

## LIPSCHITZ STABILITY OF AN INVERSE BOUNDARY VALUE PROBLEM FOR A SCHRÖDINGER TYPE EQUATION

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**Abstract.** In this paper we study the inverse boundary value problem of determining the potential in the Schrödinger equation from the knowledge of the Dirichlet-to-Neumann map, which is commonly accepted as an ill-posed problem in the sense that, under general settings, the optimal stability estimate is of logarithmic type. In this work, a Lipschitz type stability is established assuming a priori that the potential is piecewise constant with a bounded known number of unknown values.

**1. Introduction.** In this paper, we investigate the stability for the inverse boundary value problem of a Schrödinger equation with complex potential,  $q(x)$  say. This encompasses the Helmholtz equation with attenuation, when  $q(x) = \omega^2 c^{-2}(x)$ , where  $c$  denotes the speed of propagation and  $\omega$  is the frequency, which can be complex. In fact, the imaginary part of  $\omega c^{-1}(x)$  characterizes the attenuation of waves in the medium.

We begin with formulating the direct problem. Let  $u \in H^1(\Omega)$  be the weak solution to the boundary value problem,

$$(1.1) \quad \begin{cases} (-\Delta + q(x))u = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  is a bounded connected domain,  $q \in L^\infty(\Omega)$  is a complex-valued function and  $g$  is prescribed in the trace space  $H^{1/2}(\Omega)$ . The Dirichlet-to-Neumann map is the operator  $\Lambda_q : H^{1/2}(\Omega) \rightarrow H^{-1/2}(\Omega)$  given by

$$(1.2) \quad g \rightarrow \Lambda_q g = \left. \frac{\partial u}{\partial \nu} \right|_{\partial\Omega},$$

where  $\nu$  is the exterior unit normal vector to  $\partial\Omega$ .

The inverse problem that we consider, consists in determining  $q$  when  $\Lambda_q$  is known. This problem arises in geophysics, for example, in reflection seismology assuming a description in terms of time-harmonic scalar waves. The topic of this paper is the issue of continuous dependence of  $q$  from the Dirichlet-to-Neumann map  $\Lambda_q$ . The continuous dependence is of fundamental importance for the robustness of any reconstruction, as well as for the development of convergent iterative reconstruction procedures starting not too far from the solution (cf. [4]). More precisely, it has been proved that Landweber iteration reconstruction methods converge if the continuous dependence for the inverse problem is of Hölder or Lipschitz type.

From the work of [9], it is evident that for arbitrary potentials  $q$ , Lipschitz stability cannot hold. Motivated by, and following analogous results in electrical impedance tomography (EIT, cf. [2, 3]), here we study conditional stability when a-priori information on  $q$  is assumed. We consider models with discontinuous potentials to accommodate realistic reflectors. Specifically, we consider the space spanned by linear combinations of  $N$  characteristic functions. More precisely we consider potentials of the form

$$q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x),$$

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where  $q_j, j = 1, \dots, N$  are unknown complex numbers and  $D_j$  are known open Lipschitz sets in  $\mathbb{R}^n$ . Moreover, we consider the case of partial boundary data, that is, we can restrict the collection of measurements to only a part of the boundary. We refer to [11] for a review of recent uniqueness results. Here, we prove Lipschitz stability with a uniform constant, which depends on  $N$  and on the other a-priori parameters of the problem. We will show that the Lipschitz constant grows exponentially with the dimension,  $N$ , of the space of potentials. The method of proof follows the ideas introduced in Alessandrini and Vessella and relies on quantitative estimates of unique continuation of solutions to elliptic systems and on the use of singular solutions and of their asymptotic behaviour near the discontinuity interfaces. Compared to the case of the real or complex conductivity equation in the case of the Schrödinger equation we are able to derive our result relaxing the assumptions of regularity on  $\partial D_j$  that are assumed to be Lipschitz. Furthermore, taking advantage of the regularity of solutions and of its gradient inside the domain  $\Omega$  we find a better dependence of the stability constant on  $N$ .

The outline of the paper is as follows. In the next section we state all the assumptions and the main result. In Section 3, we give a summary of known regularity results connected to Schrödinger equation with complex potential, and some preparatory lemmas concerning the existence and asymptotics behaviour of singular solutions. Section 4 contains the proof of our main theorem. In Section 5 we demonstrate by an example that the Lipschitz constant grows exponentially with the dimension of the space of potentials. This example is constructed from its analogue in electrical impedance tomography [10].

## 2. Main result.

**2.1. Notation and definitions.** We denote by  $n$  the space dimension. For every  $x \in \mathbb{R}^n$ , we set  $x = (x', x_n)$  where  $x' \in \mathbb{R}^{n-1}$  for  $n \geq 2$ . With  $B_R(x)$ ,  $B'_R(x')$  and  $Q_R(x)$  we denote the open ball in  $\mathbb{R}^n$  centered at  $x$  of radius  $R$ , the ball in  $\mathbb{R}^{n-1}$  centered at  $x'$  of radius  $R$ , and the cylinder  $B'_R(x') \times (x_n - R, x_n + R)$ , respectively. For simplicity of notation,  $B_R(0)$ ,  $B'_R(0)$  and  $Q_R(0)$  are denoted by  $B_R$ ,  $B'_R$  and  $Q_R$ .

