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LIPSCHITZ STABILITY OF AN INVERSE BOUNDARY VALUE PROBLEM FOR TIME-HARMONIC ELASTIC WAVES, PART I: RECOVERY OF THE DENSITY

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Abstract. We study the inverse boundary value problem for the time-harmonic elastic waves. We focus on the recovery of the density and assume the stiffness tensor to be isotropic. The data are the Dirichlet-to-Neumann map. We establish a Lipschitz type stability estimate assuming that the density is piecewise constant with an underlying domain partitioning consisting of a finite number of subdomains.

1. Introduction. We study the inverse boundary value problems for time-harmonic elastic seismic waves. We consider isotropic elasticity. Assume that Ω is a connected open set in $\mathbb{R}^3$ and $\{D_m\}$ is a non-overlapping domain partitioning of Ω. The Lamé coefficients are assumed to be constants and satisfy

$$\mu \geq \lambda_0 > 0, \qquad \lambda + 2\mu \geq \lambda_0 > 0. \tag{1.1}$$

The density is piecewise constant.

The time-harmonic isotropic elasticity wave equation is given by

$$\begin{cases} \operatorname{div} \sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = 0, & x \in \Omega, \\ \mathbf{u} = \mathbf{g}(x), & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where the stress tensor operator σ is described by Hooke's law,

$$\sigma(\mathbf{u}) = \lambda \operatorname{div} \mathbf{u}\, \mathrm{I}_3 + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \tag{1.3}$$

Throughout this paper, we denote the 3×3 identity matrix by I_3. We absorb ω^2 in ρ for simplicity of notation.

The Dirichlet-to-Neumann map Λ_ρ is given by

$$\Lambda_\rho : \mathbf{u}\,|_{\partial\Omega} \mapsto \sigma(\mathbf{u})\nu\,|_{\partial\Omega}, \tag{1.4}$$

where ν is the exterior unit normal vector to $\partial\Omega$. The Dirichlet-to-Neumann map maps the displacement at the boundary to the corresponding traction, that is, normal component of the stress at the boundary.

The analysis presented here is part of a comprehensive study of the seismic inverse boundary value problem with a view to establishing conditional Lipschitz stability. We separate this study into the recovery of the density with known Lamé parameter, of the Lamé parameters with known density, and the density and Lamé parameters combined. The different cases require different analyses. In this paper, we consider the first case and thus focus on the role of density. The uniqueness of this inverse problems with partial data and density in L^∞ is still open.

The outline of this paper is as follows. In Section 2, we state the main result. In Section 3, we discuss the energy estimates, the three-sphere inequality and construct singular solutions. In Section 4, we analysis the propagation of smallness. In Section 5, we prove the main result. We study the asymptotic behavior of the singular solutions near the interfaces of the subdomains using the fundamental solution as a reference. This approach was developed by Alessandrini and Vessella for the inverse conductivity problem with real conductivities [2], and generalized by Beretta and Francini to complex conductivity case [4], and by Beretta, de Hoop and Qiu to the inverse boundary value problem for the Helmholtz equation [3].

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2. Main result.

2.1. Notation and definitions. For every $x \in \mathbb{R}^3$ we write $x = (x_1, x_2, x_3)$ and $x = (x', x_3)$ where $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$. For every $x \in \mathbb{R}^3$, r and L positive real numbers we will denote by $B_r(x)$, $B'_r(x')$ and $Q_{r,L}(x)$ the open ball in $\mathbb{R}^3$ centered at x of radius r, the open ball in $\mathbb{R}^2$ centered at x' of radius r and the cylinder $B'_r(x') \times (x_3 - Lr, x_3 + Lr)$, respectively. In the sequel $B_r(0)$, $B'_r(0)$ and $Q_{r,L}(0)$ will be denoted by B_r, B'_r and $Q_{r,L}$, respectively. We also introduce the notation,

$$\mathbb{R}^3_+ = \{(x', x_3) \in \mathbb{R}^3 \,:\, x_3 > 0\},$$
$$\mathbb{R}^3_- = \{(x', x_3) \in \mathbb{R}^3 \,:\, x_3 < 0\},$$

$B^+_r = B_r \cap \mathbb{R}^3_+$, and $B^-_r = B_r \cap \mathbb{R}^3_-$. We will also make use of the following notation for matrices and tensors: for any 3×3 matrices A and B, we set $\hat{A} = \frac{1}{2}(A + A^T)$; the Frobenius inner product, denoted by $A : B$, is the component-wise inner product of two matrices as though they are vectors, i.e., $A : B = \sum_{i,j=1}^3 A_{ij} B_{ij}$.

Definition 2.1. *Let Ω be an open and bounded subset of $\mathbb{R}^3$. Let $\partial\Omega$ denote the boundary of Ω. We say that a portion Σ of $\partial\Omega$ is of Lipschitz class with constants $r_0, L > 0$ if, for any $P \in \Sigma$, there exists a rigid transformation of coordinates such that $P = 0$ and*

$$\Omega \cap Q_{r_0,L} = \{(x', x_3) \in Q_{r_0,L} \mid x_3 > \phi(x')\}$$

where ϕ is a Lipschitz continuous function on B'_{r_0} with $\phi(0) = 0$ and

$$\|\phi\|_{C^{0,1}(B'_{r_0})} \le L r_0.$$

We shall say that Ω is of Lipschitz class with constants r_0 and L, if $\partial\Omega$ is of Lipschitz class with the same constants.

