant 0, \quad 0 \leqslant y \leqslant 1, \\ \tilde{u}(x, y) &= u(x-, y), \quad 0 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 0, \end{aligned} \tag{2.3}$$

and so on. We shall not distinguish between u and its extension $\tilde{u}$. A necessary condition for f to be a minimizer of (2.2) is found using standard variational calculus (compare with [13]): for all functions φ (in some suitable function space defined in terms of appropriate regularity and boundary conditions) we have

$$\begin{aligned} 0 =& \frac{d}{dt}\psi(f+t\varphi)\Big|_{t=0} \\ =& 2\int_\Omega (1-\lambda)(f-f_0)\varphi dxdy \\ &+ \int_\Omega \lambda\Big(2(L_a(T_f))K*(f_x\varphi_x) \\ &\qquad + (L_b(T_f))K*(f_x\varphi_y+f_y\varphi_x) + 2(L_c(T_f))K*(f_y\varphi_y)\Big)dxdy. \end{aligned} \tag{2.4}$$

Let I denote the integral in the right-hand side. Since K is radial, we have $\int K*u(x,y)v(x,y)dxdy = \int u(x,y)K*v(x,y)dxdy$ for all u and v, which together with a partial integration argument gives

$$\begin{aligned} I =& \int_\Omega (2(K*(\lambda(L_a\circ T_f)))f_x\varphi_x + (K*(\lambda(L_b\circ T_f)))(f_x\varphi_y+f_y\varphi_x) \\ &\qquad + 2(K*(\lambda(L_c\circ T_f)))f_y\varphi_y)dxdy \\ =& -\int_\Omega \Big(\frac{\partial}{\partial x}\Big(2(K*(\lambda(L_a\circ T_f)))f_x + (K*(\lambda(L_b\circ T_f)))f_y\Big) \\ &\qquad + \frac{\partial}{\partial y}\Big((K*(\lambda(L_b\circ T_f)))f_x + 2(K*(\lambda(L_c\circ T_f)))f_y\Big)\Big)\varphi dxdy. \end{aligned}$$

Here, the integrand in the last integral can be written as the product of φ and the factor

$$2\,\mathrm{div}\left(\begin{pmatrix} K*(\lambda(L_a\circ T_f)) & \frac{1}{2}K*(\lambda(L_b\circ T_f)) \\ \frac{1}{2}K*(\lambda(L_b\circ T_f)) & K*(\lambda(L_c\circ T_f)) \end{pmatrix}\nabla f\right).$$

Dividing by a factor 2, and working within the distributional framework to restrict our attention to test functions $\varphi \in C_0^\infty(\Omega)$, we can therefore express the necessary condition (2.4) for f to be a minimizer of the functional (2.2) in the sense indicated by saying that f must be a solution to the equation

$$(1-\lambda)f - \mathrm{div}\left(\begin{pmatrix} K*(\lambda(L_a\circ T_f)) & \frac{1}{2}K*(\lambda(L_b\circ T_f)) \\ \frac{1}{2}K*(\lambda(L_b\circ T_f)) & K*(\lambda(L_c\circ T_f)) \end{pmatrix}\nabla f\right) = (1-\lambda)f_0 \tag{2.5}$$

in $\mathscr{D}'(\Omega)$.

For simplicity, let us now consider the case $\lambda = 1$, although the arguments used treating this case are valid also for any λ of the type introduced above. For technical reasons, we will also replace the structure tensor T_f with a regularized version utilizing the Gaussian filter G_σ introduced above, which we shall also denote by T_f, but this should not cause any confusion. Let f_σ denote the regularization $G_\sigma * f$, and consider the structure tensor

$$T_f = \begin{pmatrix} K*|G_\sigma * f_x|^2 & K*((G_\sigma*f_x)(G_\sigma*f_y)) \\ K*((G_\sigma*f_x)(G_\sigma*f_y)) & K*|G_\sigma*f_y|^2 \end{pmatrix}.$$

We shall thus consider the differential equation

$$\mathrm{div}\left(\begin{pmatrix} K*(L_a\circ T_f) & \frac{1}{2}K*(L_b\circ T_f) \\ \frac{1}{2}K*(L_b\circ T_f) & K*(L_c\circ T_f) \end{pmatrix}\nabla f\right) = 0$$

in $\mathscr{D}'(\Omega)$. One way to solve this problem is to solve the nonlinear diffusion problem

$$\begin{aligned} u_t(t,x,y) - \mathrm{div}\left(\begin{pmatrix} K*(L_a\circ T_{u(t)}) & K*(L_b\circ T_{u(t)}) \\ K*(L_b\circ T_{u(t)}) & K*(L_c\circ T_{u(t)}) \end{pmatrix}\nabla u(t)\right) = 0, \\ u(0) = u_0, \end{aligned} \tag{2.6}$$

where $u(t) : \Omega \to \mathbb{R}$ plays the role of the function f above, and then to consider the steady state solution

$$f(x,y) \approx \lim_{t\to\infty} u(t,x,y).$$

Comparison with the problem studied by Weickert in [12, Chapter 2] shows that we have replaced the diffusion tensor $D : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ acting on a structure tensor of the form $K_\rho * (\nabla f_\sigma \otimes \nabla f_\sigma)$, that is, a map

$$\nabla f_\sigma \mapsto D(K_\rho * (\nabla f_\sigma \otimes \nabla f_\sigma)), \tag{2.7}$$

by the matrix valued map

$$\nabla f_\sigma \mapsto \begin{pmatrix} K * (L_a \circ T_f) & K * (L_b \circ T_f) \\ K * (L_b \circ T_f) & K * (L_c \circ T_f) \end{pmatrix}. \tag{2.8}$$

Here we also wish to mention the study by Hahn and Lee [6], which concerns a problem similar to ours but with a different setup.

