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INVERSE BOUNDARY VALUE PROBLEM FOR THE HELMHOLTZ EQUATION WITH MULTI-FREQUENCY DATA

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Abstract. We study the inverse boundary value problem for the Helmholtz equation using the Dirichlet-to-Neumann map at selected frequencies as the data. We develop an explicit iterative reconstruction of the wavespeed using a multi-level nonlinear projected steepest descent iterative scheme in Banach spaces. We consider wavespeeds containing conormal singularities. A conditional Lipschitz estimate for the inverse problem holds for wavespeeds of the form of a linear combination of piecewise constant functions with an underlying domain partitioning, and gives a framework in which the scheme converges. The stability constant grows exponentially as the number of subdomains in the domain partitioning increases. To mitigate this growth of the stability constant, we introduce a hierarchy of compressive approximations of the solution to the inverse problem with piecewise constant functions. We establish an upper bound of the stability constant, which constrains the compression rate of the solution. Then, tracking the frequency dependencies through the approximation errors, we arrive at a procedure to select the frequencies such that convergence from level to level of our scheme is guaranteed.

1. Introduction. In this paper, we study the inverse boundary value problem for the Helmholtz equation using the Dirichlet-to-Neumann map at selected frequencies as the data. We focus on developing an explicit iterative reconstruction of the wavespeed. This inverse problem arises, for example, in reflection seismology and inverse obstacle scattering problems for electromagnetic waves [6, 24, 7]. We consider wavespeeds containing conormal singularities.

Uniqueness of the mentioned inverse boundary value problem was established by Sylvester & Uhlmann [23] assuming that the wavespeed is a strictly positive bounded measurable function. From an optimization point of view this inverse problem has been extensively studied. We mention, in particular, the work of [8]. Multi-frequency data and so-called frequency progression have been introduced to intuitively stabilize the iterative schemes used in optimization [12, 22, 5, 14].

We give a complete characterization of a multi-level, multi-frequency projected steepest descent method guaranteeing convergence. The convergence is derived from stability estimates. Conditional Lipschitz stability is obtained by assuming that the wavespeed is a linear combination of piecewise constant functions with an underlying domain partitioning [11]; one can accommodate partial data. Here, we establish Fréchet differentiability, obtain the frequency dependencies of the constants, and prove (using complex geometrical optics solutions) an upper bound for the constant in the conditional Lipschitz stability estimate. Here we assume full data. It follows that the stability constant behaves exponentially with respect to the refinement of the domain, Ω say, that is, as the number of subdomains in the domain partitioning increases.

Conditional Lipschitz stability estimates have been extensively studied in Electrical Impedance Tomography (EIT). We mention the work of [1, 10]. Usually, the Calderón type problem yields only logarithmic or weaker type stability estimates. The logarithmic stability has been shown to be optimal assuming sufficient regularity and boundedness of the coefficient [19]. Selected techniques developed for EIT carry over in the analysis presented here.

To mitigate the growth of the stability constant, we introduce a hierarchy of compressive approximations of the unique solution to the inverse problem with piecewise constant functions. We arrive at a multi-level scheme while progressively increasing the number of subdomains subject to a condition which couples the approximation errors and stability constants between neighboring levels [16]. We use the projected steepest descent iteration proposed in [16]. For transparency, we

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restrict ourselves to the normalized duality mapping. To avoid the use of $L^\infty(\Omega)$, which is neither a convex, smooth nor reflexive Banach space, we consider $L^\infty(\Omega)$ as a convex subset of $L^p(\Omega)$ or a limit of the approximations using l^p. We can then choose p to be sufficiently large to obtain estimates relevant to applications. The mentioned upper bound of the stability constant constrains the minimal compression rate of the unique solution using piecewise constant functions. Tracking the frequency dependencies through the approximation errors, we arrive at a procedure to select the frequencies such that convergence from level to level of our scheme is guaranteed.

Pöschol, Resmerita and Scherzer [20] discussed whether, given a direct problem formulated in L^∞, the inverse problem should be considered in L^∞ or whether is it more appropriate to formulate the problem in some L^p-space with $1 < p < \infty$ (in fact, $p = 2$ in [20])? One conclusion there was that for numerical realizations L^p approximations are quite advantageous because in the L^∞ case, one can construct examples of weak star convergent sequences, which are not even convergent in L^1. From this perspective a numerical analysis is preferable in L^p-spaces, where the L^∞ space is embedded and a topology τ_d is introduced by a pseudometric

$$d(u,v) = \left\|{}^F u v^F\right\|_p + |\,\|u\|_\infty \;\; \|v\|_\infty\,|.$$

For instance, there exist piecewise constant Ansatz functions, which approximate an L^∞-function in the L^p-sense but also the L^∞-norm (not the function).

Multi-frequency data. The multi-frequency data are obtained from solutions to the corresponding boundary value problem for the wave equation by applying a Fourier transform. Let Ω be a (bounded) Lipschitz domain in $\mathbb{R}^3$ and $c = c(x)$ be a strictly positive bounded measurable function. We consider the boundary value problem for the wave equation

$$\begin{cases} \partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = f(x,t), & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = 0,\ \partial_t u(x,0) = 0, & x \in \Omega. \end{cases}$$

The hyperbolic Dirichlet-to-Neumann map, $\check{\Lambda}_{c^{-2}}$, is given by

$$\begin{aligned} \check{\Lambda}_{c^{-2}} :\ H \to &\quad L^2(\partial\Omega \times \mathbb{R}^+), \\ f \mapsto &\quad \partial_\nu u^f\,|_{\partial\Omega\times(0,\infty)}, \end{aligned}$$

where ∂_ν denotes the normal derivative at $\partial\Omega$ and $H = \{f \in H^1(\partial\Omega \times \mathbb{R}^+) \mid f(x,0) = 0\}$. One, indeed, may take the Fourier transform of $\partial_\nu u^f$, since it is a tempered distributions [18], and thus obtain multi-frequency data.

2. Direct problem. We describe the direct problem and some properties of the data, that is, the Dirichlet-to-Neumann map. We consider the boundary value problem,

$$\left\{ \begin{array}{rcll} (-\Delta - \omega^2 c(x)^{-2})u & = & 0, & \text{in } \Omega, \\ u & = & g & \text{on } \partial\Omega. \end{array} \right. \tag{2.1}$$

If the boundary value g is in $H^{3/2}(\partial\Omega)$ and $c(x)$ is bounded and measurable, then the unique solution to (2.1) belongs to $H^2(\Omega)$. Therefore ∇u is in $H^1(\Omega)$ and as a consequence $\nabla u\,|_{\partial\Omega}$ belongs to $H^{1/2}(\partial\Omega)$. One can then introduce the Dirichlet-to-Neumann map,

$$\Lambda_{\omega^2 c^{-2}}\, g = \nabla u \cdot \nu\,|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\,|_{\partial\Omega} \in H^{-1/2}(\partial\Omega). \tag{2.2}$$

We summarize some results which we will use in the proofs of the properties of the Dirichlet-to-Neumann map.

PROPOSITION 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, $c^{-2}(x)$ be a strictly positive bounded measurable function and $f \in L^p(\Omega)$, $g \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with $1 < p < \infty$. Assume that 0 is not a Dirichlet eigenvalue of $-\Delta - \omega^2 c^{-2}$ in Ω. Then, there exists a unique solution $u \in W^{2,p}(\Omega)$ to the problem*

$$\begin{cases} (-\Delta - \omega^2 c(x)^{-2})u &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial\Omega. \end{cases} \tag{2.3}$$

Moreover, assume that $\omega_0^2 \|c^{-2}\|_{L^\infty(\Omega)}$ is strictly less than the first Dirichlet eigenvalue of the Laplacian on Ω. Then, for any $0 < \omega < \omega_0$,

$$\|u\|_{W^{2,p}(\Omega)} \le C(\|g\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|f\|_{L^p(\Omega)}) \tag{2.4}$$

where C depends on Ω.

The proof makes use of the existence of a $W^{2,p}(\Omega)$ function w such that $w = g$ on $\partial\Omega$ and $\|w\|_{W^{2,p}(\Omega)} \le C\|g\|_{W^{2-\frac{1}{p},p}(\partial\Omega)}$, and of the Fredholm alternative; see for example Theorem 3.5.8 in Feldman and Uhlmann's notes [17]). For the reader's convenience, we also mention the following Proposition 2.2 without proof, which we use for the case of L^2.

PROPOSITION 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, $c^{-2}(x)$ be a strictly positive bounded measurable function and $f \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$. Assume that 0 is not a Dirichlet eigenvalue of the operator $-\Delta - \omega^2 c^{-2}$ in Ω. Then there exists a unique solution $u \in H^1(\Omega)$ of (2.3). Moreover, assume that $\omega_0^2 \|c^{-2}\|_{L^\infty(\Omega)}$ is strictly less than the first Dirichlet eigenvalue of the Laplacian on Ω. Then, for any $0 < \omega < \omega_0$,*

$$\|u\|_{H^1(\Omega)} \le C\left(\|g\|_{H^{1/2}(\partial\Omega)} + \|f\|_{H^{-1}(\Omega)}\right), \tag{2.5}$$

where C depends on Ω.