**DEFINITION 2.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . We say that a portion  $\Sigma$  of  $\partial\Omega$  is of Lipschitz class with constants  $r_0, L > 0$  if, for any  $P \in \Sigma$ , there exists a rigid transformation of coordinates such that  $P = 0$  and*

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \phi(x')\}$$

where  $\phi$  is a Lipschitz continuous function on  $B'_{r_0}$  with  $\phi(0) = 0$  and

$$\|\phi\|_{C^{0,1}(B'_{r_0})} \leq L.$$

We shall say that  $\Omega$  is of Lipschitz class with constants  $r_0$  and  $L$ , if  $\partial\Omega$  is of Lipschitz class with the same constants.

**DEFINITION 2.2.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  and of Lipschitz class and  $\Sigma$  be a open portion of  $\partial\Omega$ . We define  $H_{co}^{1/2}(\Sigma)$  as*

$$H_{co}^{1/2}(\Sigma) = \{g \in H^{1/2}(\partial\Omega) \mid \text{supp } g \subset \Sigma\}$$

and  $H_{co}^{-1/2}(\Sigma)$  as the topological dual of  $H_{co}^{1/2}(\Sigma)$ ; we denote by  $\langle \cdot, \cdot \rangle$  the dual pairing between  $H_{co}^{1/2}(\Sigma)$  and  $H_{co}^{-1/2}(\Sigma)$ .

**DEFINITION 2.3.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  and of Lipschitz class,  $\Sigma$  be a open portion of  $\partial\Omega$  and  $q \in L^\infty(\Omega)$ . Assume that 0 is not an eigenvalue of  $(-\Delta + q)$  with Dirichlet boundary conditions in  $\Omega$ , i.e.,*

$$\{u \in H_0^1(\Omega) \mid (-\Delta + q)u = 0\} = \{0\}.$$

For any  $g \in H_{co}^{1/2}(\Sigma)$ , let  $u \in H^1(\Omega)$  be the weak solution to the Dirichlet problem

$$(2.1) \quad \begin{cases} (-\Delta + q(x))u = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases}$$

We define the local Dirichlet-to-Neumann map  $\Lambda_q^{(\Sigma)}$  as

$$\Lambda_q^{(\Sigma)} : H_{co}^{1/2}(\Sigma) \rightarrow H_{co}^{-1/2}(\Sigma) \\ g \mapsto \left. \frac{\partial u}{\partial \nu} \right|_{\Sigma},$$

where  $\nu$  is the exterior unit normal vector to  $\partial\Omega$ .

With  $\Omega$  being a bounded open set, with  $C^{0,1}$  boundary, the set of the eigenvalues of  $(-\Delta + q)$  with Dirichlet boundary conditions is a discrete subset of  $\mathbb{C}$ , and hence can be avoided.

We observe that  $\Lambda_q^{(\Sigma)}$  can be identified with the sesquilinear form on  $H_{co}^{1/2}(\Sigma) \times H_{co}^{1/2}(\Sigma)$ , defined by

$$\langle \Lambda_q^{(\Sigma)} g, f \rangle = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} + qu\bar{v}) dx, \quad \forall f, g \in H_{co}^{1/2}(\Sigma),$$

where  $u$  is the solution to (2.1) and  $v$  is any function in  $H^1(\Omega)$  such that  $v|_{\partial\Omega} = f$ . This definition is independent of the choice of  $v$ : Let  $v_1, v_2$  be two different functions in  $H^1(\Omega)$  such that  $v_1|_{\partial\Omega} = v_2|_{\partial\Omega} = f$ . Then, since  $w = v_1 - v_2 \in H_0^1(\Omega)$ , and  $u$  is a solution, we have

$$\int_{\Omega} (\nabla u \cdot \nabla \bar{v}_1 + qu\bar{v}_1) dx - \int_{\Omega} (\nabla u \cdot \nabla \bar{v}_2 + qu\bar{v}_2) dx = \int_{\Omega} (\nabla u \nabla \bar{w} + qu\bar{w}) dx = 0,$$

using integration by parts. We denote by  $\|\cdot\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))}$  the norm defined as

$$\|\Lambda_q^{(\Sigma)}\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))} = \sup_{f, g \in H_{co}^{1/2}(\Sigma)} \{ \langle \Lambda_q^{(\Sigma)} g, f \rangle \mid \|g\|_{H_{co}^{1/2}(\Sigma)} = \|f\|_{H_{co}^{-1/2}(\Sigma)} = 1 \}.$$

**2.2. Main assumptions.** Our assumptions on  $\Omega$  and  $q(x)$  are

ASSUMPTION 2.4.  $\Omega \subset \mathbb{R}^n$  is a bounded domain satisfying

$$|\Omega| \leq A$$

Here and in the sequel  $|\Omega|$  denotes the Lebesgue measure of  $\Omega$ . We assume that  $\partial\Omega$  is of Lipschitz class and we fix an open portion  $\Sigma$  of  $\partial\Omega$  which is of Lipschitz class with constants  $r_0$  and  $L$ .

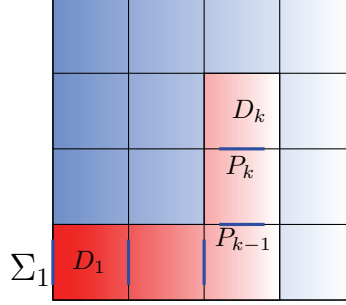
ASSUMPTION 2.5. The complex-valued function  $q(x)$  satisfies

$$\|q\|_{L^\infty(\Omega)} \leq B,$$

where  $B$  is a positive constant, and is of the form

$$q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x),$$

where  $q_j, j = 1, \dots, N$  are unknown complex numbers and  $D_j$  are known open sets in  $\mathbb{R}^n$  which satisfy the following assumption. Moreover, we assume that 0 is not an eigenvalue of  $(-\Delta + q)$  with Dirichlet boundary conditions in  $\Omega$ .