To find a suitable function g in the definition (2.1) of L, we now digress to discuss the map (2.8). Without loss of generality, we may assume that

$$L(a,b,c) = g(s_1) + s_2, \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} = U \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} U^T$$

with $s_1 \geqslant s_2$ and

$$U = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

for $\theta = \theta(a,b,c)$. It is straightforward to check that

$$L(a,b,c) = g(a\sin^2\theta - 2b\cos\theta\sin\theta + c\cos^2\theta) + a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta,$$

where $\theta = \theta(a,b,c)$ is a solution to the equation

$$(c-a)\cos\theta\sin\theta + b(\cos^2\theta - \sin^2\theta) = 0. \tag{2.9}$$

Introduce the auxiliary functions ψ and φ given by

$$\begin{aligned} s_1 = \psi(a,b,c) &= a\sin^2\theta - 2b\cos\theta\sin\theta + c\cos^2\theta, \quad \theta = \theta(a,b,c), \\ s_2 = \varphi(a,b,c) &= a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta, \quad \theta = \theta(a,b,c). \end{aligned}$$

By virtue of (2.9), a straightforward computation shows that the partial derivatives of ψ and φ are given by

$$\begin{aligned} &\psi_a = \sin^2\theta, \quad \psi_b = -2\cos\theta\sin\theta, \quad \psi_c = \cos^2\theta, \\ &\varphi_a = \cos^2\theta, \quad \varphi_b = 2\cos\theta\sin\theta, \qquad \varphi_c = \sin^2\theta. \end{aligned}$$

The partial derivatives L_a, L_b and L_c of L are therefore found to be

$$\begin{aligned} L_a(a,b,c) &= \cos^2\theta + g'(\psi(a,b,c))\sin^2\theta, \\ L_b(a,b,c) &= 2\cos\theta\sin\theta(1 - g'(\psi(a,b,c))), \\ L_c(a,b,c) &= \sin^2\theta + g'(\psi(a,b,c))\cos^2\theta, \end{aligned}$$

where $\theta = \theta(a,b,c)$. Hence, we have

$$\operatorname{div}\left(\left(\begin{array}{cc} K*(L_a\circ T_f) & \frac{1}{2}K*(L_b\circ T_f) \\ \frac{1}{2}K*(L_b\circ T_f) & K*(L_c\circ T_f)\end{array}\right)\nabla f\right) = \operatorname{div}(S_f\nabla f)$$

with the matrix S_f given by

$$S_f = K*\left(\left(\begin{array}{cc}\cos^2\omega & \cos\omega\sin\omega \\ \cos\omega\sin\omega & \sin^2\omega\end{array}\right) + \left(\begin{array}{cc} g'(\psi\circ T_f)\sin^2\omega & -g'(\psi\circ T_f)\cos\omega\sin\omega \\ -g'(\psi\circ T_f)\cos\omega\sin\omega & g'(\psi\circ T_f)\cos^2\omega\end{array}\right)\right)$$

where

$$\omega = \theta\circ T_f. \tag{2.10}$$

Note that if $(\psi\circ T_f)(\xi,\eta)$ and $(\varphi\circ T_f)(\xi,\eta)$ are the largest and the smallest eigenvalues of $T_{f(\xi,\eta)}$, respectively, then

(2.11)

$$S_{f(x,y)} = K*\left(\left(\begin{array}{cc} -\sin\theta\circ T_f & \cos\theta\circ T_f \\ \cos\theta\circ T_f & \sin\theta\circ T_f\end{array}\right)\left(\begin{array}{cc} g'\circ\psi\circ T_f & 0 \\ 0 & 1\end{array}\right)\left(\begin{array}{cc} -\sin\theta\circ T_f & \cos\theta\circ T_f \\ \cos\theta\circ T_f & \sin\theta\circ T_f\end{array}\right)\right)(x,y).$$

Proposition 1. *Let $\| \ \|_F$ be the Frobenius norm on the space $\mathscr{S}$ of symmetric 2×2 matrices. Let $\mathscr{V}$ be the set of maps $\mathbb{R}^2\to\mathscr{S}$. Let Υ be the map from a subset of $\mathscr{V}$ into $\mathscr{V}$ given by $\Upsilon : T_f\mapsto S_f$ for $f\in L^\infty(\mathbb{R}^2)$, so that $\Upsilon T_f(x,y) = S_{f(x,y)}$. If $g' : \mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, then there is a constant C depending only on K and g' such that*

$$\|\Upsilon T_u(x,y) - \Upsilon T_v(x,y)\|_F = \|S_{u(x,y)} - S_{v(x,y)}\|_F \leqslant C\sup_{(\xi,\eta)\in\Omega}\|T_{u(\xi,\eta)} - T_{v(\xi,\eta)}\|_F$$

for all u and v in $L^\infty(\mathbb{R}^2)$.

Proof. First note that the map on $\mathscr{S}$ given by

$$\hat{\Upsilon} : \left(\begin{array}{cc} a & b \\ b & c\end{array}\right)\mapsto U\left(\begin{array}{cc} g'(s_1) & 0 \\ 0 & 1\end{array}\right)U^T, \quad \left(\begin{array}{cc} a & b \\ b & c\end{array}\right) = U\left(\begin{array}{cc} s_1 & 0 \\ 0 & 1\end{array}\right)U^T,$$

is Lipschitz continuous by [9, Theorem 5.2]. Therefore, there is a constant C_{Lip} depending only on g' such that

$$\|\hat{\Upsilon}A - \hat{\Upsilon}B\|_F \leqslant C_{\mathrm{Lip}}\|A - B\|_F$$

for all A and B in $\mathscr{S}$. With

$$\hat{S}_{f(x,y)} = \left(\left(\begin{array}{cc} -\sin\theta\circ T_f & \cos\theta\circ T_f \\ \cos\theta\circ T_f & \sin\theta\circ T_f\end{array}\right)\left(\begin{array}{cc} g'\circ\psi\circ T_f & 0 \\ 0 & 1\end{array}\right)\left(\begin{array}{cc} -\sin\theta\circ T_f & \cos\theta\circ T_f \\ \cos\theta\circ T_f & \sin\theta\circ T_f\end{array}\right)\right)(x,y),$$

so that $S_{f(x,y)} = K*\hat{S}_f(x,y)$, this means that the map on $\mathscr{S}$ given by $T_{f(x,y)}\mapsto\hat{S}_{f(x,y)}$ satisfies

$$\|\hat{S}_{u(x,y)} - \hat{S}_{v(x,y)}\|_F \leqslant C_{\mathrm{Lip}}\|T_{u(x,y)} - T_{v(x,y)}\|_F, \quad u,v\in L^\infty(\mathbb{R}^2).$$