LEMMA 2.3 (Fréchet differentiability). *Assume that 0 is not a Dirichlet eigenvalue of $-\Delta - \omega^2 c^{-2}$ in Ω. The operator, F_ω, given by*

$$\begin{aligned} F_\omega : L^p(\Omega) \cap L^\infty(\Omega) \to \quad & \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)), \\ c^{-2}(x) \mapsto \quad & \Lambda_{\omega^2 c^{-2}}, \end{aligned}$$

is Fréchet differentiable at c^{-2}.

Proof. We start from Alessandrini's identity,

$$\int_\Omega \omega^2 (c_1^{-2} - c_2^{-2}) u_1 u_2 \,\mathrm{d}x = \langle (\Lambda_{\omega^2 c_1^{-2}} - \Lambda_{\omega^2 c_2^{-2}}) u_1 \,,\, u_2 \rangle \tag{2.6}$$

where u_1 and u_2 are the solutions of the Helmholtz equation with Dirichlet boundary condition and coefficient c_1 and c_2, respectively. Let $\delta c^{-2} \in L^\infty(\Omega)$. We observe, while substituting c^{-2} and $c^{-2} + \delta c^{-2}$ for c_1^{-2} and c_2^{-2}, that

$$\langle (\Lambda_{\omega^2 (c^{-2} + \delta c^{-2})} - \Lambda_{\omega^2 c^{-2}}) g \,,\, h \rangle = \omega^2 \int_\Omega \delta c^{-2}\, uv \,\mathrm{d}x, \tag{2.7}$$

where u and v solve the boundary value problems,

$$\begin{cases} (-\Delta - \omega^2 (c^{-2} + \delta c^{-2}))u &= 0, \quad x \in \Omega, \\ u &= g, \quad x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} (-\Delta - \omega^2 c^{-2})v &= 0, \quad x \in \Omega, \\ v &= h, \quad x \in \partial\Omega, \end{cases}$$

respectively. We show that

$$\langle DF_\omega(c^{-2})(\delta c^{-2})g\,,\,h\rangle = \omega^2 \int_\Omega \delta c^{-2}\,\tilde{u}v\,\mathrm{d}x, \tag{2.8}$$

where $\tilde{u}$ solves the equation

$$\begin{cases} (-\Delta - \omega^2 c^{-2})\tilde{u} &= 0, \quad x \in \Omega,\\ \tilde{u} &= g, \quad x \in \partial\Omega. \end{cases}$$

In fact, by (2.7), we have that

$$\langle (\Lambda_{\omega^2(c^{-2}+\delta c^{-2})} - \Lambda_{\omega^2 c^{-2}})g\,,\,h\rangle - \omega^2 \int_\Omega \delta c^{-2}\,\tilde{u}v\,\mathrm{d}x = \omega^2 \int_\Omega \delta c^{-2}\,(u-\tilde{u})v\,\mathrm{d}x. \tag{2.9}$$

By using the Hölder inequality twice and the Sobolev embedding theorem, we obtain that

$$\left|\omega^2 \int_\Omega \delta c^{-2}\,(u-\tilde{u})v\,\mathrm{d}x\right| \leq\ \omega^2 \|\delta c^{-2}\|_{L^p(\Omega)} \|u-\tilde{u}\|_{L^{2q}(\Omega)} \|v\|_{L^{2q}(\Omega)}, \tag{2.10}$$

where $\frac{1}{p}+\frac{1}{q}=1$. We note that $u-\tilde{u}$ solves the equations

$$\begin{cases} (-\Delta - \omega^2 c^{-2})(u-\tilde{u}) &= -\omega^2 \delta c^{-2}\,u, \qquad x \in \Omega,\\ u-\tilde{u} &= 0, \qquad\qquad\quad\ x \in \partial\Omega. \end{cases}$$

Therefore, for $3/2 \leq p \leq 9/4$, by the Sobolev embedding theorem and Proposition 2.1, we find that

$$\begin{aligned} \|u-\tilde{u}\|_{L^{2q}(\Omega)} &\leq C\|u-\tilde{u}\|_{W^{2,\frac{6q}{4q+3}}(\Omega)}\\ &\leq C\|\omega^2\delta c^{-2}\,u\|_{L^{\frac{6q}{4q+3}}(\Omega)} \leq C\omega^2\|\delta c^{-2}\|_{L^p(\Omega)}\|u\|_{L^{\frac{6q}{9-2q}}(\Omega)}, \end{aligned} \tag{2.11}$$

with $2 \leq \frac{6q}{9-2q} \leq 6$. The right-most inequality is obtained by using the Hölder inequality. Similarly, for $9/4 < p < 9/2$, we have that

$$\begin{aligned} \|u-\tilde{u}\|_{L^{2q}(\Omega)} &\leq C\|u-\tilde{u}\|_{W^{1,\frac{6q}{2q+3}}(\Omega)}\\ &\leq C\|\omega^2\delta c^{-2}\,u\|_{L^{\frac{6q}{2q+3}}(\Omega)} \leq C\omega^2\|\delta c^{-2}\|_{L^p(\Omega)}\|u\|_{L^{\frac{6q}{9-4q}}(\Omega)}, \end{aligned} \tag{2.12}$$

with $2 \leq \frac{6q}{9-4q} \leq 6$. For $p \geq 9/2$, we get

$$\|u-\tilde{u}\|_{L^{2q}(\Omega)} \leq \|u-\tilde{u}\|_{W^{2,2q}(\Omega)} \leq C\|\omega^2\delta c^{-2}\,u\|_{L^{2q}(\Omega)} \leq C\omega^2\|\delta c^{-2}\|_{L^p(\Omega)}\|u\|_{L^{\frac{2q}{3-2q}}(\Omega)}, \tag{2.13}$$

with $2 < \frac{2q}{3-2q} \leq 6$. By using the interpolation of L^p spaces,

$$\|u\|_{L^{p_\theta}(\Omega)} \leq \|u\|^{1-\theta}_{L^2(\Omega)}\,\|u\|^{\theta}_{L^6(\Omega)}, \quad \forall\theta \in [0,1],$$

with p_θ defined by $1/p_\theta = (1-\theta)/2 + \theta/6$, we conclude that, for any $p \geq 3/2$,

$$\|u-\tilde{u}\|_{L^{2q}(\Omega)} \leq C\omega^2\|\delta c^{-2}\|_{L^p(\Omega)}\|u\|^{1-\theta}_{L^2(\Omega)}\,\|u\|^{\theta}_{L^6(\Omega)}, \tag{2.14}$$

for some $\theta \in [0,1]$.

Then, upon substituting (2.14) into (2.10) and applying the Sobolev embedding theorem and Proposition 2.2 to u and v, we conclude that

$$\begin{aligned}
&\left|\langle(\Lambda_{\omega^2(c^{-2}+\delta c^{-2})}-\Lambda_{\omega^2c^{-2}})g\,,\,h\rangle-\omega^2\int_\Omega \delta c^{-2}\,\tilde{u}v\,\mathrm{d}x\right| \\
&\qquad= \left|\omega^2\int_\Omega \delta c^{-2}\,(u-\tilde{u})v\,\mathrm{d}x\right| \\
&\qquad\le \omega^2\|\delta c^{-2}\|_{L^p(\Omega)}\|u-\tilde{u}\|_{L^{2q}(\Omega)}\|v\|_{L^{2q}(\Omega)} \\
&\qquad\le C\omega^4\|\delta c^{-2}\|^2_{L^p(\Omega)}\|u\|^{1-\theta_1}_{L^2(\Omega)}\|u\|^{\theta_1}_{L^6(\Omega)}\|v\|^{1-\theta_2}_{L^2(\Omega)}\|v\|^{\theta_2}_{L^6(\Omega)} \\
&\qquad\le C\omega^4\|\delta c^{-2}\|^2_{L^p(\Omega)}\|u\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)},
\end{aligned}$$

for some constant C and $\theta_1,\theta_2\in[0,1]$. This leads to the Fréchet differentiability of F_ω at c^{-2}. ◻

We introduce a uniform constant C_0 such that any wavespeed function in the analysis satisfies

$$\|c^{-2}\|_{L^\infty(\Omega)}\le C_0.$$

We assume that $C_0\omega_0^2$ is less than the first Dirichlet eigenvalue of the Laplacian on Ω. The use of a uniform constant can be relaxed.

LEMMA 2.4. *There exists a constant $\hat{\mathfrak{L}}_0$, which depends on Ω, such that*

$$\|DF_\omega(c^{-2})\|_{\mathcal{L}(L^p(\Omega),\mathcal{L}(H^{\frac12}(\partial\Omega),H^{-\frac12}(\partial\Omega)))}\le\hat{\mathfrak{L}}_0\omega^2. \tag{2.15}$$

for $\omega\le\omega_0$.