FIG. 1. Sketch of  $\Omega$ 

ASSUMPTION 2.6. The  $D_j, j = 1, \dots, N$ , are connected and pairwise non-overlapping open sets such that  $\cup_{j=1}^N \overline{D_j} = \overline{\Omega}$  and  $\partial D_j$  are of Lipschitz class. We also assume that there exists one set, say  $D_1$ , such that  $\partial D_1 \cap \partial \Omega$  contains an open portion  $\Sigma_1$  of Lipschitz class with constants  $r_0$  and  $L$ . For every  $j \in \{2, \dots, N\}$  there exist  $j_1, \dots, j_M \in \{1, \dots, N\}$  such that

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every  $k = 1, \dots, M$ ,

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a non-empty open portion  $\Sigma_k$  of Lipschitz class with constants  $r_0$  and  $L$  such that

$$\begin{aligned} \Sigma_1 &\subset \Sigma, \\ \Sigma_k &\subset \Omega, \quad \forall k = 2, \dots, M. \end{aligned}$$

Furthermore, there exists  $P_k \in \Sigma_k$ , at which  $D_{j_{k-1}}$  satisfies the interior ball condition with radius  $\frac{3r_0}{16}$ , and a rigid transformation of coordinates such that  $P_k = 0$  and

$$\begin{aligned} \Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n = \phi_k(x')\}, \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n > \phi_k(x')\}, \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n < \phi_k(x')\}, \end{aligned}$$

where  $\phi_k$  is a  $C^{0,1}$  function on  $B'_{r_0/3}$  satisfying

$$\phi_k(0) = 0$$

and

$$\|\phi_k\|_{C^{0,1}(B'_{r_0/3})} \leq L.$$

For simplicity, we call  $D_{j_1}, \dots, D_{j_M}$  a chain of domains connecting  $D_1$  to  $D_j$ .

In the further analysis, for simplicity of notation, we also use the constant  $r_1 = \frac{r_0}{16}$ .

**2.3. Statement of the main result.** The main result of this paper is stated as follows.

THEOREM 2.7. Let  $\Omega$  satisfy Assumption 2.4 and  $q^{(k)}, k = 1, 2$  be two complex piecewise constant functions of the form

$$q^{(k)}(x) = \sum_{j=1}^N q_j^{(k)} \chi_{D_j}(x), \quad k = 1, 2$$

which satisfy Assumption 2.5 and  $D_j, j = 1, \dots, N$  satisfy Assumption 2.6. Then, there exists a constant  $C = C(n, r_0, L, A, B, N)$ , such that

$$(2.2) \quad \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))},$$

where  $\Lambda_k^{(\Sigma)} = \Lambda_{q^{(k)}}^{(\Sigma)}$  for  $k = 1, 2$ .

**3. Preliminary results.** In this section, we state some results which will be used in the proof of our main stability result

**PROPOSITION 3.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $q \in L^\infty(\Omega)$  complex valued potential,  $f \in H^{-1}(\Omega)$  and  $g \in H^{1/2}(\partial\Omega)$ . Assume that 0 is not an eigenvalue for the operator  $-\Delta + q$ . Then there exists a unique solution  $u \in H^1(\Omega)$  to the problem*

$$(3.1) \quad \begin{cases} (-\Delta + q(x))u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases}$$

Moreover,

$$(3.2) \quad \|u\|_{H^1(\Omega)} \leq C (\|g\|_{H^{1/2}(\partial\Omega)} + \|f\|_{H^{-1}(\Omega)})$$

where  $C$  depends on  $n, \Omega$  and  $\|q\|_{L^\infty(\Omega)}$ .

The proof is a consequence of the existence of an  $H^1(\Omega)$  function  $w$  such that  $w = g$  on  $\partial\Omega$  and such that  $\|w\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}$  and of the Fredholm alternative; see for example Theorem 3.5.8 in Feldman and Uhlmann’s notes [6].

Our approach follows the one of Beretta and Francini[3], which is for the EIT problem with complex conductivity, of constructing singular solutions and of studying their asymptotic behavior when the singularity approaches the interfaces  $\Sigma_k$ . This method was originally introduced by Alessandrini and Vessella in the real-valued conductivity case [2]. To construct singular solutions for the EIT problems, the Green’s function plays a crucial role. In our case, we also use the Green’s function to treat the case of high dimension ( $n \geq 4$ ) and a first order derivative of Green’s function needs to be used for lower dimension ( $n = 2, 3$ ). In the following propositions, we discuss the existence and behavior of the Green’s functions ( $n \geq 4$ ) and a first order derivative of the Green’s function ( $n = 2, 3$ ) when  $q$  satisfies Assumption 2.5. We are especially interested in their asymptotic behavior near the  $C^{0,1}$  interface  $\Sigma_k$ .