Now

$$\|S_{u(x,y)} - S_{v(x,y)}\|_F = \|K*(\hat{S}_u - \hat{S}_v)(x,y)\|_F \leqslant K*\|\hat{S}_u - \hat{S}_v\|_F(x,y),$$

which can be seen to hold by approximating the convolution by a Riemann sum, convergent in $\mathscr{C}^\infty$. Estimation of the right-hand side gives

$$\begin{aligned}\|S_{u(x,y)} - S_{v(x,y)}\|_F &\leqslant \|K\|_{L^1(\mathbb{R}^2)}\sup_{(\xi,\eta)\in\mathbb{R}^2}\|\hat{S}_{u(\xi,\eta)} - \hat{S}_{v(\xi,\eta)}\|_F \\ &\leqslant \|K\|_{L^1(\mathbb{R}^2)}C_{\mathrm{Lip}}\sup_{(\xi,\eta)\in\mathbb{R}^2}\|T_{u(\xi,\eta)} - T_{v(\xi,\eta)}\|_F,\end{aligned}$$

which completes the proof. $\square$

We shall now estimate $\|\psi \circ T_f\|_{L^\infty(\Omega)}$. Note that

$$\psi \circ T_f = (K * |G_\sigma * f_x|^2) \sin^2 \omega - 2(K * ((G_\sigma * f_x)(G_\sigma * f_y))) \cos\omega \sin\omega + (K * |G_\sigma * f_y|^2) \cos^2 \omega.$$

By trivially estimating the sine and cosine in the expression above, and applying Hölder's inequality, we have

$$\begin{aligned} \|\psi \circ T_f\|_{L^\infty(\Omega)} \leqslant \|K\|_{L^1(\mathbb{R}^2)} (&\|\partial_x G_\sigma * f\|^2_{L^\infty(\mathbb{R}^2)} \\ &+ 2\|\partial_x G_\sigma * f\|_{L^\infty(\mathbb{R}^2)} \|\partial_y G_\sigma * f\|_{L^\infty(\mathbb{R}^2)} + \|\partial_y G_\sigma * f\|^2_{L^\infty(\mathbb{R}^2)}). \end{aligned} \tag{2.12}$$

A change of variables shows that

$$|\partial_x G_\sigma * f(x,y)| \leqslant \sum_{(k,\ell)\in\mathbb{Z}^2} \int_\Omega |\partial_x G_\sigma(x-\xi-k, y-\eta-\ell)||f(\xi+k,\eta+\ell)| d\xi d\eta.$$

By virtue of the definition (2.3) of the extension of f used to define the convolution with G_σ, it is straightforward to check that the Cauchy-Schwartz inequality then gives the estimate

$$|\partial_x G_\sigma * f(x,y)| \leqslant \sum_{(k,\ell)\in\mathbb{Z}^2} \|\partial_x G_\sigma(-(\cdot+k-x), -(\cdot+\ell-y))\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}.$$

Given $(x,y) \in \mathbb{R}^2$, there are integers M and N such that $[x, x+1] \subset [M-1, M+1]$ and $[y, y+1] \subset [N-1, N+1]$. To simplify notation, let $\tau_{(x,y)}$ be the translation operator $\tau_{(x,y)} u(\xi,\eta) = u(\xi - x, \eta - y)$, and introduce the function $F(\xi,\eta) = \partial_x G_\sigma(-\xi,-\eta)$ so that

$$\|\tau_{(x-k,y-\ell)} F\|_{L^2(\Omega)} = \|\partial_x G_\sigma(-(\cdot+k-x), -(\cdot+\ell-y))\|_{L^2(\Omega)}.$$

By a change of variables, this norm satisfies

$$\begin{aligned} \|\tau_{(x-k,y-\ell)} F\|^2_{L^2(\Omega)} &\leqslant \int_{M-1}^{M+1} |\xi - k|^2 \mathrm{e}^{-\frac{|\xi-k|^2}{2\sigma}} d\xi \int_{N-1}^{N+1} \mathrm{e}^{-\frac{|\eta-\ell|^2}{2\sigma}} d\eta \\ &\leqslant 2(1+|k-M|)^2 \mathrm{e}^{-\frac{(|k-M|-1)^2}{2\sigma}} 2\mathrm{e}^{-\frac{|\ell-N|^2}{2\sigma}} \end{aligned}$$

for all $|k-M| \geqslant 1$ and $|\ell - N| \geqslant 1$. Trivially we also have

$$\int_{M-1}^{M+1} |\xi - M|^2 \mathrm{e}^{-\frac{|\xi-M|^2}{2\sigma}} d\xi \leqslant 2 \quad \text{and} \quad \int_{N-1}^{N+1} \mathrm{e}^{-\frac{|\eta-N|^2}{2\sigma}} d\eta \leqslant 2.$$

A straightforward computation then gives that

$$\sum_{(k,\ell)\in\mathbb{Z}^2} \|\tau_{(x-k,y-\ell)} F\|_{L^2(\Omega)} \leqslant \left(\sqrt{2} + 2\sqrt{2} \sum_{n=1}^{\infty} (1+n) \mathrm{e}^{-\frac{(n-1)^2}{4\sigma}}\right) \left(\sqrt{2} + 2\sqrt{2} \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{(n-1)^2}{4\sigma}}\right),$$

which implies that

$$\|\partial_x G_\sigma * f\|_{L^\infty(\mathbb{R}^2)} \leqslant C_\sigma \|f\|_{L^2(\Omega)} \tag{2.13}$$

where the constant only depends on G_σ. We have the same estimates for the terms involving y derivatives in (2.12), from which we therefore obtain

$$\|\psi \circ T_f\|_{L^\infty(\mathbb{R}^2)} \leqslant 4C_\sigma^2 \|K\|_{L^1(\mathbb{R}^2)} \|f\|^2_{L^2(\Omega)}.$$

By comparing the definitions of ψ and φ, it is also clear that

$$\|\varphi \circ T_f\|_{L^\infty(\mathbb{R}^2)} \leqslant 4C_\sigma^2 \|K\|_{L^1(\mathbb{R}^2)} \|f\|^2_{L^2(\Omega)}.$$

Recall that $\psi \circ T_f$ is the largest eigenvalue of the matrix T_f. Since the entries on the diagonal in T_f are nonnegative, it follows that $\psi \circ T_f \geqslant 0$ for all $f \in C^1(\mathbb{R}^2)$. Hence, if g' is a positive and decreasing on the positive half-axis with $\lim_{t \to \infty} g'(t) = 0$ then for all f in a bounded subset of $L^2(\Omega)$ it follows that $g'(\psi \circ T_f)$ is bounded away from zero from below, uniformly with respect to f. If in addition $g'(0) = 1$ we have

$$0 < \nu_0 \leqslant g'(\psi \circ T_f) \leqslant 1 \tag{2.14}$$

for all f in a bounded subset of $L^2(\Omega)$. This allows us to conclude that the matrix S_f is positive and uniformly bounded away from zero from below with respect to $f \in B$, where $B \subset L^2(\Omega)$ is bounded.