Definition 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^3$ and of Lipschitz class and Σ be a open portion of $\partial\Omega$. We define $H^{1/2}_{co}(\Sigma)$ as*

$$H^{1/2}_{co}(\Sigma) = \{g \in H^{1/2}(\partial\Omega) \mid \operatorname{supp} g \subset \Sigma\}$$

and $H^{-1/2}_{co}(\Sigma)$ as the topological dual of $H^{1/2}_{co}(\Sigma)$; we denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $H^{1/2}_{co}(\Sigma)$ and $H^{-1/2}_{co}(\Sigma)$.

Definition 2.3 (Local Dirichlet-to-Neumann map). *Let Ω be a bounded open subset of $\mathbb{R}^3$ and of Lipschitz class, Σ be an open portion of $\partial\Omega$ and $\rho \in L^\infty(\Omega)$. Assume that 0 is not an eigenvalue of $\operatorname{div}\sigma + \rho$ with Dirichlet boundary condition in Ω, i.e.,*

$$\{\mathbf{u} \in H^1_0(\Omega) \mid \operatorname{div}\sigma(\mathbf{u}) + \rho\mathbf{u} = 0\} = \{0\}.$$

For any $\mathbf{g} \in H^{1/2}_{co}(\Sigma)$, let $\mathbf{u} \in H^1(\Omega)$ be the unique weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}\sigma(\mathbf{u}) + \rho\mathbf{u} = 0, & x \in \Omega, \\ \mathbf{u} = \mathbf{g}(x), & x \in \partial\Omega, \end{cases} \tag{2.1}$$

We define the local Dirichlet-to-Neumann map $\Lambda^{(\Sigma)}_\rho$ as

$$\begin{aligned} \Lambda^{(\Sigma)}_\rho : \quad H^{1/2}_{co}(\Sigma) &\to \quad H^{-1/2}_{co}(\Sigma) \\ \mathbf{g} &\mapsto \quad \nu\sigma(\mathbf{u})|_\Sigma, \end{aligned}$$

where ν is the exterior unit normal vector to $\partial\Omega$.

The map $\Lambda^{(\Sigma)}_\rho$ can be identified with the bilinear form $\tilde{\Lambda}^{(\Sigma)}_\rho$ on $H^{1/2}_{co}(\Sigma) \times H^{1/2}_{co}(\Sigma)$,

$$\tilde{\Lambda}^{(\Sigma)}_\rho[\mathbf{g}, \mathbf{f}] := \langle \Lambda^{(\Sigma)}_\rho \mathbf{g}, \mathbf{f} \rangle = \int_\Omega \lambda \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v} + 2\mu \widehat{\nabla\mathbf{u}} : \widehat{\nabla\mathbf{v}} + \rho\mathbf{u}\cdot\mathbf{v} \,\mathrm{d}x, \tag{2.2}$$

for all $\mathbf{g}, \mathbf{f} \in H^{1/2}_{co}(\Sigma)$ and where $\mathbf{u}$ solves (1.2) and $\mathbf{v}$ is any $H^1(\Omega)$ function such that $\mathbf{v} = \mathbf{f}$ on $\partial\Omega$. Note that this bilinear form does not depend on the choice of $\mathbf{v}$.

We shall denote by $\|\cdot\|_\star$ the norm in $\mathcal{L}\left(H^{1/2}_{co}(\Sigma), H^{-1/2}_{co}(\Sigma)\right)$ defined by

$$\|T\|_\star = \sup\{\langle T\mathbf{g}, \mathbf{f}\rangle \,|\, \mathbf{g}, \mathbf{f} \in H^{1/2}_{co}(\Sigma), \quad \|\mathbf{g}\|_{H^{1/2}_{co}(\Sigma)} = \|\mathbf{f}\|_{H^{1/2}_{co}(\Sigma)} = 1\}$$

for every $T \in \mathcal{L}(H^{1/2}_{co}(\Sigma), H^{-1/2}_{co}(\Sigma))$.

2.2. Main assumptions. Our assumptions on the domain Ω and density ρ are

Assumption 2.4. *$\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying*

$$|\Omega| \leq A$$

Here and in the sequel $|\Omega|$ denotes the Lebesgue measure of Ω. We assume that $\partial\Omega$ is of Lipschitz class and we fix an open portion Σ of $\partial\Omega$ which is of Lipschitz class with constants r_0 and L.