Before doing this, we need to extend our original domain. We consider  $\Sigma_1$  and recall that up to a rigid transformation of coordinates we can assume that  $P_1 = 0$  and

$$(\mathbb{R}^n \setminus \Omega) \cap B_{r_0} = \{(x', x_n) \in B_{r_0} \mid x_n < \phi(x')\}$$

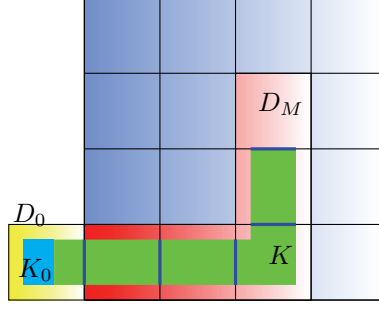
where  $\phi$  is a Lipschitz function such that  $\phi(0) = 0$  and  $\|\phi\|_{C^{0,1}(B'_{r_0})} \leq L$ . Then we extend  $\Omega$  to  $\Omega_0 = \Omega \cup D_0$  by adding an open set  $D_0$  defined as

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \mid \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6}r_0, |x_i| < \frac{2}{3}r_0, i = 1, \dots, n - 1 \right\}.$$

It turns out that  $\Omega_0$  is of Lipschitz class with constants  $\frac{r_0}{3}$  and  $L_1$ , where  $L_1$  depends on  $L$  only. We define

$$K_0 = \left\{ x \in D_0 \mid \text{dist}(x, \Sigma_1) \geq \frac{r_0}{3} \right\}$$

with  $\text{dist}(K_0, \partial\Omega) > \frac{r_0}{3}$ . We extend  $q(x)$  defined on  $\Omega$  by setting it equal to 1 in  $D_0$ . For simplicity of notation we still denote this extension by  $q(x)$ .

FIG. 2. Sketch of  $\Omega_0$ 

We consider any subdomain in  $\Omega$  and the chain of domains connecting it to  $D_1$ . For simplicity let us rearrange the indices of subdomains so that this chain corresponds to  $D_0, D_1, \dots, D_M$ ,  $M \leq N$ . Let  $S = \cup_{j=0}^M \overline{D_j}$  and  $K$  be a connected subset of  $S$  with Lipschitz boundary such that  $\overline{K} \cap \partial D_j = \Sigma_j \cup \Sigma_{j+1}$  for  $j = 1, 2, \dots, M$ ,  $K_0 \subset K$  and  $\text{dist}(K, \partial S \setminus \{\Sigma_{M+1} \cup \Sigma_1\}) > \frac{r_0}{16}$ .

In the following, we shall use  $C$  to denote positive constants. The value of the constants may change from line to line, but we shall specify their dependence everywhere where they appear. For  $n \geq 4$ , let  $\Gamma$  denote the fundamental solution associated with the Laplace operator.

PROPOSITION 3.2. *Let the complex-valued function  $q \in L^\infty(\Omega_0)$  satisfy Assumption 2.5 and  $n \geq 4$ . For  $y \in \Omega_0$ , there exists a unique function  $G(\cdot, y)$  continuous in  $\Omega_0 \setminus \{y\}$  such that*

$$(3.3) \quad \int_{\Omega_0} \nabla G(\cdot, y) \nabla \phi + qG(\cdot, y)\phi = \phi(y), \quad \forall \phi \in C_0^\infty(\Omega).$$

Furthermore, we have that  $G(x, y)$  is symmetric, that is,

$$(3.4) \quad G(x, y) = G(y, x), \quad x, y \in \Omega_0,$$

and the following estimates

$$(3.5) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{1-\frac{n}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega_0)$$

and

$$(3.6) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C, \quad \text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16},$$

hold, where the constant  $C$  depends on the constant in Proposition 3.1.

In both Beretta & Francini's proof [3] and Alessandrini & Vessella's proof [2], the blow-up property of a singular function,

$$\int_{U_k} \nabla G_1(y, x) \nabla G_2(x, y) \, dx,$$

where  $U_k = \Omega \setminus \cup_{j=1}^k D_j$  and  $G_1, G_2$  are functions defined by (3.3) for potentials  $q^{(1)}, q^{(2)}$ , respectively, when  $y$  approaches the interfaces, is essential. However, in the case of the Schrödinger equation, this does not happen if  $n = 2, 3$ . Therefore, for  $n = 2, 3$ , we will introduce a derivative in the point source. For  $n = 3$ , let

$$\Gamma(x, y) = -\frac{x_3 - y_3}{4\pi|x - y|^3},$$

which is the solution to the equation

$$(3.7) \quad -\Delta \Gamma(x, y) = \frac{\partial}{\partial x_3} \delta_y(x).$$

PROPOSITION 3.3. *Let  $n = 3$  and  $q \in L^\infty(\Omega_0)$ . For  $y \in \Omega_0$ , there exists a unique function  $G(\cdot, y)$  continuous in  $\Omega_0 \setminus \{y\}$  such that*

$$(3.8) \quad \int_{\Omega_0} \nabla G(\cdot, y) \cdot \nabla \phi + qG(\cdot, y)\phi = \frac{\partial}{\partial x_n} \phi(y), \quad \forall \phi \in C_0^\infty(\Omega).$$

Furthermore, we have that  $G(x, y)$  is symmetric, i.e.,

$$(3.9) \quad G(x, y) = G(y, x), \quad x, y \in \Omega_0,$$

and the following estimates

$$(3.10) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{-\frac{1}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega)$$

and

$$(3.11) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C, \quad \text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}.$$

hold, where the constant  $C$  depends on the constant in Proposition 3.1.