Lemma 2. *The smallest eigenvalue of the matrix S_f is positive and uniformly bounded away from zero from below with respect to $f \in B$, where B is a bounded subset of $L^2(\Omega)$.*

Proof. Let A and B be two (symmetric) positive semidefinite matrices. Then

$$\min_{\|z\|=1} z^T(A+B)z \geqslant \left(\min_{\|z\|=1} z^T A z\right) + \left(\min_{\|z\|=1} z^T B z\right) \tag{2.15}$$

with equality only if the eigenvectors corresponding to the smallest eigenvalue of A and B, respectively, are parallel. Note that the smallest eigenvalue of

$$\begin{pmatrix} \cos^2\omega & \cos\omega\sin\omega \\ \cos\omega\sin\omega & \sin^2\omega \end{pmatrix} + g'(\psi \circ T_f)\begin{pmatrix} \sin^2\omega & -\cos\omega\sin\omega \\ -\cos\omega\sin\omega & \cos^2\omega \end{pmatrix}$$

is equal to $g'(\psi \circ T_f)$, which follows from (2.14) and since both the two matrices above are of rank 1. From applying (2.15) to the Riemann integral definition defining the convolution with K we then conclude that the smallest eigenvalue of S_f is larger than $\nu_0\|K\|_1$. □

We remark that Weickert [12] introduces a condition on the diffusion tensor D in (2.7) which he calls "uniformly positively definiteness", similar to that given by Lemma 2; compare with condition (C) on p. 58 in [12].

The arguments used in the proof of Lemma 2 also show that

$$(\nabla v)^T S_f \nabla u \leqslant C_B |\nabla u||\nabla v| \tag{2.16}$$

for all $f \in B$, where $B \subset L^2(\Omega)$ is bounded. With the notation introduced above, we have the following result.

Theorem 3. *Let $u_0 \in L^2(\Omega)$. Then there is a unique function $u = u(t,x)$ in $\mathscr{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ such that u is a solution to*

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(S_u \nabla u) &= 0 \quad \text{in } (0,T] \times \Omega, \\ (S_u \nabla u, n)_{\mathbb{R}^2} &= 0 \quad \text{on } (0,T] \times \Gamma \\ u(0) &= u_0, \end{aligned} \tag{2.17}$$

where Γ is the boundary of Ω, n is the outward normal direction on Γ and $(\ ,\ )_{\mathbb{R}^2}$ is the usual scalar product in $\mathbb{R}^2$. Moreover, this solution is in $\mathscr{C}^\infty((0,T) \times \bar{\Omega})$.

Here, $H^k(\Omega)$ for nonnegative integers $k \in \mathbb{N}$ is the Sobolev space of distributions with weak derivatives of order $\leqslant k$ belonging to $L^2(\Omega)$. This is a Hilbert space with the norm

$$\|v\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha v\|^2_{L^2(\Omega)}\right)^{1/2}$$

where we use the standard multi-index notation $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ for $\alpha \in \mathbb{N} \times \mathbb{N}$, with $|\alpha| = \alpha_1 + \alpha_2$. Moreover, $L^p(0,T; H^k(\Omega))$ is the space consisting of all distributions u such that $u(t) \in H^k(\Omega)$ for almost every $t \in (0,T)$, equipped with the norm

$$\|u\|_{L^p(0,T;H^k(\Omega))} = \left(\int_0^T \|u(t)\|^p_{H^k(\Omega)}dt\right)^{1/p}, \quad p > 1, \quad k \in \mathbb{N}.$$

When $p = \infty$, this is to understood as the essential supremum norm $\operatorname{ess\,sup}_{0<t<T} \|u(t)\|_{H^k(\Omega)}$. Note also that the type of boundary conditions imposed in Theorem 3 affects the definition of the convolution $G_\sigma * u$, since one needs to extend the domain of definition of u as done in (2.3).

Proof. The idea is to adapt the proof of [3, Theorem 2.1] to our situation. We first determine the existence of a weak solution to (2.17). To this end, let $(H^1(\Omega))^*$ denote the dual of $H^1(\Omega)$, and introduce the subspace $W(0,T)$ of $L^2(0,T;H^1(\Omega))$ defined by

$$W(0,T) = \left\{w \in L^2(0,T;H^1(\Omega)) : \frac{dw}{dt} \in L^2(0,T;(H^1(\Omega))^*)\right\}.$$

Let $w \in W(0,T) \cap L^\infty(0,T;L^2(\Omega))$ satisfy

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \|u_0\|_{L^2(\Omega)} \tag{2.18}$$

and consider the auxiliary problem (E_w) given by

$$(E_w) \qquad \begin{aligned} (\partial u(t)/\partial t, v)_{L^2(\Omega)} + \int_\Omega (\nabla v)^T S_{w(t)} \nabla u(t) dxdy = 0 \quad \text{for all } v \in H^1(\Omega) \quad \text{a.e. in } [0,T], \\ u(0) = u_0, \end{aligned}$$

where $(\ ,\ )_{L^2(\Omega)}$ denotes the usual inner product on $L^2(\Omega)$. Let $\mathscr{A}(t;u,v) = \int_\Omega (\nabla v)^T S_{w(t)} \nabla u dxdy$ for u and v in $H^1(\Omega)$. By Lemma 2 it follows by virtue of (2.18) that

$$\mathscr{A}(t;v,v) \geqslant \nu \int_\Omega |\nabla v|^2 dxdy = \nu\|v\|_{H^1(\Omega)} - \nu\|v\|_{L^2(\Omega)} \quad \text{for almost every } t \in (0,T),$$

where ν is a positive constant independent of w (but depending on u_0). Similarly, property (2.16) together with an application of the Cauchy-Schwartz inequality implies that $\mathscr{A}(t;u,v) \leqslant C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$ for almost every $t \in (0,T)$, where the constant is independent of w. We may then apply a result due to J. L. Lions, see Brezis [2, Théorème X.9], and conclude that the problem (E_w) has a unique solution $U_w \in L^2(0,T;H^1(\Omega)) \cap \mathscr{C}([0,T];L^2(\Omega))$ with $dU_w/dt \in L^2(0,T;(H^1(\Omega))^*)$. In particular, we have $U_w \in W(0,T)$.