Proof. We start from Alessandrini's identity (2.6). By applying the Hölder inequality twice, we find that

$$\begin{aligned}
|\langle(\Lambda_{\omega^2c_1^{-2}}-\Lambda_{\omega^2c_2^{-2}})u_1\,,\,u_2\rangle| &= \left|\int_\Omega \omega^2(c_1^{-2}-c_2^{-2})u_1u_2\,\mathrm{d}x\right| \\
&\le \omega^2\|c_1^{-2}-c_2^{-2}\|_{L^p(\Omega)}\|u_1\|_{L^{2q}(\Omega)}\|u_2\|_{L^{2q}(\Omega)}.
\end{aligned}$$

By the interpolation of L^p spaces, the Sobolev embedding theorem and Proposition 2.2, we obtain

$$\|u_i\|_{L^{2q}(\Omega)}\le C\|u_i\|_{H^1(\Omega)}\le C\|u_i\|_{H^{1/2}(\partial\Omega)},\quad i=1,2.$$

Hence,

$$\|\Lambda_{\omega^2c_1^{-2}}-\Lambda_{\omega^2c_2^{-2}}\|_{\mathcal{L}(H^{\frac12}(\partial\Omega),H^{-\frac12}(\partial\Omega))}\le\hat{\mathfrak{L}}_0\omega^2\|c_1^{-2}-c_2^{-2}\|_{L^p(\Omega)}$$

from which (2.15) follows. ◻

LEMMA 2.5. *For any $c_1^{-2},c_2^{-2}\in L^\infty(\Omega)$ strictly positive and bounded and $0\le\omega\le\omega_0$, there exists a constant $\mathfrak{L}_0$, which depends on Ω, such that*

$$\|DF_\omega(c_1^{-2})-DF_\omega(c_2^{-2})\|_{\mathcal{L}(L^p(\Omega),\mathcal{L}(H^{\frac12}(\partial\Omega),H^{-\frac12}(\partial\Omega)))}\le\mathfrak{L}_0\omega^4\|c_1^{-2}-c_2^{-2}\|_{L^p(\Omega)}.$$

Proof. Let $g,h\in H^{1/2}(\Omega)$ and $u_i,v_i,\ i=1,2$, solve the boundary value problems,

$$\left\{\begin{array}{rcll}(-\Delta-\omega^2c_i^{-2})u_i&=&0,&x\in\Omega,\\ u_i&=&h,&x\in\partial\Omega,\end{array}\right.$$

$$\left\{\begin{array}{rcll}(-\Delta-\omega^2c_i^{-2})v_i&=&0,&x\in\Omega,\\ v_i&=&g,&x\in\partial\Omega,\end{array}\right.$$

resepcectively. By usingidentity (2.8) and applying the Hölder inequality twice, we have

$$
\begin{aligned}
&|\langle (DF_\omega(c_1^{-2})(\delta c^{-2}) - DF_\omega(c_2^{-2})(\delta c^{-2}))g\,,\, h\rangle| \\
&\qquad\qquad = \left|\omega^2 \int_\Omega \delta c^{-2}\,(u_1 v_1 - u_2 v_2)\,\mathrm{d}x\right| \\
&\qquad\qquad \le \omega^2 \|\delta c^{-2}\|_{L^p(\Omega)} (\|u_1 - u_2\|_{L^{2q}(\Omega)} \|v_1\|_{L^{2q}(\Omega)} + \|u_2\|_{L^{2q}(\Omega)} \|v_1 - v_2\|_{L^{2q}(\Omega)}).
\end{aligned}
$$

We note that $u_1 - u_2$ solves the equations

$$
\begin{cases}
(-\Delta - \omega^2 c_1^{-2})(u_1 - u_2) = \omega^2 (c_1^{-2} - c_2^{-2})\, u_2, & x \in \Omega, \\
u_1 - u_2 = 0, & x \in \partial\Omega.
\end{cases}
$$

Using an argument similar to the one in the proof of Lemma 2.3, we derive that

$$
\|u_1 - u_2\|_{L^{2q}(\Omega)} \le C\omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \|u_2\|_{L^{2q}(\Omega)} \le C\omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}
$$

and, analogously,

$$
\|v_1 - v_2\|_{L^{2q}(\Omega)} \le C\omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \|v_2\|_{L^{2q}(\Omega)} \le C\omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \|h\|_{H^{\frac{1}{2}}(\partial\Omega)}.
$$

Hence

$$
\begin{aligned}
&|\langle (DF_\omega(c_1^{-2})(\delta c^{-2}) - DF_\omega(c_2^{-2})(\delta c^{-2}))g\,,\, h\rangle| \\
&\qquad\qquad\qquad \le C\omega^4 \|\delta c^{-2}\|_{L^p(\Omega)} \|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \|h\|_{H^{\frac{1}{2}}(\partial\Omega)},
\end{aligned}
$$

which gives that

$$
\|DF_\omega(c_1^{-2}) - DF_\omega(c_2^{-2})\|_{\mathcal{L}(L^p(\Omega), \mathcal{L}(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega)))} \le C\omega^4 \|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)}.
$$

□

3. Stability of the inverse problem. We invoke the following

ASSUMPTION 3.1. *$\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying*

$$
|\Omega| \le A
$$

Here and in the sequel $|\Omega|$ denotes the Lebesgue measure of Ω. We assume that $\partial\Omega$ is of Lipschitz class and we fix an open portion Σ of $\partial\Omega$ which is of Lipschitz class with constants r_0 and L.

A domain partitioning of Ω is given by

$$
\mathcal{D}_N \triangleq \{\{D_1, D_2, \ldots, D_N\} \mid \bigcup_{j=1}^N \overline{D}_j = \Omega\ ,\ (D_j \cap D_{j'})^\circ = \emptyset\}. \tag{3.1}
$$

ASSUMPTION 3.2. *The wavespeed function $c(x)$ satisfies*

$$
\|c\|_{L^\infty(\Omega)} \le B_1, \quad \|c^{-1}\|_{L^\infty(\Omega)} \le B_2,
$$

where B_1 and B_2 are positive constants, and is of the form

$$
c(x) = \sum_{j=1}^N c_j \chi_{D_j}(x),
$$

where $c_j, j = 1, \ldots N$ are unknown numbers and D_j are known open sets in $\mathbb{R}^n$ which satisfy the following assumption.

Assumption 3.3. *The $D_j, j = 1, \dots, N$ are connected and ∂D_j are of Lipschitz class. There exists one set, say D_1, such that $\partial D_1 \cap \partial\Omega$ contains an open portion Σ_1 of Lipschitz class with constants r_0 and L. For every $j \in \{2, \dots, N\}$ there exist $j_1, \dots, j_M \in \{1, \dots, N\}$ such that*

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \dots, M$,

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a non-empty open portion Σ_k of Lipschitz class with constants r_0 and L such that

$$\begin{gathered} \Sigma_1 \subset \Sigma, \\ \Sigma_k \subset \Omega, \quad \forall k = 2, \dots, M. \end{gathered}$$

Furthermore there exists $P_k \in \Sigma_k$, at which D_{k-1} satisfies the interior ball condition with radius $\frac{3r_0}{16}$, and a rigid transformation of coordinates such that $P_k = 0$ and

$$\begin{aligned} \Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n = \phi_k(x')\}, \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n > \phi_k(x')\}, \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n < \phi_k(x')\}, \end{aligned}$$

where ϕ_k is a $C^{0,1}$ function on $B'_{r_0/3}$ satisfying

$$\phi_k(0) = 0$$

and

$$\|\phi_k\|_{C^{0,1}(B'_{r_0/3})} \le L.$$

For simplicity, we call $D_{j_1}, \dots, D_{j_M}$ a chain of domains connecting D_1 to D_j.

Definition 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^3$ and of Lipschitz class and Σ be a open portion of $\partial\Omega$. We define $H^{1/2}_{co}(\Sigma)$ as*

$$H^{1/2}_{co}(\Sigma) = \{g \in H^{1/2}(\partial\Omega) \mid \operatorname{supp} g \subset \Sigma\}$$

and $H^{-1/2}_{co}(\Sigma)$ as the topological dual of $H^{1/2}_{co}(\Sigma)$; we denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $H^{1/2}_{co}(\Sigma)$ and $H^{-1/2}_{co}(\Sigma)$.

We define the local Dirichlet-to-Neumann map $\Lambda_q^{(\Sigma)}$ as

$$\begin{aligned} \Lambda_q^{(\Sigma)} : \quad H^{1/2}_{co}(\Sigma) \to &\quad H^{-1/2}_{co}(\Sigma) \\ g \mapsto &\quad \left.\frac{\partial u}{\partial \nu}\right|_{\Sigma}, \end{aligned}$$

where u solves (2.1) and ν is the exterior unit normal vector to $\partial\Omega$.

Note that Lemma 2.3, 2.4, 2.5, 4.1, 4.2 and 4.3 also hold true for the local Dirichlet-to-Neumann map.