*Proof.* Consider  $G(x, y) = \Gamma(x, y) + \omega(x, y)$ , where  $\omega$  solves

$$(3.12) \quad \begin{cases} (-\Delta + q)\omega = q\Gamma, & \text{in } \Omega_0, \\ \omega = -\Gamma, & \text{on } \partial\Omega_0. \end{cases}$$

Since  $\Gamma(\cdot, y) \in W^{5/4, 4/3}(\partial\Omega_0)$ ,  $q\Gamma \in L^{4/3}(\Omega_0)$  and  $-\Gamma(\cdot, y) \in H^{1/2}(\partial\Omega_0)$ , by Proposition 3.1, (3.12) has a unique solution  $\omega \in H^1(\Omega_0)$  and  $\omega = G - \Gamma$  satisfies the estimate

$$(3.13) \quad \|\omega(\cdot, y)\|_{H^1(\Omega_0)} \leq C (\|\Gamma(\cdot, y)\|_{H^{1/2}(\partial\Omega_0)} + \|q(\cdot)\Gamma(\cdot, y)\|_{H^{-1}(\Omega_0)}) \leq C,$$

when  $\text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}$ . Hence,

$$(3.14) \quad \begin{aligned} \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} &= \|\omega(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \\ &\leq \|\omega(\cdot, y)\|_{H^1(\Omega_0 \setminus B_r(y))} \leq \|\omega(\cdot, y)\|_{H^1(\Omega_0)} \leq C. \end{aligned}$$

With the fact that

$$(3.15) \quad \|\Gamma(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{-\frac{1}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega),$$

(3.14) gives the desired estimate

$$(3.16) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{-\frac{1}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega).$$

Finally, again by (3.13) we immediately get

$$(3.17) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C.$$

□

For  $n = 2$ , let

$$\Gamma(x, y) = -\frac{2\pi(x_2 - y_2)}{|x - y|^2}$$

which is the solution to the equation

$$(3.18) \quad -\Delta\Gamma(x, y) = \frac{\partial}{\partial x_2}\delta_y(x).$$

PROPOSITION 3.4. *Let  $n = 2$  and  $q \in L^\infty(\Omega_0)$ . For  $y \in \Omega_0$ , there exists a unique function  $G(\cdot, y)$  continuous in  $\Omega_0 \setminus \{y\}$  such that*

$$(3.19) \quad \int_{\Omega_0} (\nabla G(\cdot, y) \cdot \nabla \phi + qG(\cdot, y)\phi) = \frac{\partial}{\partial x_n}\phi(y), \quad \forall \phi \in C_0^\infty(\Omega).$$

Furthermore, we have that  $G(x, y)$  is symmetric, that is,

$$(3.20) \quad G(x, y) = G(y, x), \quad x, y \in \Omega_0,$$

and the estimates

$$(3.21) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq C|\ln r|^{\frac{1}{2}}, \quad r \leq \frac{1}{2} \min\left(\text{dist}(y, \partial\Omega_0), \frac{1}{2}\right)$$

and

$$(3.22) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C, \quad \text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}$$

hold, where the constant  $C$  depends on the constant in Proposition 3.1.

We omit the proof here, because it follows from an adaption of the proof of Proposition 3.3. The symmetry of  $G$  follows by standard arguments based on integration by parts (see for example [5]).

In the sequel we will derive estimates of unique continuation in  $K$  for solutions to our equation. A key ingredient to obtain these estimates is the Three Spheres Inequality that we will state below and that was proved by [1, Theorem 3.1]. The next two propositions concern Three Sphere Inequalities for our equation. To prove it, one interprets the equation  $(-\Delta + q)u = 0$  for a complex function  $q(x)$  as a weakly coupled system of equations with Laplacian principal part

$$(3.23) \quad -\Delta U + QU = 0,$$

where  $U$  is a vector with components the real and imaginary parts of  $u$ , that is,  $u^{(1)} = \Re u$ ,  $u^{(2)} = \Im u$ , and  $Q$  is a two by two tensor with elements the real and complex part of the potential  $q$ , that is,  $q^{(1)} = \Re q$  and  $q^{(2)} = \Im q$ . We can also write the system in the form

$$\begin{cases} -\Delta u^{(1)} + q^{(1)}u^{(1)} - q^{(2)}u^{(2)} & = 0, \\ -\Delta u^{(2)} + q^{(1)}u^{(2)} + q^{(2)}u^{(1)} & = 0. \end{cases}$$

In [1, Theorem 3.1] the authors prove the validity of the Three Spheres Inequality for elliptic systems with Laplacian principal part (see Theorem 3.1). In particular it applies to solutions  $U$  of (3.23) and hence also to solutions of  $(-\Delta + q)u = 0$ .



PROPOSITION 3.5. *Let  $u$  be a solution to the equation*

$$(-\Delta + q)u = 0 \quad \text{in } B_R.$$

*Then, for every  $\rho_1, \rho_2, \rho_3$ , with  $0 < \rho_1 < \rho_2 < \rho_3 \leq R$ ,*

$$(3.24) \quad \|u\|_{L^2(B_{\rho_2})} \leq Q_2 \|u\|_{L^2(B_{\rho_1})}^\alpha \|u\|_{L^2(B_{\rho_3})}^{1-\alpha},$$

*where  $\alpha = \frac{\ln \frac{\rho_3}{\rho_2}}{\ln \frac{\rho_3}{\rho_1}} \in (0, 1)$  and  $Q_2 \geq 1$  depends on  $\|q\|_{L^\infty(B_R)}$ ,  $\frac{\rho_2}{\rho_1}$  and  $\frac{\rho_3}{\rho_2}$ .*

Also, we have

COROLLARY 3.6. *Let  $u$  be a solution to the equation*

$$(-\Delta + q)u = 0 \quad \text{in } B_R.$$

*Then, for every  $\rho_1, \rho_2, \rho_3$ , with  $0 < \rho_1 < \rho_2 < \rho_3 \leq R$ ,*

$$(3.25) \quad \|u\|_{L^\infty(B_{\rho_2})} \leq Q_\infty \|u\|_{L^\infty(B_{\rho_1})}^\beta \|u\|_{L^\infty(B_{\rho_3})}^{1-\beta},$$