Assumption 2.5. *The complex-valued function $\rho(x)$ satisfies*

$$\|\rho\|_{L^\infty(\Omega)} \leq B,$$

where B is a positive constant strictly less then the first Dirichlet eigenvalue of $\operatorname{div}\sigma + \rho$ in Ω, and is of the form

$$\rho(x) = \sum_{j=1}^{N} \rho_j \chi_{D_j}(x)$$

where $\rho_j, j = 1, \ldots N$ are unknown complex numbers and D_j are known open sets in $\mathbb{R}^n$ which satisfy the following assumption.

Assumption 2.6. *The $D_j, j = 1, \ldots, N$, are connected and pairwise non-overlapping open sets such that $\cup_{j=1}^N \overline{D}_j = \overline{\Omega}$ and ∂D_j are of Lipschitz class. We also assume that there exists one set, say D_1, such that $\partial D_1 \cap \partial\Omega$ contains a flat open portion Σ_1. For every $j \in \{2, \ldots, N\}$ there exist $j_1, \ldots, j_M \in \{1, \ldots, N\}$ such that*

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \ldots, M$,

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a non-empty flat open portion Σ_k such that

$$\begin{gathered}\Sigma_1 \subset \Sigma, \\ \Sigma_k \subset \Omega, \quad \forall k = 2, \ldots, M.\end{gathered}$$

Furthermore, there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates such that $P_k = 0$ and

$$\begin{aligned}\Sigma_k \cap Q_{r_0/3,L} =& \{x \in Q_{r_0/3,L} \mid x_3 = 0\}, \\ D_{j_k} \cap Q_{r_0/3,L} =& \{x \in Q_{r_0/3,L} \mid x_3 > 0\}, \\ D_{j_{k-1}} \cap Q_{r_0/3,L} =& \{x \in Q_{r_0/3,L} \mid x_3 < 0\}.\end{aligned}$$

For simplicity, we call $D_{j_1}, \ldots, D_{j_M}$ a chain of domains connecting D_1 to D_j.

In the further analysis, for simplicity of notation, we also use the constant $r_1 = \frac{r_0}{16}$.

2.3. Statement of the main result. The main result of this paper is stated as follows.

THEOREM 2.7. *Let Ω satisfy Assumption 2.4 and $\rho^{(k)}, k = 1, 2$ be two complex piecewise constant functions of the form*

$$\rho^{(k)}(x) = \sum_{j=1}^{N} \rho_j^{(k)} \chi_{D_j}(x), \quad k = 1, 2$$

which satisfy Assumption 2.5 and the domain partitioning $\{D_j\}_{j=1}^N$ satisfies Assumption 2.6. Then, there exists a constant $C = C(r_0, L, A, B, N)$, such that

$$\|\rho^{(1)} - \rho^{(2)}\|_{L^\infty(\Omega)} \leq C\|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_\star, \tag{2.3}$$

where $\Lambda_1^{(\Sigma)}$ and $\Lambda_2^{(\Sigma)}$ denote the local Dirichlet-to-Neumann map corresponding to $\rho^{(1)}$ and $\rho^{(2)}$, respectively.

Note that Theorem 2.7 indicates that F is injective and its inverse is Lipschitz continuous.

3. Preliminaries. In this section, we summarize some facts, which we use in our proof of the Lipschitz stability. We consider the vector-valued function $\mathbf{u} = (u_1, u_2, u_3)$.

PROPOSITION 3.1. *Let $\mathbf{u}$ solves the boundary value problem*

$$\begin{cases} \operatorname{div}\sigma(\mathbf{u}) + \rho\mathbf{u} = \displaystyle\sum_{n=1}^{3} \frac{\partial}{\partial x_n}\mathbf{f}^{(n)} + \mathbf{f}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

in a bounded Lipschitz domain Ω of $\mathbb{R}^3$ with $\mathbf{f}, \mathbf{f}^{(n)} \in L^2(\Omega)$, $n = 1, 2, 3$ and $\mathbf{g} \in H^{1/2}(\partial\Omega)$. Assume that the Lamé coefficients satisfy (1.1) *and $\rho \in L^\infty(\Omega)$. Then, we have the following estimate,*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C(\|\mathbf{u}\|_{L^2(\Omega)} + \sum_{n=1}^{3} \|\mathbf{f}^{(n)}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}), \tag{3.2}$$

where C depends on Ω and λ_0.

In several cases, the term $\|\mathbf{u}\|_{L^2(\Omega)}$ in the right hand side of (3.2) can be eliminated. We state the following proposition, which is a special case.

PROPOSITION 3.2. *Let us assume in addition to the hypotheses of Proposition 3.1, that ρ satisfy Assumption 2.5. Then, the system* (3.1) *has a unique solution $\mathbf{u} \in H^1(\Omega)$ and*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C(\sum_{n=1}^{3} \|\mathbf{f}^{(n)}\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}), \tag{3.3}$$

where C depends on Ω and λ_0.

For the proofs of Proposition 3.1 and 3.2, we refer to [6].