We define the nonlinear operator-valued map $F_{\omega,\mathcal{D}_N}$ as

$$\begin{aligned} F_{\omega,\mathcal{D}_N} : \operatorname{span}(\chi_{D_1}, \dots, \chi_{D_N}) &\to \mathcal{L}(H^{\frac{1}{2}}(\Sigma), H^{-\frac{1}{2}}(\Sigma)) \\ c &\mapsto \Lambda^{(\Sigma)}_{\omega^2 c^{-2}}. \end{aligned} \tag{3.2}$$

The convergence of our multi-level (frequency, domain partitioning) iterative scheme relies on the Fréchet differentiability of this map, its boundedness, and the stability of its inverse.

LEMMA 3.5 ([9]). *Let Ω satisfy Assumption 3.1 and $c_k, k = 1,2$ be two piecewise constant functions of the form*

$$c_k(x) = \sum_{j=1}^{N} c_{k,j}\chi_{D_j}(x), \quad k = 1,2$$

which satisfy Assumption 3.2 and $D_j, j = 1,\dots,N$ satisfy Assumption 3.3. Then, there exists a constant $\underline{\mathfrak{C}}_0 = \underline{\mathfrak{C}}_0(r_0, L, A, B, N, \omega_0)$, such that

$$\|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \le \underline{\mathfrak{C}}_0 \, \frac{\|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_{\mathcal{L}(H^{1/2}_{co}(\Sigma), H^{-1/2}_{co}(\Sigma))}}{\omega^2}, \tag{3.3}$$

where $\Lambda_k^{(\Sigma)} = \Lambda^{(\Sigma)}_{\omega^2 c_k^{-2}}$ for $k = 1,2$.

We note that $\omega^{-2}\|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_{\mathcal{L}(H^{1/2}_{co}(\Sigma), H^{-1/2}_{co}(\Sigma))}$ does not blow up as ω tends to zero.

REMARK 3.6. *The lower bound of the Fréchet derivative, DF_ω, has been used to study the stability properties [4, 2, 3, 11]. A standard proposition states the following. Let M_1 and M_2 be positive numbers and N be a positive integer. Let $\mathcal{A}$ and $\mathcal{K}$ be an open subset and a compact subset of $\mathbb{R}^N$, respectively. Assume that $\mathcal{K} \subset \mathcal{A}$,*

$$\operatorname{dist}\big(\mathcal{K}, \mathbb{R}^N \setminus \mathcal{A}\big) \ge M_1 \quad \text{and } \mathcal{K} \subset B_{M_2}(0).$$

Let $\mathcal{B}$ be a Banach space and let $F: \mathcal{A} \to \mathcal{B}$ be such that:

(i) F is Frechét differentiable;
(ii) the Frechét derivative $DF: \mathcal{A} \to \mathcal{L}(\mathbb{R}^N, \mathcal{B})$ is uniformly continuous with a modulus of continuity $\sigma_1(\cdot)$;
(iii) $F_{|\mathcal{K}}$ is injective;
(iv) $(F_{|\mathcal{K}})^{-1} : F(\mathcal{K}) \to \mathcal{K}$ is uniformly continuous with a modulus of continuity $\sigma_2(\cdot)$;
(v) DF is injective in $\mathcal{K}$, namely, there is a positive number C_0 such that

$$\min_{x\in\mathcal{K}, |h|=1} \|DF(x)(h)\|_{\mathcal{B}} \ge C_0;$$

then

$$\|x_1 - x_2\|_{\mathbb{R}^N} \le C\|F(x_1) - F(x_2)\|_{\mathcal{B}} \quad \text{for every } x_1, x_2 \in \mathcal{K},$$

where $C = \max\{\frac{2M_1}{\sigma_2^{-1}(\delta_1)}, \frac{2}{C_0}\}$, for $\delta_1 = \frac{1}{2}\min\{\delta_0, M_2\}$ with $\delta_0 = \sigma_1^{-1}(\frac{C_0}{2})$.

Thus a positive estimate of the lower bound of the Fréchet derivative implies a Lipschitz type stability estimate for the inverse. It holds true also in infinite dimensional spaces. For a proof in finite dimensional spaces, we refer to [3].

PROPOSITION 3.7. *Let c_k, $k = 1,2$ satisfy the assumptions in Lemma 3.5. Then there exist constants $\mathfrak{C}_0$ and K_1, such that*

$$\|c_1^{-2} - c_2^{-2}\|_{L^p(\Omega)} \le \mathfrak{C}_0 \, e^{K_1 N^{7/2}} \, \frac{\|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}}{\omega^2}, \tag{3.4}$$

where $\Lambda_k = \Lambda^{(\partial\Omega)}_{\omega^2 c_k^{-2}}$ for $k = 1,2$ and $\mathfrak{C}_0$ and K_1 depend only on A, L, r_0, B_1 and B_2.

Proof. We start the proof with two facts. The first fact is that, if the bounded measurable function $c^{-2}(x)$ is of the form

$$c(x) = \sum_{j=1}^{N} c_j\chi_{D_j}(x),$$

then its $H^{s'}$ norm can be bounded by its L^∞ norm for every $0 < s' < 1/2$. That is, $\|c\|_{H^{s'}} \le C$, where the constant C depends only on A, B_1, B_2, L, r_0 and N. The second fact is on the existence of so-called complex geometrical optics (CGO) solutions of the Helmholtz equation. Assume that $c^{-2} \in L^\infty(\Omega)$, $\|c^{-2}\|_{L^\infty(\Omega)} \le B_2$, then the equation

$$-\Delta u - \omega^2 c^{-2} u = 0$$

has a solution u, which is of the form

$$u(x) = \exp^{\mathrm{i}x\cdot\zeta}(1 + r(x)),$$

where $r \in H^1(\Omega)$ satisfies

$$\|r\|_{L^2(\Omega)} \le \frac{C}{|\zeta|}\omega^2 \|c^{-2}\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \le C\omega^2 \|c^{-2}\|_{L^2(\Omega)},$$

for a constant C which only depends on Ω. For a proof, we refer to [23].

Now, we claim that

$$\omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)} \le CEN^{1/2}\sigma\left(\frac{\|\Lambda_1 - \Lambda_2\|}{E}\right) \tag{3.5}$$

where

$$\sigma(t) = \begin{cases} |\log t|^{-\frac{1}{7}} & \text{for} \quad 0 < t < \frac{1}{e}, \\ et & \text{for} \quad t \ge \frac{1}{e}, \end{cases} \tag{3.6}$$

and where $E = \omega^2 \|c_1^{-2} - c_2^{-2}\|_{L^\infty(\Omega)}$. To proof (3.5), we start from Alessandrini's identity

$$\int_\Omega \omega^2 (c_1^{-2} - c_2^{-2}) u_1 u_2 = \langle (\Lambda_1 - \Lambda_2) u_1|_{\partial\Omega}, u_2|_{\partial\Omega}\rangle,$$

for any $u_j \in H^1(\Omega)$ solution of $-\Delta u_j - \omega^2 c_j^{-2} u_j = 0$ for $j = 1, 2$. Let $\xi \in \mathbb{R}^3$ and let ζ_1 and ζ_2 be unit vectors of $\mathbb{R}^3$ such that $\{\zeta_1, \zeta_2, \xi\}$ is an orthogonal set. We take

$$\zeta_1 = \frac{s}{\sqrt{2}}\left(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\zeta_1 + \frac{1}{\sqrt{2}s} + \mathrm{i}\zeta_2\right),$$
$$\zeta_2 = -\frac{s}{\sqrt{2}}\left(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\zeta_1 - \frac{1}{\sqrt{2}s} + \mathrm{i}\zeta_2\varphi\right).$$

Then, $\zeta_j \cdot \zeta_j = 0$ for $j = 1, 2$ and $|\zeta_1| = |\zeta_2| = s$ and $\zeta_1 + \zeta_2 = \xi$. Assume that u_j are solutions to $-\Delta u_j - \omega^2 c_j^{-2} u_j = 0$ for $j = 1, 2$ of the form

$$u_1(x) = \mathrm{e}^{\mathrm{i}x\cdot\zeta_1}(1 + r_1(x)), u_2(x) = \mathrm{e}^{\mathrm{i}x\cdot\zeta_2}(1 + r_2(x))$$

provided that $|\zeta| = s \ge \max(C_0 B_2, 1)$ with

$$\|r_j\|_{L^2(\Omega)} \le \frac{C_0}{s} B_2$$

for $j = 1, 2$ and $C_0 = C_0(\Omega)$. Inserting the solutions u_1 and u_2 in Alessandrini's identity we derive

$$\begin{aligned} &\left|\int_\Omega \omega^2 (c_1^{-2} - c_2^{-2}) \mathrm{e}^{\mathrm{i}\xi\cdot x} \mathrm{d}x\right| \\ &\quad\le \|\Lambda_1 - \Lambda_2\| \|u_1\|_{H^{1/2}(\partial\Omega)} \|u_2\|_{H^{1/2}(\partial\Omega)} + \left|\int_\Omega \omega^2 (c_1^{-2} - c_2^{-2}) \mathrm{e}^{\mathrm{i}\xi\cdot x}(r_1 + r_2 + r_1 r_2)\mathrm{d}x\right| \\ &\quad\le \|\Lambda_1 - \Lambda_2\| \|u_1\|_{H^1(\Omega)} \|u_2\|_{H^1(\Omega)} + CE(\|r_1\|_{L^2(\Omega)} + \|r_2\|_{L^2(\Omega)} + \|r_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)}). \end{aligned} \tag{3.7}$$