*where  $\beta = \frac{\ln \frac{2\rho_3}{\rho_2 + \rho_3}}{\ln \frac{\rho_3}{\rho_1}} \in (0, 1)$  and  $Q_\infty \geq 1$  depends on  $\|q\|_{L^\infty(B_R)}$ ,  $\frac{\rho_2}{\rho_1}$  and  $\frac{\rho_3}{\rho_2}$ .*

*Proof.* We use the local boundedness estimate for  $u^{(1)}$  and  $u^{(2)}$ , weak solutions of elliptic equations (see for instance [7, Theorem 8.17]), to obtain that there exists a constant  $C$ , which only depends on  $n$  and  $\|q\|_{L^\infty(B_R)}$ , such that

$$(3.26) \quad \|u\|_{L^\infty(B_{\rho_2})} \leq \frac{C}{(\rho_3 - \rho_2)^{n/2}} \|u\|_{L^2(B_{\rho_3})}.$$

Then, by Corollary 3.6,

$$(3.27) \quad \begin{aligned} \|u\|_{L^\infty(B_{\rho_2})} &\leq \frac{C}{\left(\frac{\rho_2 + \rho_3}{2} - \rho_2\right)^{n/2}} \|u\|_{L^2(B_{\frac{\rho_2 + \rho_3}{2}})} \\ &\leq \frac{CQ_2}{\left(\frac{\rho_2 + \rho_3}{2} - \rho_2\right)^{n/2}} \|u\|_{L^2(B_{\rho_1})}^\alpha \|u\|_{L^2(B_{\rho_3})}^{1-\alpha} \\ &\leq \frac{CQ_2}{\left(\frac{\rho_2 + \rho_3}{2} - \rho_2\right)^{n/2}} |B_{\rho_1}|^{\alpha/2} |B_{\rho_3}|^{(1-\alpha)/2} \|u\|_{L^\infty(B_{\rho_1})}^\alpha \|u\|_{L^\infty(B_{\rho_3})}^{1-\alpha}. \end{aligned}$$

□

As a consequence of the Three Spheres Inequality stated in Corollary 3.6, we derive the following quantitative estimate for unique continuation of solutions to our equation.

PROPOSITION 3.7. *Let  $K$  and  $K_0$  be defined as before, and let  $v \in H^1(K)$  be a weak solution to the equation*

$$(-\Delta + q(x))v = 0 \quad \text{in } K.$$

*Assume that, for given positive numbers  $\varepsilon_0$  and  $E_0$ ,  $v$  satisfies*

$$(3.28) \quad \|v\|_{L^\infty(K_0)} \leq \varepsilon_0,$$

and

$$(3.29) \quad |v(x)| \leq (\varepsilon_0 + E_0) \operatorname{dist}(x, \Sigma_{M+1})^{1-\frac{n}{2}}, \quad x \in K.$$

Then the following inequality holds true for every  $0 < r < 2r_1$ ,

$$(3.30) \quad |v(\tilde{x})| \leq C \left( \frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E_0) r^{(1-\frac{n}{2})(1-\tau_r)},$$

where  $\tilde{x} = P_{M+1} - r\nu(P_{M+1})$  with  $\nu$  being the exterior unit normal vector to  $\Sigma_{M+1}$  at  $P_{M+1}$ ,  $\beta = \frac{\ln(8/7)}{\ln 4}$ ,  $\tau_r = \frac{\ln\left(\frac{12r_1-2r}{12r_1-3r}\right)}{\ln\left(\frac{6r_1-r}{2r_1}\right)} \in (0, 1)$  and the constants  $N_1$  and  $C$  depend on  $r_0, L, A, B$  and  $n$ .

*Proof.* We construct a chain of spheres of radius  $r_1$  with centers  $x_0, x_1, \dots, x_k$  such that the first is  $B_{r_1}(x_0) \subset B_{4r_1}(x_0) \subset K_0$ , all the spheres are externally tangent, and the last one is centered at  $x_k = P_{M+1} - 3r_1\nu(P_{M+1})$ . We choose this chain so that the spheres of radius  $4r_1$  concentric with those of the chain, except the last one, are contained in  $K$  and have a distance greater than  $r_1$  away from  $\Sigma_{M+1}$ . Such a chain has a finite number of spheres that is smaller than  $N_1 = \frac{A}{|B_{r_1}|} + 1$ .

By Corollary 3.6 and (3.29), we have

$$\begin{aligned} \|v\|_{L^\infty(B_{r_1}(x_1))} &\leq \|v\|_{L^\infty(B_{3r_1}(x_0))} \\ &\leq Q_\infty \|v\|_{L^\infty(B_{r_1}(x_0))}^\beta \|v\|_{L^\infty(B_{4r_1}(x_0))}^{1-\beta} \\ &\leq C \left( \frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^\beta (\varepsilon_0 + E_0), \end{aligned}$$

where  $C$  depends on  $Q_\infty$  and  $r_1$ . By iterated application of Corollary 3.6 to  $v$  with radii  $r_1, 3r_1$  and  $4r_1$  over the chain of spheres, we have, by (3.28),

$$\begin{aligned} \|v\|_{L^\infty(B_{r_1}(x_k))} &\leq Q_\infty \|v\|_{L^\infty(B_{r_1}(x_{k-1}))}^\beta \|v\|_{L^\infty(B_{4r_1}(x_{k-1}))}^{1-\beta} \\ &\leq C \left( \frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\beta^{N_1}} (\varepsilon_0 + E_0), \end{aligned}$$