LEMMA 3.3 (Alessandrini identity). *Assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions to*

$$\operatorname{div}\sigma(\mathbf{u}) + \rho\mathbf{u} = 0, \quad \text{in } \Omega,$$

subject to replacing ρ by ρ_1 and ρ_2, respectively. Let Λ_{ρ_1} and Λ_{ρ_2} denote the local Dirichlet-to-Neumann maps corresponding to ρ_1 and ρ_2, respectively. Then

$$\langle(\Lambda_{\rho_1} - \Lambda_{\rho_2})\mathbf{u}_1, \mathbf{u}_2\rangle = \int_\Omega (\rho_1 - \rho_2)\mathbf{u_1} \cdot \mathbf{u_2} \, \mathrm{d}x. \tag{3.4}$$

In the following, we use C to denote positive constants. The value of the constants may change from line to line, but we shall specify their dependence everywhere where they appear.

Fundamental solution for constant density. In our proof of Lipschitz stability estimates, we construct singular solutions and study their asymptotic behavior near the interface of subdomains, using the fundamental solution as a reference. In the scalar case, we use the explicit form of the fundamental solution for the Laplace equation or the Helmholtz equation. In order to construct the singular solutions for the case of systems, we make use of the following fundamental solution for the operator $\operatorname{div}\sigma + \rho_0$, where ρ_0 is a constant:

$$\tilde{\mathbf{\Gamma}}(x,y) = \frac{1}{\mu}\left(\frac{\mathrm{e}^{\mathrm{i}\kappa_s|x-y|}}{4\pi|x-y|}\mathrm{I}_3 + \frac{1}{\kappa_s^2}\nabla_x\nabla_x^T\left(\frac{\mathrm{e}^{\mathrm{i}\kappa_s|x-y|}}{4\pi|x-y|} - \frac{\mathrm{e}^{\mathrm{i}\kappa_p|x-y|}}{4\pi|x-y|}\right)\right), \tag{3.5}$$

with the longitudinal and transversal wave numbers given by

$$\kappa_p^2 = \frac{\rho_0}{\lambda + 2\mu}\ , \quad \kappa_s^2 = \frac{\rho_0}{\mu}.$$

In the construction of the singular solution and the proof of the main theorem, we use a first order derivative of the above fundamental solution. We define $\mathbf{\Gamma}(x,y) = \frac{\partial}{\partial x_1}\tilde{\mathbf{\Gamma}}(x,y), \quad x,y \in \mathbb{R}^3$.

For the proof of the main result, we need to extend our original domain. We consider Σ_1 and recall that up to a rigid transformation of coordinates we can assume that $P_1 = 0$ and

$$(\mathbb{R}^3\backslash\Omega) \cap B_{r_0} = \{(x', x_n) \in B_{r_0} \mid x_3 < 0\}.$$

Then we extend Ω to $\Omega_0 = \Omega \cup D_0$ by adding an open set D_0 defined as

$$D_0 = \left\{x \in (\mathbb{R}^n\backslash\Omega) \cap B_{r_0} \mid \left|x_n - \frac{r_0}{6}\right| < \frac{5}{6}r_0,\ |x_i| < \frac{2}{3}r_0,\ i = 1,\dots,n-1\right\}.$$

It turns out that Ω_0 is of Lipschitz class with constants $\frac{r_0}{3}$ and L_1, where L_1 depends on L only. We define

$$K_0 = \left\{x \in D_0 \mid \operatorname{dist}(x, \Sigma_1) \geq \frac{r_0}{3}\right\},$$

with $\operatorname{dist}(K_0, \partial\Omega) > \frac{r_0}{3}$. We extend ρ defined on Ω by setting it equal to 1 in D_0 and extend the Lamé coefficients λ and μ using the same constants as in Ω. For simplicity of notation we still denote this extension by $\rho = \rho(x)$.

We consider any subdomain in Ω and the chain of domains connecting it to D_1. For simplicity let us rearrange the indices of subdomains so that this chain corresponds to $D_0, D_1, \dots, D_M$, $M \leq N$. Let $S = \cup_{j=0}^{M}\overline{D}_j$ and K be a connected subset of S with Lipschitz boundary such that $\overline{K} \cap \partial D_j = \Sigma_j \cup \Sigma_{j+1}$ for $j = 1, 2, \dots, M$, $K_0 \subset K$ and $\operatorname{dist}(K, \partial S\backslash\{\Sigma_{M+1} \cup \Sigma_1\}) > \frac{r_0}{16}$.

In the following proposition, we discuss the existence and some estimates of the Green's matrix with zero Dirichlet boundary condition.