If $\Omega \subset B_R(0)$ then

$$\|u_j\|_{H^1(\Omega)} \leq Cs\mathrm{e}^{Rs}, \quad j = 1, 2.$$

Let s be large enough so that $s \leq \mathrm{e}^{Rs}$. Then, for $s \geq C'$, we have

$$|\omega^2(c_1^{-2} - c_2^{-2})\hat{}(\xi)| \leq C\left(\mathrm{e}^{4Rs}\|\Lambda_1 - \Lambda_2\| + \frac{E}{s}\right) \tag{3.8}$$

where the $\omega^2 c_j^{-2}$'s have been extended to all $\mathbb{R}^3$ by zero. Hence we get

$$\|\omega^2(c_1^{-2} - c_2^{-2})\hat{}\|^2_{L^2(\mathbb{R}^3)} \leq C\rho^3(\mathrm{e}^{8Rs}\|\Lambda_1 - \Lambda_2\|^2 + \frac{E^2}{s^2}) + \int_{|\xi|\geq\rho} |\omega^2(c_1^{-2} - c_2^{-2})\hat{}(\xi)|^2 \,\mathrm{d}\xi \tag{3.9}$$

The we have that

$$\|\omega^2(c_1^{-2} - c_2^{-2})\|^2_{H^{s'}(\Omega)} \leq CNE^2,$$

where C depends on A, L, r_0 and on s' and hence

$$\begin{aligned}\rho^{2s'} \int_{|\xi|\geq\rho} |\omega^2(c_1^{-2} - c_2^{-2})\hat{}(\xi)|^2 \,\mathrm{d}\xi &\leq \int_{|\xi|\geq\rho} |\xi|^{2s'} |\omega^2(c_1^{-2} - c_2^{-2})\hat{}(\xi)|^2 \,\mathrm{d}\xi \\ &\leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^{s'} |\omega^2(c_1^{-2} - c_2^{-2})\hat{}(\xi)|^2 \,\mathrm{d}\xi \leq CNE^2.\end{aligned}$$

Hence

$$\int_{|\xi|\geq\rho} |\omega^2(c_1^{-2} - c_2^{-2})\hat{}(\xi)|^2 \, d\xi \leq \frac{CNE^2}{\rho^{2s'}}$$

for any $s' \in (0, 1/2)$. Inserting last bound in (3.9) we derive

$$\|\omega^2(c_1^{-2} - c_2^{-2})\hat{}\|^2_{L^2(\mathbb{R}^3)} \leq C\rho^3(\mathrm{e}^{8Rs}\|\Lambda_1 - \Lambda_2\|^2 + \frac{E^2}{s^2}) + \frac{CNE^2}{\rho^{2s'}}.$$

Picking up $\rho = s^{\frac{2}{3+2s'}}$ and inserting it into last relation we obtain

$$\|\omega^2(c_1^{-2} - c_2^{-2})\|^2_{L^2(\Omega)} \leq CNE^2\left(\mathrm{e}^{16Rs}\left(\frac{\|\Lambda_1 - \Lambda_2\|}{E}\right)^2 + \frac{1}{s^{\frac{4s'}{3+2s'}}}\right)$$

for $s \geq C'$. Let us choose

$$s = \frac{1}{16R}\left|\ln \frac{\|\Lambda_1 - \Lambda_2\|}{E}\right|$$

where we have assumed that

$$\frac{\|\Lambda_1 - \Lambda_2\|}{E} < c$$

so that $s \geq C'$. Under this assumption,

$$\|\omega^2(c_1^{-2} - c_2^{-2})\|_{L^2(\Omega)} \leq CN^{1/2}E\left(\left(\frac{\|\Lambda_1 - \Lambda_2\|}{E}\right) + \left|\ln \frac{\|\Lambda_1 - \Lambda_2\|}{E}\right|^{-\frac{2s'}{3+2s'}}\right)$$

On the other hand if

$$\frac{\|\Lambda_1 - \Lambda_2\|}{E} \geq c,$$

then

$$\|\omega^2(c_1^{-2} - c_2^{-2})\|_{L^2(\Omega)} \leq N^{1/2}E \leq CN^{1/2}E\frac{\|\Lambda_1 - \Lambda_2\|}{E}$$

Finally, choosing $s' = 1/4$, the claim (3.5) follows.

Now observing that

$$\|c_1^{-2} - c_2^{-2}\|_{L^\infty(\Omega)} \leq C(r_0)\|c_1^{-2} - c_2^{-2}\|_{L^2(\Omega)},$$

we derive

$$\|\omega^2(c_1^{-2} - c_2^{-2})\|_{L^\infty(\Omega)} \leq CN^{1/2}E\sigma\left(\frac{\|\Lambda_1 - \Lambda_2\|}{E}\right),$$

so that

$$E \leq CN^{1/2}E\sigma\left(\frac{\|\Lambda_1 - \Lambda_2\|}{E}\right),$$

which gives

$$E \leq \frac{1}{\sigma^{-1}(CN^{-1/2})}\|\Lambda_1 - \Lambda_2\|,$$

and finally recalling the expression of σ we get

$$E \leq \mathfrak{C}_0\, \mathrm{e}^{K_1 N^{7/2}}\|\Lambda_1 - \Lambda_2\|,$$

where $\mathfrak{C}_0$ and K_1 depends only on A, B_1, B_2, L, r_0. □

Approximation error estimates. For a given domain partitioning, we introduce the function

$$\varphi :\ N \mapsto \mathfrak{C}_0^{-1}\, \mathrm{e}^{-K_1 N^{7/2}},$$

which will play the role of compression rate.

DEFINITION 3.8. *For a family of given domain partitionings $\{\mathcal{D}_N\}$, the corresponding family of admissible sets $\{\mathcal{A}_N\}$ is given by*

$$\mathcal{A}_N \triangleq \{c \in L^\infty(\Omega) \mid \mathrm{dist}_{L^p(\Omega)}(c^{-2}, \mathrm{span}(\chi_{D_1}, \chi_{D_2}, \ldots, \chi_{D_N})) \leq \varphi(N)\}.$$

DEFINITION 3.9. *For a given admissible set $\mathcal{A}_N$, the approximation error $\eta_{\omega,\mathcal{D}_N}$ is defined by*

$$\begin{array}{rcl} \eta_{\omega,\mathcal{D}_N} : Y & \to & \overline{\mathbb{R}}^+ \\ y & \mapsto & \mathrm{dist}(y, F_{\omega,\mathcal{D}_N}(\mathcal{A}_N)). \end{array}$$

For any given data, y, from Lemma 2.4, it immediately follows that

$$\eta_{\omega,\mathcal{D}_N}(y) \leq \hat{\mathfrak{L}}_0\omega^2\varphi(N). \tag{3.10}$$

4. Formulation in l^p. Here, we describe the direct problem and some properties of the Dirichlet-to-Neumann map in terms of a coefficient space identified as a subspace embedded in l^p. We revisit the forward operator-valued map and define

$$\begin{aligned} F_{\omega,\mathcal{D}_N}:\ & l^p && \to \mathcal{L}(H^{\frac{1}{2}}_{co}(\Sigma), H^{-\frac{1}{2}}_{co}(\Sigma)) \\ & (c_1^{-2},\dots,c_N^{-2},0,\dots) && \mapsto \Lambda^{(\Sigma)}_{\omega^2\sum_{j=1}^N c_j^{-2}\chi_{D_j}}. \end{aligned} \tag{4.1}$$

The following lemma is an analogue of Lemma 2.3.

LEMMA 4.1 (Fréchet differentiability). *Let Ω satisfy Assumption 3.1 and $c, \delta c^{-2}$ be functions of the form*

$$c(x) = \sum_{j=1}^N c_j \chi_{D_j}(x), \quad \delta c^{-2}(x) = \sum_{j=1}^N \delta c_j^{-2}\chi_{D_j}(x),$$

which satisfy Assumption 3.2 and $D_j, j = 1,\dots,N$ satisfy Assumption 3.3. Assume that 0 is not a Dirichlet eigenvalue of $-\Delta - \omega^2 c^{-2}$ in Ω. Then, the map $F_{\omega,\mathcal{D}_N}$ given in (4.1) is Fréchet differentiable at c^{-2}.