Following [3, pp. 188-189], we now deduce that $U : w \mapsto U_w$ preserves a nonempty, convex and weakly compact subset W_0 of $W(0,T)$, defined by

$$\begin{aligned} W_0 = \{w \in W(0,T) : w(0) = u_0,\ \|w\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \|u_0\|_{L^2(\Omega)}, \\ \|w\|_{L^2(0,T;H^1(\Omega))} \leqslant C_1,\ \|dw/dt\|_{L^2(0,T;(H^1(\Omega))^*)} \leqslant C_2\}. \end{aligned}$$

By considerations of compact inclusions of Sobolev spaces if follows that U is weakly continuous on W_0, see the theorem of Rellich and Kondrachov in Brezis [2, Théorème IX.16]. Since $W(0,T)$ is compactly embedded in $L^2(0,T;L^2(\Omega))$, the Schauder fixed point theorem can then be applied to conclude the existence of an element $u = U_u$ in W_0, see for example Friedman [5, p. 189]. Thus we have found a weak solution u to the problem (2.17).

Regularity of solutions. By a bootstrap argument follows that $u(t) \in H^1(\Omega)$ for all $t > 0$, from which we conclude that $u(t) \in H^2(\Omega)$ for all $t > 0$. Iterating the argument and using the general theory of parabolic equations [2, 7], we conclude that $u \in \mathscr{C}^\infty((0,T) \times \bar{\Omega})$, and that u is a strong solution to (2.17). In particular, the boundary condition in (2.17) must hold; compare for example with [2, Étape D, p. 180].

Uniqueness of solutions. Let u and v be two solutions to (2.17). They will then satisfy

$$du(t)/dt - \operatorname{div}(S_{u(t)}\nabla u(t)) = 0,$$
$$dv(t)/dt - \operatorname{div}(S_{v(t)}\nabla v(t)) = 0.$$

The difference of the two equations above can be written

$$d(u(t)-v(t))/dt - \operatorname{div}(S_{u(t)}(\nabla u(t)-\nabla v(t))) = \operatorname{div}((S_{u(t)}-S_{v(t)})\nabla v(t)). \tag{2.19}$$

Multiplying (2.19) with $u(t)-v(t)$ and integrating over Ω gives after partial integrations

$$\begin{aligned}\int_\Omega \frac{1}{2}\frac{d}{dt}|u(t)-v(t)|^2 dxdy = &\int_\Omega (\nabla u(t)-\nabla v(t))^T(S_{u(t)}-S_{v(t)})\nabla v(t)dxdy \\ &- \int_\Omega (\nabla u(t)-\nabla v(t))^T S_{u(t)}(\nabla u(t)-\nabla v(t))dxdy.\end{aligned} \tag{2.20}$$

Recall that if u is a solution then $t \mapsto \|u(t)\|_{L^2(\Omega)}$ is continuous, and that $\|u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \|u_0\|_{L^2(\Omega)}$. Since v is also a solution, Lemma 2 implies that we can find a constant ν such that

$$\int_\Omega (\nabla u(t)-\nabla v(t))^T S_{u(t)}(\nabla u(t)-\nabla v(t))dxdy \geqslant \nu\|\nabla u(t)-\nabla v(t)\|^2_{L^2(\Omega)}$$

for all $t\in[0,T]$. Hence,

$$\frac{1}{2}\frac{d}{dt}\|u(t)-v(t)\|^2_{L^2(\Omega)}+\nu\|\nabla u(t)-\nabla v(t)\|^2_{L^2(\Omega)} \leqslant \int_\Omega (\nabla u(t)-\nabla v(t))^T(S_{u(t)}-S_{v(t)})\nabla v(t)dxdy, \tag{2.21}$$

where, by an application of the Cauchy-Schwartz inequality, the right-hand side is bounded by

$$\sup_\Omega \|S_{u(t)}-S_{v(t)}\|_{\mathrm{spec}}\|\nabla u(t)-\nabla v(t)\|_{L^2(\Omega)}\|\nabla v(t)\|_{L^2(\Omega)}.$$

Here, $\|A\|_{\mathrm{spec}}$ is the spectral radius norm of the matrix A, which satisfies $\|A\|_{\mathrm{spec}} \leqslant \|A\|_F$, the second norm being Frobenius norm. We now claim that

$$\sup_{(x,y)\in\Omega} \|S_{u(t,x,y)}-S_{v(t,x,y)}\|_F \leqslant C\|u(t)-v(t)\|_{L^2(\Omega)}, \tag{2.22}$$

where the constant depends only on K, G_σ, g' and the initial value u_0. Admitting this for the moment, the previous discussion and an application of Young's inequality then gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)-v(t)\|^2_{L^2(\Omega)}+\nu\|\nabla u(t)-\nabla v(t)\|^2_{L^2(\Omega)} \leqslant &\frac{\nu}{2}\|\nabla u(t)-\nabla v(t)\|^2_{L^2(\Omega)} \\ &+ \frac{C^2}{2\nu}\|u(t)-v(t)\|^2_{L^2(\Omega)}\|\nabla v(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

After subtracting $\nu\|\nabla u(t)-\nabla v(t)\|^2_{L^2(\Omega)}$ from both sides, another estimation gives

$$\frac{d}{dt}\|u(t)-v(t)\|^2_{L^2(\Omega)} \leqslant \frac{C^2}{\nu}\|u(t)-v(t)\|^2_{L^2(\Omega)}\|\nabla v(t)\|^2_{L^2(\Omega)}.$$

Together with the initial condition $u(0)-v(0)=0$, an application of Gröwall's inequality to this estimate now shows that $\|u(t)-v(t)\|^2_{L^2(\Omega)}$ is constant on $[0,T]$. Since $u(0)-v(0)=0$, uniqueness follows.