where  $C$  depends on  $Q_\infty$  and  $r_1$ . Now, we let  $\tilde{x} = P_{M+1} - r\nu(P_{M+1})$  where  $r < 2r_1$ . Using Corollary 3.6 again for spheres centered at  $x_k$  of radii  $r_1, 3r_1 - r$  and  $3r_1 - \frac{r}{2}$ , we obtain that

$$\begin{aligned} \|v\|_{L^\infty(B_{3r_1-r}(x_k))} &\leq Q_\infty \|v\|_{L^\infty(B_{r_1}(x_k))}^{\tau_r} \|v\|_{L^\infty(B_{3r_1-\frac{r}{2}}(x_k))}^{1-\tau_r} \\ &\leq C \left( \frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E_0) r^{(1-\frac{n}{2})(1-\tau_r)}, \end{aligned}$$

which completes the proof.  $\square$

**4. Proof of the main result.** Assume that  $D_{M+1}$  is the subdomain of the partition of  $\Omega$  where the maximum of  $\|q^{(1)} - q^{(2)}\|$  is realized and let us denote

$$(4.1) \quad E = \|q^{(1)} - q^{(2)}\|_{L^\infty(D_M)} = \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)}.$$

We consider the chain of domains,  $D_0, D_1, \dots, D_M$ , as before;  $S, K$  and  $K_0$  are defined as in the previous section. We set

$$U_0 = \Omega, \quad U_k = \Omega \setminus \cup_{j=1}^k D_j, \quad k = 1, \dots, M \text{ and } W_k = \cup_{j=0}^k D_j.$$

Let  $y \in K$ . For dimension  $n \geq 4$ , let  $G_1(x, y)$  and  $G_2(x, y)$  be the Green's function related to  $q^{(1)}$  and  $q^{(2)}$ , respectively, the existence and behavior of which was shown in Proposition 3.2. For

dimension  $n = 2, 3$ , let  $G_1(x, y)$  and  $G_2(x, y)$  be a first order derivative of the Green's function, the existence and behavior of which was shown in Propositions 3.4 and 3.3, respectively. We define

$$(4.2) \quad S_k(y, z) = \int_{U_k} (q^{(1)} - q^{(2)})(x)G_1(x, y)G_2(x, z) dx.$$

We focus on  $n \geq 3$  first; we will discuss the adaptation of the proof for the case  $n = 2$  at the end of the proof. By Proposition 3.2 and 3.3, there exist a constant  $C$  such that

$$(4.3) \quad |S_k(y, z)| \leq CE (\text{dist}(y, U_k) \text{dist}(z, U_k))^{1-\frac{n}{2}}, \quad y, z \in K \cap W_k.$$

LEMMA 4.1. *For every  $y, z \in K \cap W_k$ , we have  $S_k(\cdot, z), S_k(y, \cdot) \in H^1(K \cap W_k)$  and*

$$(4.4) \quad (-\Delta + q^{(1)})S_k(\cdot, z) = 0, \quad (-\Delta + q^{(2)})S_k(y, \cdot) = 0 \quad \text{in } K \cap W_k.$$

The proof of this Lemma follows from the symmetry of  $G_i$  ( $i = 1, 2$ ) and changing the order of integration and differentiation.

LEMMA 4.2. *If for some  $\varepsilon_0 > 0$  and  $k \in \{1, \dots, M - 1\}$  we have that*

$$(4.5) \quad |S_k(y, z)| \leq \varepsilon_0, \quad \forall y, z \in K_0,$$

then

$$(4.6) \quad |S_k(y_r, y_r)| \leq C \left( \frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E) r^{2-n},$$

where  $y_r = P_{k+1} - r\nu(P_{k+1})$ ,  $r$  is small,  $\nu(P_{k+1})$  is the outer unit normal vector to  $\partial D_k$  at  $P_{k+1}$  and the positive constant  $C$  depends on  $r_0, L, A, B$  and  $n$ .

*Proof.* We fix  $z \in K_0$  first and consider  $v(y) = S_k(y, z)$ . By Lemma 4.1,  $v$  solves the equation  $(-\Delta + q^{(1)})v = 0$  in  $K \cap W_k$ . Moreover, by (3.5), we have

$$(4.7) \quad |v(y)| \leq CE \text{dist}(y, \Sigma_{k+1})^{1-\frac{n}{2}}, \quad y \in K \cap W_k.$$

Then, by Proposition 3.7, we have, for  $0 < r < 2r_1$ ,

$$(4.8) \quad |S_k(y_r, z)| \leq C \left( \frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E) r^{1-\frac{n}{2}}.$$

Next, we consider

$$(4.9) \quad \tilde{v}(z) = S_k(y_r, z), \quad z \in K \cap W_k,$$

which solves the equation  $(-\Delta + q^{(2)})\tilde{v} = 0$  in  $K \cap W_k$ , and, by (3.5), satisfies

$$(4.10) \quad |\tilde{v}(z)| \leq CE (r \text{dist}(z, \Sigma_{k+1}))^{1-\frac{n}{2}}, \quad z \in K \cap W_k.$$