PROPOSITION 3.4. *Let the Lamé coefficients λ and μ be constant satisfying* (1.1) *and the density function ρ satisfy Assumption 2.5. For any $y \in \Omega_0$, there exists a unique matrix-valued function $\mathbf{G}(\cdot, y)$, which is continuous in $\Omega_0\backslash\{y\}$, such that*

$$\int_\Omega -\sigma(\mathbf{G}(\cdot,y)) : \nabla\mathbf{v} + \rho\mathbf{G}(\cdot,y)\mathbf{v}\,\mathrm{d}x = \frac{\partial}{\partial y_1}\mathbf{v}(y), \quad \forall \mathbf{v} \in C_0^\infty(\Omega_0),$$

and

$$\mathbf{G}(x,y) = 0, \quad x \in \partial\Omega_0.$$

Furthermore, we have that

$$\mathbf{G}(x,y) = \mathbf{G}(y,x)^T, \quad \forall x,y \in \Omega_0,$$

and the following estimates,

$$\|\mathbf{G}(\cdot,y)-\mathbf{\Gamma}(\cdot,y)\|_{H^1(\Omega_0)}\le C,\quad \operatorname{dist}(y,\partial\Omega_0)\ge\frac{r_0}{16}, \tag{3.6}$$

and

$$\|\mathbf{G}(\cdot,y)\|_{H^1(\Omega_0\setminus B_r(y))}\le Cr^{-1/2},\quad r\le\frac{1}{2}\operatorname{dist}(y,\partial\Omega_0), \tag{3.7}$$

hold true, where the constant C depends on Ω_0.

Proof. Consider $\mathbf{G}(\cdot,y)=\mathbf{\Gamma}(\cdot,y)+\omega(\cdot,y)$, where ω solves

$$\begin{cases}\operatorname{div}\sigma(\omega)+\rho\omega=(\rho_0-\rho)\mathbf{\Gamma}, & \text{in } \Omega_0,\\ \omega=-\mathbf{\Gamma}, & \text{on } \partial\Omega_0.\end{cases} \tag{3.8}$$

Note that $\tilde{\mathbf{\Gamma}}(\cdot,y)\in L^2(\Omega_0)$, hence $\mathbf{\Gamma}(\cdot,y)\in H^{-1}(\Omega_0)$, and $\mathbf{\Gamma}(\cdot,y)\mid_{\partial\Omega_0}\in H^{1/2}(\partial\Omega_0)$. Then the system (3.8) has a unique solution $\omega\in H^1(\Omega_0)$ and $\omega=\mathbf{G}-\mathbf{\Gamma}$ satisfies the estimate

$$\|\omega(\cdot,y)\|_{H^1(\Omega_0)}\le C(\|\mathbf{\Gamma}(\cdot,y)\|_{H^{1/2}(\partial\Omega_0)}+\|(\rho-\rho_0)(\cdot)\mathbf{\Gamma}(\cdot,y)\|_{H^{-1}(\Omega_0)})\le C,$$

when $\operatorname{dist}(y,\partial\Omega_0)\ge r_0/16$. Then, (3.7) follows from

$$\begin{aligned}\|\mathbf{G}(\cdot,y)\|_{H^1(\Omega_0\setminus B_r(y))}&\le\|\mathbf{G}(\cdot,y)-\mathbf{\Gamma}(\cdot,y)\|_{H^1(\Omega_0\setminus B_r(y))}+\|\mathbf{\Gamma}(\cdot,y)\|_{H^1(\Omega_0\setminus B_r(y))}\\&\le\|\omega(\cdot,y)\|_{H^1(\Omega_0)}+\|\tilde{\mathbf{\Gamma}}(\cdot,y)\|_{L^2(\Omega_0\setminus B_r(y))},\end{aligned}$$

and the explicit form of $\tilde{\mathbf{\Gamma}}(x,y)$, shown in (3.5).

The symmetry of $\mathbf{G}$ follows by standard arguments based on integration by parts (see for example [5]). ◻

Three-sphere inequality. To prove the propagation of smallness, we need the quantitative strong unique continuation for the Lamé system with lower order terms. The following proposition is from [1, 7], in which a more general result is given. Here, we give the statement for our special case.

PROPOSITION 3.5 (Quantitative strong unique continuation [1, 7]). *Assume that the Lamé coefficients λ and μ are constant satisfying* (1.1) *and ρ satisfies Assumption 2.5. There exists a positive number $\tilde{R}<1$, depending only on λ_0, such that, for any $x_0\in\Omega$ if $0<R_1<R_2<R_3\le$ $\operatorname{dist}(x_0,\partial\Omega)$ and $R_1/R_3<R_2/R_3<\tilde{R}$, then*

$$\int_{B_{R_2}(x_0)}|\mathbf{u}|^2dx\le C\left(\int_{B_{R_1}(x_0)}|\mathbf{u}|^2dx\right)^{\tau}\left(\int_{B_{R_3}(x_0)}|\mathbf{u}|^2dx\right)^{1-\tau} \tag{3.9}$$

for $u\in H^1_{loc}(\Omega)$ satisfying

$$\operatorname{div}\sigma(\mathbf{u})+\rho\mathbf{u}=0$$

in $B_{R_3}(x_0)$, where the constant C depends on R_2/R_3, λ_0, and $0<\tau<1$ depends on R_1/R_3, R_2/R_3, λ_0. Moreover, for fixed R_2 and R_3, the exponent τ behaves like $1/(-\log R_1)$ when R_1 is sufficiently small.