It remains to prove (2.22). By Proposition 1 we have that

$$\|S_{u(t,x,y)}-S_{v(t,x,y)}\|_F \leqslant C \sup_{(\xi,\eta)\in\Omega}\|T_{u(t,\xi,\eta)}-T_{v(t,\xi,\eta)}\|_F, \tag{2.23}$$

where C only depends on K and g'. To simplify notation, let us for the moment suppress the parameter t. Note that $T_u - T_v$ is the symmetric matrix

$$T_u - T_v = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{2.24}$$

where a straightforward computation shows that a, b and c can be written as

$$\begin{aligned} a &= K * ((\partial_x G_\sigma * (u - v))(\partial_x G_\sigma * (u + v))), \\ b &= K * ((\partial_y G_\sigma * u)(\partial_x G_\sigma * (u - v)) + (\partial_x G_\sigma * v)(\partial_y G_\sigma * (u - v))), \\ c &= K * ((\partial_y G_\sigma * (u - v))(\partial_y G_\sigma * (u + v))). \end{aligned}$$

Let as before $\psi(a, b, c)$ and $\varphi(a, b, c)$ denote the largest and smallest eigenvalues of the symmetric matrix given by (2.24). In view of the expressions for a, b and c, estimates similar to (2.12) and (2.13) show that

$$\|T_{u(x,y)} - T_{v(x,y)}\|_F \leqslant C(K, G_\sigma)(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)})\|u - v\|_{L^2(\Omega)}.$$

Reintroducing the parameter t, recall that since u and v are solutions to (2.17), we have that the norms $\|u(t)\|_{L^2(\Omega)}$ and $\|v(t)\|_{L^2(\Omega)}$ are bounded by $\|u_0\|_{L^2(\Omega)}$ for all $t \in [0, T]$. By virtue of (2.23) this gives (2.22), which completes the proof. □

3. Numerical aspects. For solving the problem (2.5) we consider the steady state solution to

$$u_t'(t, x, y) = \lambda(x) \operatorname{div}(S_(u(t, x, y))\nabla u(t, x, y)) + (1 - \lambda(x))(u(t, x, y) - f_0(x, y)). \tag{3.1}$$

Written out explicitly, this is an parabolic equation containing non-zero contribution from the mixed derivatives, i.e.,

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + (1 - \lambda(x))(u - f_0), \tag{3.2}$$

where the coefficients a, b, c, d and e depend on the solution u. Due to the averaging in the computation of S_u they vary slowly compared to u which heuristically makes feasible to instead solve the linearized equations. The fixed point argument in the existence part of Theorem **??** formalizes this claim. For the solution of (3.1), we therefore compute a, b, c, d and e given the solution u at time t. As t increases we then recompute the coefficient functions. Due to the averaging effect, the coefficient function does not in practice need to be recomputed at each time step, but multiple time steps can be taken before it is needed to update them.

For the solution of parabolic differential equations, we can choose between using an explicit or implicit method. Explicit methods have the advantage that they are easy to implement, and that each time step can easily be computed in a fast manner. However, they require that the time steps taken are small in order to be stable. If large timesteps are desirable, it is advantageous to instead use an implicit method. We provide details for how to deal with both explicit and implicit implementations.

First, let us consider the computation of the coefficient functions a, b, c, d and e for a fixed time t. The mollified version of the derivatives $G_\sigma u_x(t, x, y)$ and $G_\sigma u_y(t, x, y)$ can be computed rapidly be means of FFT. Also by using the FFT, we can rapidly compute the functions $K * |G_\sigma u_x(t, x, y)|^2$, $K * G_\sigma u_x(t, x, y) G_\sigma u_y(t, x, y)$ and $K * |G_\sigma u_y(t, x, y)|^2$. For the computation of $\hat{S}$, let

$$\hat{a} = K * |G_\sigma u_x(t, x, y)|^2, \quad \hat{b} = K * (G_\sigma u_x(t, x, y)|G_\sigma u_y(t, x, y)|), \quad \hat{c} = K * |G_\sigma u_y(t, x, y)|^2.$$

The angle ω in (2.10) is then given by

$$\omega = \arctan\left(\frac{\hat{c} - \hat{a} + \sqrt{\hat{c}^2 - 2\hat{a}\hat{c} + \hat{a}^2 + 4\hat{b}^2}}{2\hat{b}}\right)$$

at each point (x, y), and the largest eigenvector s_1 of the matrix

$$\begin{pmatrix} \hat{a} & \hat{b} \\ \hat{b} & \hat{c} \end{pmatrix}$$

it holds that

$$s_1 = \frac{\hat{a} + \hat{c} + \sqrt{\hat{a}^2 + \hat{c}^2 - 2\hat{a}\hat{c} + +4\hat{b}^2}}{2}.$$

Using the formulas above, we can compute the $\theta(x, y)$ and $s1(x, y)$ explicitly, without having looping through each point (x, y) and make eigenvalue decompositions of 2×2 matrices.

We now have explicit formulas for the computation of elements

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{pmatrix} = \left(\begin{pmatrix} -\sin\theta \circ T_f & \cos\theta \circ T_f \\ \cos\theta \circ T_f & \sin\theta \circ T_f \end{pmatrix} \begin{pmatrix} g' \circ \psi \circ T_f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\theta \circ T_f & \cos\theta \circ T_f \\ \cos\theta \circ T_f & \sin\theta \circ T_f \end{pmatrix} \right),$$

and we may thus obtain

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = S_{u(t)},$$

by using (2.11) and FFT to compute the element-wise convolutions with K.

Now, let us consider an explicit discretization. The simplest version is to use an scheme which is first order approximate in time and second order approximate in the spatial discretization. We thus approximate

$$\frac{\partial u}{\partial t}(t, x, y) \approx \frac{u(t + \delta_t, x, y) - u(t, x, y)}{\delta_t}$$

and use standard stencils for the spatial discretizations, i.e.