Proof. We follow and adapt the proof of Lemma 2.3. We continue from (2.9). By using the Hölder inequality twice and the Sobolev embedding theorem, we find that

$$\begin{aligned} &\left|\omega^2 \int_\Omega \delta c^{-2}\,(u-\tilde u)v\,\mathrm{d}x\right| \\ &\qquad \le \omega^2 \left\|\delta c^{-2}\sum_{j=1}^N (|D_j|\chi_{D_j})^{-1/p}\right\|_{L^p(\Omega)} \left\|\sum_{j=1}^N (|D_j|\chi_{D_j})^{1/p}(u-\tilde u)v\right\|_{L^q(\Omega)} \\ &\qquad \le \omega^2 \max_j |D_j|^{1/p}\,\|\delta c^{-2}\|_{l^p}\|u-\tilde u\|_{L^{2q}(\Omega)}\|v\|_{L^{2q}(\Omega)}. \end{aligned} \tag{4.2}$$

We note that $u - \tilde u$ solves

$$\begin{cases} (-\Delta - \omega^2 c^{-2})(u-\tilde u) = -\omega^2\delta c^{-2}\,u, & x\in\Omega, \\ u-\tilde u = 0, & x\in\partial\Omega. \end{cases}$$

Again, following the proof of Lemma 2.3, we obtain the estimate

$$\|u-\tilde u\|_{L^{2q}(\Omega)} \le C\omega^2 \max_j |D_j|^{1/p}\|\delta c^{-2}\|_{l^p}\|u\|^{1-\theta}_{L^2(\Omega)}\,\|u\|^{\theta}_{L^6(\Omega)}, \tag{4.3}$$

for some $\theta\in[0,1]$. Then, substituting (4.3) into (4.2) and applying the Sobolev embedding theorem and Proposition 2.2 to u and v, we conclude that

$$\begin{aligned} &\left|\langle(\Lambda_{\omega^2(c^{-2}+\delta c^{-2})} - \Lambda_{\omega^2 c^{-2}})g\,,\,h\rangle - \omega^2\int_\Omega \delta c^{-2}\,\tilde u v\,\mathrm{d}x\right| \\ &\qquad = \left|\omega^2\int_\Omega \delta c^{-2}\,(u-\tilde u)v\,\mathrm{d}x\right| \\ &\qquad \le \omega^2\max_j |D_j|^{1/p}\,\|\delta c^{-2}\|_{l^p}\|u-\tilde u\|_{L^{2q}(\Omega)}\|v\|_{L^{2q}(\Omega)} \\ &\qquad \le C\omega^4 \max_j |D_j|^{2/p}\,\|\delta c^{-2}\|^2_{l^p}\|u\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

for some constant C. This implies the Fréchet differentiability of $F_{\omega,\mathcal{D}_N}$ at c^{-2}. □

The following lemma is an analogue of Lemma 2.4.

LEMMA 4.2. *Let Ω satisfy Assumption 3.1 and c_1 and c_2 be functions of the form*

$$c_i(x) = \sum_{j=1}^{N} c_{i,j}\chi_{D_j}(x), \quad i = 1, 2$$

which satisfy Assumption 3.2 and $D_j, j = 1, \ldots, N$ satisfy Assumption 3.3. Let $0 \le \omega \le \omega_0$. Then there exists a constant $\hat{\mathfrak{L}}_0$, which depends on Ω , such that

$$\|DF_{\omega,\mathcal{D}_N}(c_1^{-2})\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega),H^{-\frac{1}{2}}(\partial\Omega))} \le \hat{\mathfrak{L}}_0\omega^2 \max_j |D_j|^{1/p}. \tag{4.4}$$

Proof. We start from the Alessandrini's identity

$$\int_\Omega \omega^2(c_1^{-2} - c_2^{-2})u_1u_2 \,\mathrm{d}x = \langle(\Lambda_{\omega^2c_1^{-2}} - \Lambda_{\omega^2c_2^{-2}})u_1 \,,\, u_2\rangle \tag{4.5}$$

where u_1 and u_2 are the solutions of the Helmholtz equation with wavespeeds c_1 and c_2, respectively, subject to the Dirichlet boundary condition. Then, by applying the Hölder inequality twice, we have that

$$\begin{aligned}
|\langle(\Lambda_{\omega^2c_1^{-2}} - \Lambda_{\omega^2c_2^{-2}})u_1 \,,\, u_2\rangle| &= \left|\int_\Omega \omega^2(c_1^{-2} - c_2^{-2})u_1u_2 \,\mathrm{d}x\right| \\
&\le \omega^2 \left\|(c_1^{-2} - c_2^{-2}) \sum_{j=1}^{N}(|D_j|\chi_{D_j})^{-1/p}\right\|_{L^p(\Omega)} \left\|\sum_{j=1}^{N}(|D_j|\chi_{D_j})^{1/p}u_1u_2\right\|_{L^q(\Omega)} \\
&\le \omega^2 \max_j |D_j|^{1/p} \,\|c_1^{-2} - c_2^{-2}\|_{l^p}\|u_1\|_{L^{2q}(\Omega)}\|u_2\|_{L^{2q}(\Omega)}.
\end{aligned}$$

By the Sobolev embedding theorem and Proposition 2.2, we obtain that

$$\|u_i\|_{L^{2q}(\Omega)} \le C\|u_i\|_{H^1(\Omega)} \le C\|u_i\|_{H^{1/2}(\partial\Omega)}, \quad i = 1, 2.$$

We conclude that

$$\|\Lambda_{\omega^2c_1^{-2}} - \Lambda_{\omega^2c_2^{-2}}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega),H^{-\frac{1}{2}}(\partial\Omega))} \le \hat{\mathfrak{L}}_0\omega^2 \max_j |D_j|^{1/p}\|c_1^{-2} - c_2^{-2}\|_{l^{\frac{3}{2}}}$$

and (4.4) follows. ☐

In the following lemma, an analogue of Lemma 2.5, we discuss the Lipschitz continuity of $DF_{\omega,\mathcal{D}_N}$.

LEMMA 4.3. *Let Ω satisfy Assumption 3.1 and c_1 and c_2 be two functions of the form*

$$c_i(x) = \sum_{j=1}^{N} c_{i,j}\chi_{D_j}(x), \quad i = 1, 2$$

which satisfy Assumption 3.2 and $D_j, j = 1, \ldots, N$ satisfy Assumption 3.3. Let $0 \le \omega \le \omega_0$. Then there exists a constant $\mathfrak{L}_0$, which depends on Ω, such that

$$\|DF_{\omega,\mathcal{D}_N}(c_1^{-2}) - DF_{\omega,\mathcal{D}_N}(c_2^{-2})\|_{\mathcal{L}(l^p,\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega),H^{-\frac{1}{2}}(\partial\Omega)))} \le \mathfrak{L}_0\omega^4 \max_j |D_j|^{2/p}\|c_1^{-2} - c_2^{-2}\|_{l^p}.$$

Proof. We let δc^{-2} have the same form as c_1 and c_2. Then

$$\begin{aligned}
&|\langle(DF_{\omega,\mathcal{D}_N}(\delta c^{-2}) - DF_{\omega,\mathcal{D}_N}(c_2^{-2})(\delta c^{-2}))u_1 \,,\, u_2\rangle| \\
&\qquad = \left|\omega^2 \int_\Omega \delta c^{-2}\,(u_1v_1 - u_2v_2)\,\mathrm{d}x\right| \\
&\qquad \le \omega^2 \max_j |D_j|^{1/p}\|\delta c^{-2}\|_{l^p}(\|u_1 - u_2\|_{L^{2q}(\Omega)}\|v_1\|_{L^{2q}(\Omega)} + \|u_2\|_{L^{2q}(\Omega)}\|v_1 - v_2\|_{L^{2q}(\Omega)}).
\end{aligned}$$

Using an argument similar to the one in the proof of Lemma 4.1, we find the estimate

$$\begin{aligned}\|u_1 - u_2\|_{L^{2q}(\Omega)} &\le C\omega^2 \max_j |D_j|^{1/p} \|c_1^{-2} - c_2^{-2}\|_{l^p} \|u_2\|_{L^{2q}(\Omega)} \\ &\le C \max_j |D_j|^{1/p} \|c_1^{-2} - c_2^{-2}\|_{l^p} \|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)}\end{aligned} \tag{4.6}$$

and analogously

$$\begin{aligned}\|v_1 - v_2\|_{L^{2q}(\Omega)} &\le C\omega^2 \max_j |D_j|^{1/p} \|c_1^{-2} - c_2^{-2}\|_{l^p} \|v_2\|_{L^{2q}(\Omega)} \\ &\le C \max_j |D_j|^{1/p} \|c_1^{-2} - c_2^{-2}\|_{l^p} \|v_2\|_{H^{\frac{1}{2}}(\partial\Omega)}.\end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned}&|\langle (DF_{\omega,\mathcal{D}_N}(c_1^{-2})(\delta c^{-2}) - DF_{\omega,\mathcal{D}_N}(c_2^{-2})(\delta c^{-2}))g\, ,\, h\rangle| \\ &\qquad \le C\omega^4 \max_j |D_j|^{2/p} \|\delta c^{-2}\|_{l^p} \|c_1^{-2} - c_2^{-2}\|_{l^p} \|u_1\|_{H^{\frac{1}{2}}(\partial\Omega)} \|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)},\end{aligned}$$

which gives that

$$\|DF_{\omega,\mathcal{D}_N}(c_1^{-2}) - DF_{\omega,\mathcal{D}_N}(c_2^{-2})\|_{\mathcal{L}(l^p,\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega),H^{-\frac{1}{2}}(\partial\Omega)))} \le C\omega^4 \max_j |D_j|^{2/p} \|c_1^{-2} - c_2^{-2}\|_{l^p}.$$

□

The key reason to follow a formulation in l^p is that the dependencies of the constants on the domains are 'weaker' in the sense that they depend on a *fractional* power of the measures of the sets in the domain partitioning only.