Singular solutions. Assume that D_M is the subdomain of the partition of Ω where the maximum of $\|\rho^{(1)}-\rho^{(2)}\|$ is realized. Let us denote

$$E=\|\rho^{(1)}-\rho^{(2)}\|_{L^\infty(D_M)}=\|\rho^{(1)}-\rho^{(2)}\|_{L^\infty(\Omega)}.$$

We consider the chain of domains, $D_0, D_1, \dots, D_M$, as before; S, K and K_0 are defined as in the previous section. We set

$$U_0 = \Omega,\, U_k = \Omega \setminus \cup_{j=1}^{k} D_j, \quad k = 1, \dots, M \text{ and } W_k = \cup_{j=0}^{k} D_j.$$

Let $\mathbf{G}_i(x,y)$ be the function related to $\rho^{(i)})$, $i = 1,2$, the existence and behavior of which was shown in Proposition 3.4. We define the matrix-valued singular solution $\mathbf{S}_k(y,z)$ as following. For $y, z \in K \subset W_k$,

$$\mathbf{S}_k^{(m,n)}(y,z) = \int_{U_k} (\rho^{(1)} - \rho^{(2)}) \mathbf{G}_1^{(\cdot,n)}(x,y) \cdot \overline{\mathbf{G}}_2^{(m,\cdot)}(x,y)\,\mathrm{d}x, \quad m,n = 1,2,3. \tag{3.10}$$

PROPOSITION 3.6. *For every $y, z \in K \cap W_k$, we have that $\mathbf{S}_k^{(\cdot,n)}(\cdot,z)$, $\mathbf{S}_k^{(m,\cdot)}(y,\cdot) \in H^1(K \cap W_k)$ and, for any $n = 1,2,3$,*

$$\begin{aligned}\operatorname{div}(\mu(\nabla \mathbf{S}_k^{(\cdot,n)}(\cdot,z) + (\nabla \mathbf{S}_k^{(\cdot,n)}(\cdot,z))^T)) \\ + \nabla(\lambda \operatorname{div} \mathbf{S}_k^{(\cdot,n)}(\cdot,z)) + \rho^{(1)} \mathbf{S}_k^{(\cdot,n)}(\cdot,z) = 0 \quad \textit{in} \quad K \cap W_k;\end{aligned}$$

for any $m = 1,2,3$,

$$\begin{aligned}\operatorname{div}(\overline{\mu}(\nabla \mathbf{S}_k^{(m,\cdot)}(y,\cdot) + (\nabla \mathbf{S}_k^{(m,\cdot)}(y,\cdot))^T)) \\ + \nabla(\overline{\lambda} \operatorname{div} \mathbf{S}_k^{(m,\cdot)}(y,\cdot)) + \overline{\rho^{(2)}} \mathbf{S}_k^{(m,\cdot)}(y,\cdot) = 0 \quad \textit{in} \quad K \cap W_k.\end{aligned}$$

The proof of Proposition 3.6 follows from the symmetry of G_i $(i = 1,2)$, as in Proposition 3.4, and changing the order of integration and differentiation.

4. Propagation of smallness. PROPOSITION 4.1. *Let K and K_0 as before and ρ satisfy the Assumption 2.5. Assume that $\mathbf{u} \in H^1(K)$ is a solution of the system*

$$\operatorname{div}(\mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)) + \nabla(\lambda \operatorname{div} \mathbf{u}) + \rho \mathbf{u} = 0 \quad \textit{in} \quad K, \tag{4.1}$$

and, for given positive numbers ε_0 and E_0, we have

$$\|\mathbf{u}\|_{L^\infty(K_0)} \le \varepsilon_0, \tag{4.2}$$

and

$$|\mathbf{u}(x)| \le (\varepsilon_0 + E_0)\operatorname{dist}(x, \Sigma_M)^{-1/2}, \quad x \in K. \tag{4.3}$$

Then the following inequality holds true for every $r \in (0, 2r_1)$,

$$|\mathbf{u}(\tilde{x})| \le C \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\tau_r \tau^{N_1}} (\varepsilon_0 + E_0) r^{-(1-\tau_r)/2}$$

where $\tilde{x} = P_M - r\nu(P_M)$ with ν being the exterior unit normal vector to ∂D_{M-1} at P_M, $\tau = \frac{\ln(8/7)}{\ln 4}$, $\tau_r = \dfrac{\ln\left(\frac{12r_1 - 2r}{12r_1 - 3r}\right)}{\ln\left(\frac{6r_1 - r}{2r_1}\right)} \in (0,1)$ and the constants N_1 and C depend on r_0 and Ω.

Proof. We construct a chain of spheres of radius r_1 with centers $x_0, x_1, \dots, x_k$ such that the first is $B_{r_1}(x_0) \subset B_{4r_1}(x_0) \subset K_0$, all the spheres are externally tangent, and the last one is centered at $x_k = P_M - 3r_1\nu(P_M)$. We choose this chain so that the spheres of radius $4r_1$ concentric with those of the chain, except the last one, are contained in K and have a distance greater than r_1 away from Σ_{M+1}. Such a chain has a finite number of spheres that is smaller than $N_1 = \frac{A}{|B_{r_1}|} + 1$.