$$\frac{\partial^2}{\partial x^2} \approx \frac{1}{\delta_x^2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial^2}{\partial x \partial y} \approx \frac{1}{4\delta_x\delta_y} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \frac{\partial^2}{\partial y^2} \approx \frac{1}{\delta_y^2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

for a second order scheme, and

$$\frac{1}{12\delta_x^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 16 & -30 & 16 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{1}{144\delta_x\delta_y} \begin{bmatrix} 1 & -8 & 0 & 8 & -1 \\ -8 & 64 & 0 & -64 & 80 \\ 0 & 0 & 0 & 0 & 0 \\ 8 & -64 & 0 & 64 & -8 \\ -1 & 8 & 0 & -8 & 1 \end{bmatrix},$$

$$\frac{1}{12\delta_y^2} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & -30 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

for a fourth order spatial scheme. The explicit discretization then reads

$$\begin{aligned} u(t + \delta_t, x, y) = & u(t, x, y) + \delta_t \bigg(a\frac{\partial^2 u}{\partial x^2}(t, x, y) + 2b\frac{\partial^2 u}{\partial x \partial y}(t, x, y) + c\frac{\partial^2 u}{\partial y^2}(t, x, y) \\ & + d\frac{\partial u}{\partial x}(t, x, y) + e\frac{\partial u}{\partial y}(t, x, y) + (1 - \lambda(x))(u(t, x, y) - f_0(x, y)) \bigg). \end{aligned}$$

Implicit methods are numerical schemes where $u(t + j\delta_t, x, y)$ for $j \in \mathbb{N}^+$ appear on the right hand side of the equation (3.2). Hence, in order to obtain $u(t+\delta_t, x, y)$ when $u(t-j\delta_t, x, y)$, $j \in \mathbb{N}^0$, it is necessary to solve a system of linear equations. Using generic methods for inverting the matrices that the stencils give rise to will prohibitively slow. However, for (short) finite difference stencils in one dimension, solving the linear system of equations can be done in linear time, since the matrix describing the linear system of equations will be diagonal dominant. For instance, for the case where the second order spatial stencil $[1 -2 1]$ for the approximation of $\frac{\partial^2}{\partial x^2}$, the matrix will the tridiagonal, and hence one can make use of for instance Thomas algorithm [11, §2.4].

However, when solving problem in two dimensions or higher, this approach will not work. Column stacking (or any other ordering of elements) of the unknowns on a two-dimensional grid, will give rise to diagonal entries far away from the main diagonal. For instance, if the $u(t, x, y)$ for fixed t is represented on an $N \times N$ lattice, then there will be contributions (at least) from the N:th off center diagonal, and standard approaches like LU-factorizations will fairly dense. A remedy to this problem when discretizing the Laplace operator $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is to make use of alternating direction implicit (ADI) methods, [4, 10]. The idea is to split the operator into two parts, one that is only acting in the x variable and one that is only acting in the y-variable. In this way, one can transfer the two-dimensional problem into solving two problems containing essentially only spatial derivatives in one variable (either x or y), and solve those problem by the one-dimensional approach discussed above (for instance using Thomas algorithm). These methods are well known and commonly used.

The problem that we are dealing with can, however, not be directly treating using ADI-methods. This is because of the presence of the mixed derivate term

$$b\frac{\partial^2}{\partial x \partial y}.$$

There are suggestions for how to generalize the ideas behind ADI to also include the presence of mixed derivatives in the literature, although this case is much less known that the case with no mixed terms. One early such reference is [8], where a two-level first order accurate (in time) unconditionally stable scheme is presented. In this paper, we will follow the path suggested in [1].

Define the forward and backward time difference operators by

$$Q^+u(t, x, y) = u(t + \delta_t, x, y) - u(t, x, y),$$

and

$$Q^-u(t, x, y) = u(t, x, y) - u(t - \delta_t, x, y),$$

respectively. A factored scheme then reads [1, pp. 19–20]

$$\begin{aligned}\left(1 - \breve{\omega}\delta_t a\frac{\partial^2}{\partial x^2}\right)&\left(1 - \breve{\omega}\delta_t c\frac{\partial^2}{\partial y^2}\right)(Q^+ - \alpha Q^-)u \\ &= \frac{\delta_t}{1+\xi^2}\left(a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2}\right)\left(1 + (\breve{\xi} - \breve{\theta} + \frac{1}{2}Q^-\right)u + 2\breve{\omega}\delta_t b\frac{\partial^2}{\partial x \partial y}Q^-u \\ &\quad + \breve{\alpha}\breve{\omega}\delta_t\left(a\frac{\partial^2}{\partial x^2} + c\frac{\partial^2}{\partial y^2}\right)Q^-u + \left(\frac{\breve{\xi}}{1+\xi} - \alpha\right)Q^-u,\end{aligned}$$

for constants $\breve{\omega}$, $\breve{\alpha}$, $\breve{\theta}$ and $\breve{\xi}$ satisfying

$$\breve{\theta} \leqslant \frac{2(1+\breve{\xi})}{2 + \sqrt{\frac{(1+\breve{\alpha})(1+2\breve{\xi})}{1+\breve{\xi}}}}, \quad \breve{\xi} \leqslant -\frac{1}{2}, \quad -1 \leqslant \breve{\alpha} \leqslant 1, \quad \breve{\omega} = \frac{\breve{\theta}}{1+\breve{\xi}}.$$

For details about the choices of parameters effect the properties of the numerical scheme, cf. [1].

The factored scheme can be implemented as follows

1. Compute the auxiliary function $u^*(x,y)$ by solving

$$\left(1-\breve{\omega}\delta_t a\frac{\partial^2}{\partial x^2}\right)u^*$$
$$=\frac{\delta_t}{1+\xi^2}\left(a\frac{\partial^2}{\partial x^2}+2b\frac{\partial^2}{\partial x\partial y}+c\frac{\partial^2}{\partial y^2}\right)\left(1+(\breve{\xi}-\breve{\theta}+\frac{1}{2}Q^-\right)u+2\breve{\omega}\delta_t b\frac{\partial^2}{\partial x\partial y}Q^-u$$
$$+\breve{\alpha}\breve{\omega}\delta_t\left(a\frac{\partial^2}{\partial x^2}+c\frac{\partial^2}{\partial y^2}\right)Q^-u+\left(\frac{\breve{\xi}}{1+\xi}-\alpha\right)Q^-u,$$

2. Compute the auxiliary function $u^\dagger(x,y)$ by solving

$$\left(1-\breve{\omega}\delta_t c\frac{\partial^2}{\partial y^2}\right)u^\dagger=u^*$$

3. Update for the solution u at time $t+\delta_t$.

$$u(t+\delta_t,x,y)=u^\dagger+(1+\alpha)u(t,x,y)-\alpha u(t-\delta_t,x,y).$$

Note that the first step can be implement fast since it only involves derivatives in the x-direction, and can thus be solved by using a tridiagonal (second order) or pentadiagonal (fourth order) solver. The same thing holds for the second step, but now the derivatives are only in the y-direction.