5. Projected steepest descent iteration for $L^p(\Omega)$. In this section, we described the projected steepest descent iteration proposed in [16] for L^p and the corresponding constants. We summarize some basic notions associated with iterative methods in Banach spaces.

We let $1 < p, q < \infty$ be conjugate exponents,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For $p > 1$, the subdifferential mapping $J_p = \partial f_p : X \to 2^{X^*}$ of the convex functional $f_p : x \mapsto \frac{1}{p}\|x\|^p$ defined by

$$J_p(x) = \{x^* \in X^* \mid \langle x, x^*\rangle = \|x\|\,\|x^*\| \text{ and } \|x^*\| = \|x\|^{p-1}\} \tag{5.1}$$

is called the duality mapping of X with gauge function $t \mapsto t^{p-1}$. Generally, the duality mapping is set-valued. For a detailed introduction to the geometry of Banach spaces and the duality mapping, we refer to [15, 21].

DEFINITION 5.1. *Let X be a uniformly smooth Banach space. The Bregman distance $\Delta_2(x,\cdot)$ of the convex functional $x \mapsto \frac{1}{2}\|x\|^2$ at $x \in X$ is defined as*

$$\Delta_2(x,\tilde{x}) = \frac{1}{2}\|\tilde{x}\|^2 - \frac{1}{2}\|x\|^2 - \langle J_2(x), \tilde{x} - x\rangle, \quad \tilde{x} \in X, \tag{5.2}$$

where J_2 denotes the normalized duality mapping of X.

In the next lemma, we summarize some facts we need here concerning the duality mapping and Bregman distance for $L^p(\Omega)$.

LEMMA 5.2. *For the Banach spaces $L^p(\Omega)$, the following holds true:*

(a) The normalized duality mapping J_2 for $L^p(\Omega)$ is single-valued and defined by

$$\begin{aligned}J_2 : L^p(\Omega) &\to L^q(\Omega) \\ f(x) &\mapsto \|f\|_{L^p(\Omega)}^{2-p} |f(x)|^{p-2} f(x),\end{aligned}$$

(b) For all $f, \tilde{f} \in L^p(\Omega)$, we have that

$$\Delta_2(f, \tilde{f}) \geq \frac{1}{2}\|f - \tilde{f}\|^p_{L^p(\Omega)}. \tag{5.3}$$

The Bregman distance Δ_2 is similar to a metric, but, in general, does not satisfy the triangle inequality nor symmetry. For the Hilbert space $L^2(\Omega)$,

$$\Delta_2(f, \tilde{f}) = \frac{1}{2}\|f - \tilde{f}\|^2_{L^2(\Omega)}.$$

To constrain the iterates to a subset where Lipschitz type stability holds, we use the non-expansive Bregman projection for L^p spaces [16]. A comprehensive introduction to this topic can be found in [13].

DEFINITION 5.3. *Let X be a uniformly smooth Banach space. Given a closed convex set $Z \subset X$ and Bregman distance Δ_2, which is defined in Definition 5.2, the Bregman projection of a point $x \in X$ onto Z is the point*

$$P_Z(x) = \underset{y \in Z}{\arg\min}\, \Delta_2(y, x). \tag{5.4}$$

In the case of $L^p(\Omega)$, the Bregman projection is given by

$$P_Z(f) = \underset{g \in Z}{\arg\min} \left\{ \frac{1}{2}\|f\|^2_{L^p(\Omega)} + \frac{1}{2}\|g\|^2_{L^p(\Omega)} - \|g\|^{2-p}_{L^p(\Omega)} \int_\Omega |g(x)|^{p-2} g(x) f(x)\, \mathrm{d}x \right\}.$$

REMARK 5.4. *Assume that $\{D_j\}_{j=1}^N$ is a domain partitioning of Ω as in Section 3. Let Z be defined by $Z = \operatorname{span}\{\chi_{D_1}, \ldots, \chi_{D_N}\}$. For $p = 2$, we have that*

$$P_Z(f) = \sum_{j=1}^N g_j\, \chi_{D_j} \quad \textit{with} \quad g_j = \frac{1}{|D_j|} \int_{D_j} f(x)\, \mathrm{d}x.$$

The following projected steepest descent iteration is taken from [16]. We identify, if we restrict our analysis to l^p,

$$\begin{aligned} \hat{\mathfrak{L}} &= \hat{\mathfrak{L}}_0 \omega^2 \max_j |D_j|^{1/p}, \\ \mathfrak{L} &= \mathfrak{L}_0 \omega^4 \max_j |D_j|^{2/p}, \\ \mathfrak{C} &= \underline{\mathfrak{C}}_0 \omega^{-2} \mathrm{e}^{K_1 N^{7/2}}, \end{aligned} \tag{5.5}$$

and $F_{\omega, \mathcal{D}_N}$ with (4.1).

ALGORITHM 5.5. *We fix some abbreviations first: For c_k^{-2}, $k = 0, 1, 2, \ldots$, fixed denote*

$$R_k = F(c_k^{-2}) - y^\delta\,, \quad T_k = DF(c_k^{-2})^* j_2(F(c_k^{-2}) - y^\delta)\,, \quad r_k = \|R_k\|\,, \quad t_k = \|T_k\|\,. \tag{5.6}$$

Moreover, we define

$$\begin{aligned} \tilde{\mathfrak{C}} &:= \mathfrak{L}\mathfrak{C}^2\,, \\ \rho &:= \frac{1}{2}(2\tilde{\mathfrak{C}}\hat{\mathfrak{L}})^{-2} \left(1 + \sqrt{1 - 8\tilde{\mathfrak{C}}\eta} - 4\eta\tilde{\mathfrak{C}}\right)^2. \end{aligned} \tag{5.7}$$

and for $k = 0, 1, \ldots$

$$\begin{aligned} u_k &:= -\tilde{\mathfrak{C}} r_k^2 + (1 - 2\tilde{\mathfrak{C}}\eta) r_k - \eta - \tilde{\mathfrak{C}}\eta^2\,, \\ v_k &:= t_k^{-2} u_k r_k^2 (r_k - \eta) - \frac{1}{2} t_k^{-2} u_k^2 r_k^2\,, \\ w_k &:= \mathfrak{L} t_k^{-2} u_k r_k^2\,, \\ \mu_k &:= t_k^{-2} u_k r_k\,. \end{aligned} \tag{5.8}$$

The algorithm is given by

($S0$) *Choose a starting point* $x_0 \in Z$ *such that*

$$\Delta_2(x_0, z^\dagger) < \rho, \tag{5.9}$$

($S1$) *Compute the new iterate via*

$$\begin{aligned} \tilde{c}_{k+1}^{-2} &= J_2^*(J_2(c_k^{-2}) - \mu_k T_k) \\ c_{k+1}^{-2} &= \mathcal{P}_Z(\tilde{c}_{k+1}^{-2}). \end{aligned} \tag{5.10}$$

Set $k \leftarrow k+1$ *and repeat step* ($S1$).

Due to the projection applied, all iterates belong to the 'stable subset' (corresponding with a domain partitioning), Z, which in general can only offer an approximation to the unique solution. While the dimension of Z should be low to ensure a large radius of convergence, the approximation should be compressive. In [16] we introduced a multi-level approach to enable a gradual refinement of the domain partitioning defining subsets Z_n where n stands for the level; Z_n is given by

$$Z_n = \{c^{-2} \in \operatorname{span}(\chi_{D_1}, \ldots, \chi_{D_{N_n}})\}. \tag{5.11}$$

As the level index n increases, the number of subdomains, N_n, grows, and hence, the n-level domain partitioning $\mathcal{D}_{N_n}$ refines. In the following algorithm, $c_{n,k}^{-2}$ denotes the k^{th} iterate at level n. We identify $F_{\omega_n, \mathcal{D}_{N_n}}$ with F_n and $\eta_{\omega_n, \mathcal{D}_{N_n}}$ with η_n. For each frequency ω_n, the noisy data are written as y^δ. We identify $\hat{\mathfrak{L}}_n$, $\mathfrak{L}_n$ and $\mathfrak{C}_n$ with (5.5) replacing ω by ω_n. See Lemma 2.4, Lemma 2.5 and Proposition 3.7. For simplicity, we omit the subscript in the operator norm. It is natural to let the frequency ω_n increase with increasing n.

ALGORITHM 5.6.