By Proposition 3.5 and (4.2), (4.3), we have

$$\begin{aligned}\|\mathbf{u}\|_{L^2(B_{r_1}(x_1))} \leq & \|\mathbf{u}\|_{L^2(B_{3r_1}(x_0))}\\ \leq & C\|\mathbf{u}\|^{\tau}_{L^2(B_{r_1}(x_0))}\|\mathbf{u}\|^{1-\tau}_{L^2(B_{4r_1}(x_0))}\\ \leq & C\left(\frac{\varepsilon_0}{\varepsilon_0+E_0}\right)^{\tau}(\varepsilon_0+E_0),\end{aligned}$$

where C and τ depend on λ_0. By iterated application of Proposition 3.5 to $\mathbf{u}$ with radii r_1, $3r_1$ and $4r_1$ over the chain of spheres, we have that

$$\begin{aligned}\|\mathbf{u}\|_{L^2(B_{r_1}(x_k))} \leq & C\|\mathbf{u}\|^{\tau}_{L^2(B_{r_1}(x_{k-1}))}\|\mathbf{u}\|^{1-\tau}_{L^2(B_{4r_1}(x_{k-1}))}\\ \leq & C\left(\frac{\varepsilon_0}{\varepsilon_0+E_0}\right)^{\tau^{N_1}}(\varepsilon_0+E_0),\end{aligned}$$

where C depends on λ_0 and N_1. Now, we let $\tilde{x} = P_{M+1} - r\nu(P_{M+1})$ where $r < 2r_1$. Using Proposition 3.5 again for spheres centered at x_k of radii r_1, $3r_1 - \frac{r}{2}$ and $3r_1 - \frac{r}{4}$, we obtain that

$$\begin{aligned}\|\mathbf{u}\|_{L^2(B_{3r_1-\frac{r}{2}}(x_k))} \leq & C\|\mathbf{u}\|^{\tau_r}_{L^2(B_{r_1}(x_k))}\|\mathbf{u}\|^{1-\tau_r}_{L^2(B_{3r_1-\frac{r}{4}}(x_k))}\\ \leq & C\left(\frac{\varepsilon_0}{\varepsilon_0+E_0}\right)^{\tau_r\tau^{N_1}}(\varepsilon_0+E_0).\end{aligned}$$

Then, by the local property of the weak solution (see [6, Chapter 7.2]), we have that

$$\sup_{B_{3r_1-r}(x_k)}|\mathbf{u}| \leq Cr^{-1/2}\|\mathbf{u}\|_{L^2(B_{3r_1-\frac{r}{2}}(x_k))} \leq Cr^{-1/2}\left(\frac{\varepsilon_0}{\varepsilon_0+E_0}\right)^{\tau_r\tau^{N_1}}(\varepsilon_0+E_0),$$

which completes the proof. □

LEMMA 4.2. *If for some $\varepsilon_0 > 0$ and $k \in \{1, \dots, M-1\}$ we have that*

$$|\mathbf{S}_k(y,z)| \leq \varepsilon_0, \quad \forall y, z \in K_0, \tag{4.4}$$

then

$$|\mathbf{S}_k(y_r,y_r)| \leq C\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r^2\tau^{2N_1}}(\varepsilon_0+E)\,r^{-1}, \tag{4.5}$$

where $y_r = P_{k+1} - r\nu(P_{k+1})$, r is small, $\nu(P_{k+1})$ is the exterior unit normal vector to ∂D_k at P_{k+1} and the positive constant C depends on λ_0 and N_1.

Proof. We first fix $z \in K_0$ and consider the function $\mathbf{v}(y) = \mathbf{S}_k^{(\cdot,n)}(y,z)$ for some fix $n \in \{1,2,3\}$. By Proposition 3.6, we know that $\mathbf{v}$ solves the system

$$\operatorname{div}\sigma(\mathbf{v}) + \rho^{(1)}\mathbf{v} = 0, \quad \text{in } K \cap W_k.$$

By the construction of $\mathbf{S}_k(y,z)$ and Proposition 3.4, we have that

$$|\mathbf{v}(y)| \leq CE\,\operatorname{dist}(y,\Sigma_{k+1})^{-1/2}, \quad \forall y \in K \cap W_k.$$

Now, we can apply Proposition 4.1 to obtain that

$$|\mathbf{v}(y_r)| \leq C \left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r \tau^{N_1}} (\varepsilon_0 + E)\, r^{-1/2}. \tag{4.6}$$

Next, let us consider

$$\tilde{\mathbf{v}}(z) = \mathbf{S}_k^{(m,\cdot)}(y_r, z), \quad z \in K \cap W_k.$$

By Proposition 3.6, we know that $\tilde{\mathbf{v}}$ solves the system

$$\operatorname{div} \sigma(\mathbf{v}) + \overline{\rho^{(2)}}\mathbf{v} = 0, \quad \text{in } K \cap W_k,$$

with λ and μ replaced by $\bar{\lambda}$ and $\overline{\mu}$, respectively. By the construction of $\mathbf{S}_k(y,z)$ and Proposition 3.4, we have that

$$|\tilde{\mathbf{v}}(y)| \leq CEr^{-1/2} \operatorname{dist}(z, \Sigma_{k+1})^{-1/2}, \quad \forall z \in K \cap W_k.$$