We briefly show some numerical results. The left upper panel of Figure 2 shows a synthetic data set consisting of sums of parabolic Gaussian wave packets of the form

$$g_j(x,y)=\mathrm{e}^{-\gamma_j((y-y_j)-\sigma(x-x_j)^2)^2+2\pi i\eta_j((y-y_j)-\sigma(x-x_j)^2)^2},\quad \gamma>0,$$

where γ_j describes the Gaussian decay,η_j wavelet frequency, σ_j the parabolic curvature, and where x_j and y_j denotes center locations. Hence

$$f(x,y)=\sum_j g_j(x,y).$$

We assume that we know f along 11 vertical lines (traces) $x=x_l$. The upper right panel of Figure 2 shows f at the know traces, and where the unknown traces have been blanked out. The traces are also depicted in the lower right panel of Figure 2. In the lower left panel of the figure, the reconstruction by solving (2.5) by means of the anisotropic diffusion problem (3.1) is shown.

4. Conclusions. We have designed a method for the interpolation of seismic data when the data is either very sparsely undersampled, or where there are large gaps in the data. We assume that in addition to knowledge of data along traces, we also know the x-derivative of f at the locations of the traces. This means that we have local knowledge of the function in a neighborhood around the traces. We then formulate the interpolation problem as a variational problem, where we minimimize the second derivative of the structure tensor. This is done in order to minimize the local presence of several directions in the reconstructed data. The solution of the variational formulation can be expressed in terms of an elliptic partial differential equation. One way to compute the solution to such an equation, is by considering the steady state solution of a parabolic equation constructed from the elliptic part. This has the advantage that it can be solved by a time stepping approach.

We show existence and uniqueness of the nonlinear parabolic (anisotropic diffusion) problem that we have derived. This is important as it reveals that we can solve the variational problem by solving the nonlinear diffusion problem. Because of the nonlinearity of the anisotropic diffusion problem, it is not obvious that solutions are unique. The kind of anisotropic diffusion that we have derived is slightly different from the one that is usually considered. The proofs are based on ideas from [3], but applied to the more complicated setting that follows from the designed variational problem.

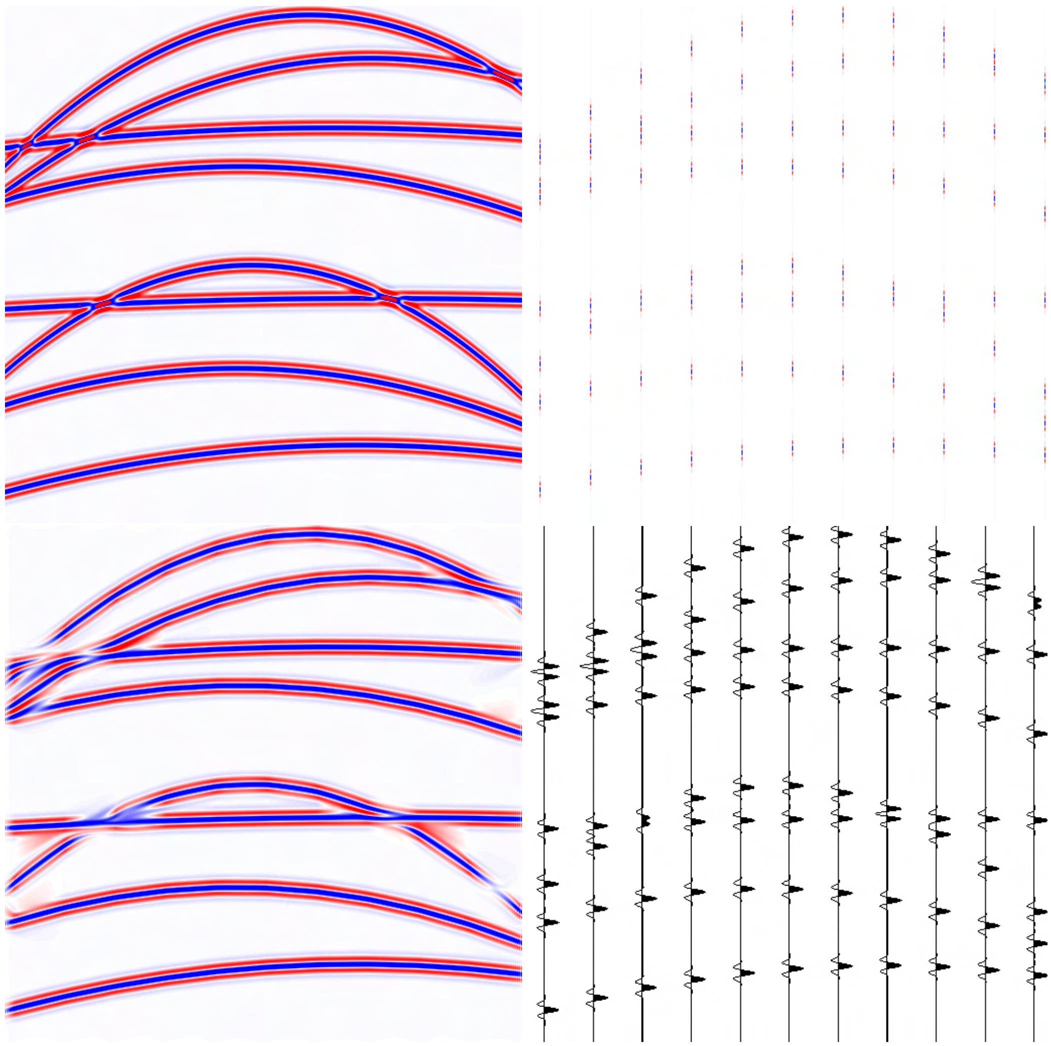

FIG. 2. *Example on a synthetic data set consisting of parabolic Gaussian wave packets. The top left panel shows the original data. The information available is depicted in the two right panels, while the result from doing interpolation by solving the variational problem is shown the lower right panel.*

5. Acknowledgements. This research has been supported by the JSPS postdoctoral fellowship program, the Craaford Foundation, the Swedish Foundation for International Cooperation in Research and Higher Education, the Swedish Research Council and the members of the Geomathematical Imaging Group: BGP, ExxonMobil, PGS, Statoil and Total.

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