($S0$) *Use* $c_{0,0}$ *as the starting point. Set* $n = 0$.

($S1$) *Iteration. Use* F_n *and* Z_n *as the modelling operator and convex subset to run Algorithm 5.5 with the discrepancy criterion given by*

$$K_n = \min\{k \in \mathbb{N} \mid \|F_n(c_{n,k}^{-2}) - y^\delta\| \leq (3+\varepsilon)\eta_n\}, \tag{5.12}$$

where $\varepsilon > 0$ *is a given uniform tolerance constant.*
STOP, if $n = N$, *a given number.*

($S2$) *Set* $c_{n+1,0} = c_{n,K_n}$; *Refine the domain partitioning to* $\mathcal{D}_{n+1}$; *Choose frequency* ω_{n+1} *such that the corresponding parameters* $\hat{\mathfrak{L}}_{n+1}$, $\mathfrak{L}_{n+1}$, $\mathfrak{C}_{n+1}$ *and the approximation error* η_{n+1} *satisfy the following inequality*

$$(3+\varepsilon)\eta_n < 2^{-1/2}(\hat{\mathfrak{L}}_{n+1}\mathfrak{C}_{n+1})^{-1}\left(\frac{1+\sqrt{1-8\tilde{\mathfrak{C}}_{n+1}\eta_{n+1}}}{2\tilde{\mathfrak{C}}_{n+1}} - 2\eta_{n+1}\right) - \eta_{n+1}; \tag{5.13}$$

Set $n = n+1$ *and go to step* ($S1$).

Main result. Our multi-level scheme starts at a low frequency. We cannot only choose frequency as the index for the levels because only one parameter appears not to be sufficient to satisfy (5.13) and arrive at a convergent scheme.

To achieve high accuracy reconstruction with multi-frequency data, we design an algorithm such that the starting point $c_{0,n+1}^{-2}$ at level $n+1$, which equals the result after the iterations at level n, is within the $(n+1)-$level convergence radius ρ_{n+1}. Therefore, the iterations can continue until the desired accuracy is obtained. In the above, (5.13) yields a sufficient multi-level condition balancing the competition between the approximation error and the convergence radius of neighboring levels [16]. Here, we establish that subject to a minimal compression rate of the unique solution using piecewise constant functions we can find frequencies such that this multi-level condition is satisfied.

THEOREM 5.7. *Let $\mathcal{D}_{N_n}, n = 1, \ldots, n_{\max}$ be a sequence of domain partitionings, and $\mathcal{A}_{N_n}$ the sequence of corresponding admissible sets, where $n_{\max}$ is a positive integer and depends on r_0, L and $|\Omega|$. Let the unique solution satisfy the compression rate,*

$$\mathrm{dist}(c_\dagger, \cup_{n=1}^{m} \mathcal{A}_{N_n}) \leq \mathfrak{C}_0^{-1}\, \mathrm{e}^{-K_1 m^{7/2}}, \quad m = 1, \ldots, n_{\max}. \tag{5.14}$$

There exists a set of selected frequencies, $\{\omega_n\}_{n=1}^{n_{\max}}$, such that the multi-level condition (5.13) *is satisfied. Then, Algorithm 5.6 has the property that the n-level iteration result, c_{n,K_n}^{-2}, satisfies*

$$\|c_{n,K_n}^{-2} - c_\dagger^{-2}\|_{L^p(\Omega)} \leq 3\mathfrak{C}_0^{-1}\, \mathrm{e}^{-K_1 N_n^{7/2}}, \quad \forall n = 1, \ldots, n_{max}, \quad \frac{3}{2} \leq p < \infty.$$

Proof. Following Proposition 3.7, we set $\varphi(N) = \mathfrak{C}_0^{-1}\, \mathrm{e}^{-K_1 N^{7/2}}$. Given the frequency, ω_n, for level n, we choose the frequency ω_{n+1} such that the following inequalities are satisfied:

$$\omega_{n+1}^6 (\varphi(N_{n+1}))^3 ((3+\varepsilon)\omega_n^2 \varphi(N_n) + \omega_{n+1}^2 \varphi(N_{n+1})) \leq (2^{3/2} \hat{\mathfrak{L}}_0^2 \mathfrak{L}_0)^{-1}, \tag{5.15}$$

and

$$\omega_{n+1}^6 \leq (8 \mathfrak{L}_0 \hat{\mathfrak{L}}_0 (\varphi(N_{n+1}))^3)^{-1}. \tag{5.16}$$

Then, on the one hand, from Lemma 2.4 and 2.5, (3.10) and (5.15), we conclude that

$$\begin{aligned} 2^{3/2} \tilde{\mathfrak{C}}_{n+1} (\hat{\mathfrak{L}}_{n+1} \mathfrak{C}_{n+1}) ((3+\varepsilon)\eta_n + \eta_{n+1}) & \\ (2^{3/2} \hat{\mathfrak{L}}_0^2 \mathfrak{L}_0) \omega_{n+1}^6 (\varphi(N_{n+1}))^3 ((3+\varepsilon)\omega_n^2 \varphi(N_n) + \omega_{n+1}^2 \varphi(N_{n+1})) & \leq 1. \end{aligned} \tag{5.17}$$

On the other hand, (5.16) with Lemma 2.5 and (3.10), gives that

$$1 - 8\tilde{\mathfrak{C}}_{n+1} \eta_{n+1} > 0.$$

Hence,

$$1 + \sqrt{1 - 8\tilde{\mathfrak{C}}_{n+1}\eta_{n+1}} - 4\tilde{\mathfrak{C}}_{n+1}\eta_{n+1} > 1,$$

which together with (5.17) gives (5.13). ◻

For any true wavespeed $c_\dagger$ and starting model $c_{0,0}$, if the first frequency is sufficiently low, then our multi-level projected steepest descent scheme converges. That is, by letting ω tend to 0, the convergence radius ρ tends to infinity. The following theorem gives a precise statement. Note that this statement does not give a lower bound of the frequency which is uniform for all $c_\dagger$ and c_0.

THEOREM 5.8. *Let the admissible set $\mathcal{A}_N$ be as defined in Definition 3.8 for some given N. Assume that $c_0 \in \mathcal{A}_N$. Then, for any $c_\dagger$ satisfying*

$$c^{-2} \in L^\infty(\Omega) \cap L^p(\Omega), \quad \|c\|_{L^\infty(\Omega)} \leq B_1, \quad \|c^{-1}\|_{L^\infty(\Omega)} \leq B_2,$$

there exists a frequency sufficiently small such that the convergence radius ρ is larger than $\|c_0^{-2} - c_\dagger^{-2}\|_{L^p(\Omega)}$.

Proof. Note that the convergence radius, derived in [16], for L^p is

$$\Delta_2(x_0, z^\dagger) < \rho := \frac{1}{2}(2\tilde{\mathfrak{C}}\hat{\mathfrak{L}})^{-2} \left(1 + \sqrt{1 - 8\tilde{\mathfrak{C}}\eta} - 4\eta\tilde{\mathfrak{C}}\right)^2. \tag{5.18}$$

The proof is divided into two steps. In the first step we provide a uniform lower bound of

$$1+\sqrt{1-8\tilde{\mathfrak{C}}\eta}-4\eta\tilde{\mathfrak{C}}.$$

Note that Lemma 2.5 and Proposition 3.7 implies that $\tilde{\mathfrak{C}}$ is uniformly bounded. Then the desired lower bound follows by noticing that

$$\lim_{\omega\to 0}\eta=0.$$

In the second step we show that

$$\lim_{\omega\to 0}\mathfrak{L}\mathfrak{C}^2\hat{\mathfrak{L}}=0.$$

The above indentity is proved by combining Lemma 2.5,2.4 and Proposition 3.7 to arrive at

$$\mathfrak{L}\mathfrak{C}^2\hat{\mathfrak{L}}\le C\omega^2,$$

for some constant C. This completes the proof. □

6. Discussion. We apply a projected steepest descent iterative method to the inverse boundary value problem of the Helmholtz equation using the Dirichlet-to-Neumann map as the data. We give explicit conditions for the convergence, derived from a conditional Lipschitz stability estimate for the inverse problem. The asymptotic behavior of the stability constant plays an important role in the convergence radius and required compression rate for the unique solution.

With sufficient regularity of the wave speed function, a Lipschitz stability estimate can be obtained as a variation of the arguments for the general logarithmic type stability estimates. For example, see [1] for the analogue of the EIT problem. To be more precise, assuming that $c_i^{-2}, i=1,2$ are the linear combinations of finitely many known C^1 functions $\psi_1,\dots,\psi_N$, we have that

$$\|c_1^{-2}-c_2^{-2}\|_{L^\infty(\Omega)}\le\mathfrak{C}_N\|\Lambda_1-\Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}.$$

However, here we have been focused on the presence of discontinuities. With the same behavior of the stability constant, we can repeat the analogous treatment for C^1 true wave speed reconstruction with $\tilde{\mathcal{A}}_N$ constructed from $\{\psi_1,\dots,\psi_N\}$.

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