By Proposition 4.1 again, we get the desired estimate (4.5). □

5. Proof of the main theorem. Let

$$\varepsilon = \|\Lambda_1 - \Lambda_2\|_\star$$

and

$$\delta_k = \|\rho^{(1)} - \rho^{(2)}\|_{L^\infty(W_k)}, \quad k = 0, 1, \ldots, M.$$

From the Alessandrini identity for the Lamé system, (3.4), we find that

$$\begin{aligned} \int_\Omega (\rho^{(1)} - \rho^{(2)})\mathbf{G}_1^{(\cdot,n)}(x,y) \cdot \overline{\mathbf{G}}_2^{(m,\cdot)}(x,y)\,\mathrm{d}x \\ = \langle (\Lambda_1 - \Lambda_2)\mathbf{G}_1^{(\cdot,n)}(x,y), \mathbf{G}_2^{(m,\cdot)}(x,y)\rangle, \quad \forall y, z \in K_0. \end{aligned} \tag{5.1}$$

We note that

$$\begin{aligned} S_{k-1}^{(m,n)}(y,z) =& \int_{U_{k-1}} (\rho^{(1)} - \rho^{(2)})\mathbf{G}_1^{(\cdot,n)}(x,y) \cdot \overline{\mathbf{G}}_2^{(m,\cdot)}(x,y)\,\mathrm{d}x \\ =& \langle (\Lambda_1 - \Lambda_2)\mathbf{G}_1^{(\cdot,n)}(x,y), \mathbf{G}_2^{(m,\cdot)}(x,y)\rangle \\ & - \int_{\cup_{j=1}^{k-1} D_j} (\rho^{(1)} - \rho^{(2)})\mathbf{G}_1^{(\cdot,n)}(x,y) \cdot \overline{\mathbf{G}}_2^{(m,\cdot)}(x,y)\,\mathrm{d}x, \end{aligned} \tag{5.2}$$

for any $m, n \in \{1,2,3\}$. Hence, by Proposition 3.4, we obtain that

$$|\mathbf{S}_{k-1}(y,z)| \leq C(\varepsilon + \delta_{k-1}), \quad y, z \in K_0, \tag{5.3}$$

for some constant C. Let $P_k \in \Sigma_k$ and $y_r = z_r = P_k - r\nu(P_k)$, where $\nu(P_k)$ is the exterior unit normal vector to ∂D_{k-1} and r is small. We write

$$\mathbf{S}_{k-1}(y_r, y_r) = I_1 + I_2 \tag{5.4}$$

with

$$I_1 = \int_{B_{r_0/6}(P_k) \cap D_k} (\rho^{(1)} - \rho^{(2)})(x)\mathbf{G}_1(x, y_r) \cdot \overline{\mathbf{G}}_2(x, y_r)\,\mathrm{d}x, \tag{5.5}$$

and

$$I_2 = \int_{U_{k-1}\setminus(B_{r_0/6}(P_k)\cap D_k)} (\rho^{(1)} - \rho^{(2)})(x)\mathbf{G}_1(x, y_r) \cdot \overline{\mathbf{G}}_2(x, y_r)\,\mathrm{d}x. \tag{5.6}$$

By Proposition 3.4, we have that

$$|I_2| \leq CE.$$

Then, by the same argument as in [3], we obtain the estimate for I_1,

$$|I_1| \geq |\rho_k^{(1)} - \rho_k^{(2)}| \left(\frac{1}{2} \int_{B_{r_0/6}(P_k)\cap D_k} |\mathbf{\Gamma}(x, y_r)|^2 \mathrm{d}x - C \right).$$

Using the explicit form of $\Gamma(x, y)$, we find that

$$\begin{aligned} |I_1| &\geq |\rho_k^{(1)} - \rho_k^{(2)}|(Cr^{-1} - C) \\ &\geq C|\rho_k^{(1)} - \rho_k^{(2)}|r^{-1} - CE. \end{aligned} \tag{5.7}$$

Now, by Lemma 4.2 and (5.3), we have

$$|\mathbf{S}_{k-1}(y_r, y_r)| \leq C \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \tau^{2N_1}} (\varepsilon + \delta_{k-1} + E) r^{-1}.$$

Hence, we have

$$|\rho_k^{(1)} - \rho_k^{(2)}|\, r^{-1} \leq C \left(E + \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \tau^{2N_1}} (\varepsilon + \delta_{k-1} + E) r^{-1} \right),$$

so that

$$|\rho_k^{(1)} - \rho_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left(\left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \tau^{2N_1}} + r \right). \tag{5.8}$$

Then, by the same argument as in [3], it follows that

$$E \leq \frac{1 - \omega_M^{-1}((C + 3^{1/4})^{-M})}{\omega_M^{-1}((C + 3^{1/4})^{-M})}\, \varepsilon, \tag{5.9}$$

which completes the proof. □

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