By Proposition 3.7, again, we then obtain estimate (4.6).  $\square$

*Proof of Theorem 2.7.* Let

$$\varepsilon = \|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})}$$

and

$$\delta_k = \|q^{(1)} - q^{(2)}\|_{L^\infty(W_k)}, \quad k = 0, 1, \dots, M.$$

From the Alessandrini identity (see for instance, Chapter 5 of [8])

$$(4.11) \quad \int_{\Omega} (q^{(1)} - q^{(2)})(x) G_1(x, y) G_2(x, z) dx = \langle (\Lambda_1 - \Lambda_2) G_1(\cdot, y), \overline{G}_2(\cdot, z) \rangle, \quad \forall y, z \in K_0$$

and Proposition 3.2, we find that

$$(4.12) \quad |S_{k-1}(y, z)| \leq C(\varepsilon + \delta_{k-1}).$$

Let  $P_k \in \Sigma_k$  and  $y_r = z_r = P_k - r\nu(P_k)$ , where  $\nu(P_k)$  is the exterior normal vector to  $\Sigma_k$  and  $r$  is small. We write

$$(4.13) \quad S_{k-1}(y_r, y_r) = I_1 + I_2$$

with

$$(4.14) \quad I_1 = \int_{B_{\rho_0}(P_k) \cap D_k} (q^{(1)} - q^{(2)})(x) G_1(x, y_r) G_2(x, y_r) dx$$

and

$$(4.15) \quad I_2 = \int_{U_{k-1} \setminus (B_{\rho_0}(P_k) \cap D_k)} (q^{(1)} - q^{(2)})(x) G_1(x, y_r) G_2(x, y_r) dx,$$

where  $\rho_0 = \frac{r_0}{6}$ .

For  $n \geq 3$ , by Proposition 3.2 and 3.3, we have

$$(4.16) \quad |I_2| \leq C E.$$

We estimate  $I_1$  as follows:

$$\begin{aligned} |I_1| &= |q_k^{(1)} - q_k^{(2)}| \left| \int_{B_{\rho_0}(P_k) \cap D_k} G_1(x, y_r) G_2(x, y_r) dx \right| \\ &\geq |q_k^{(1)} - q_k^{(2)}| \left\{ \left| \int_{B_{\rho_0}(P_k) \cap D_k} \Gamma(x, y_r) \Gamma(x, y_r) dx \right| \right. \\ &\quad - \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_1(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\ &\quad - \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_2(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\ &\quad \left. - \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_1(x, y_r) - \Gamma(x, y_r)) (G_2(x, y_r) - \Gamma(x, y_r)) dx \right| \right\}. \end{aligned}$$

By Propositions 3.2 and 3.3 and the fact that

$$\begin{aligned} &\left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_i(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\ &\leq \frac{1}{2} \int_{B_{\rho_0}(P_k) \cap D_k} \left( 2|G_i(x, y_r) - \Gamma(x, y_r)|^2 + \frac{1}{2} |\Gamma(x, y_r)|^2 \right) dx, \quad i = 1, 2 \end{aligned}$$

and

$$\left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_1(x, y_r) - \Gamma(x, y_r))(G_2(x, y_r) - \Gamma(x, y_r)) dx \right\} \\ \leq \frac{1}{2} \int_{B_{\rho_0}(P_k) \cap D_k} (|G_1(x, y_r) - \Gamma(x, y_r)|^2 + |G_2(x, y_r) - \Gamma(x, y_r)|^2) dx,$$

we obtain that

$$|I_1| \geq |q_k^{(1)} - q_k^{(2)}| \left( \frac{1}{2} \int_{B_{\rho_0}(P_k) \cap D_k} |\Gamma(x, y_r)|^2 dx - C \right).$$

Using the explicit form of  $\Gamma(x, y)$ , we find that

$$(4.17) \quad \begin{aligned} |I_1| &\geq |q_k^{(1)} - q_k^{(2)}| (Cr^{2-n} - C) \\ &\geq C|q_k^{(1)} - q_k^{(2)}| r^{2-n} - CE. \end{aligned}$$

Now, by Lemma 4.2 and (4.12), we have

$$|S_{k-1}(y_r, y_r)| \leq C \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon + \delta_{k-1} + E) r^{2-n}.$$

Hence, using (4.13), (4.16) and (4.17), we have

$$|q_k^{(1)} - q_k^{(2)}| r^{2-n} \leq C \left( E + \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon + \delta_{k-1} + E) r^{2-n} \right),$$

so that

$$(4.18) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left( \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} + r^{n-2} \right).$$

Noting that

$$\tau_r = \frac{\ln \left( \frac{12r_1 - 2r}{12r_1 - 3r} \right)}{\ln \left( \frac{6r_1 - r}{2r_1} \right)}, \quad \forall r \in (0, 2r_1)$$

implies

$$\frac{\tau_r}{r} \geq \frac{1}{12r_1 \ln 3}, \quad \forall r \in (0, 2r_1)$$

we get

$$(4.19) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left( \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\beta^{2N_1} (12r_1 \ln 3)^{-2} r^2} + r^{n-2} \right).$$

By taking  $r = \left| \ln \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-1/4}$  and noting that

$$\left( e^{-r^{-4}} \right)^{\beta^{2N_1} (12r_1 \ln 3)^{-2} r^2} \leq Cr^{n-2}, \quad \forall r > 0$$

for some constant  $C$ , we obtain that

$$(4.20) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left| \ln \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-(n-2)/4}.$$

We let

$$\omega(t) = \begin{cases} |\ln t|^{-(n-2)/4}, & 0 < t < e^{-n}, \\ n^{-(n-2)/4}, & t \geq e^{-n}. \end{cases}$$

Noting that the function  $t \mapsto t\omega(1/t)$  is increasing, we have

$$\frac{\varepsilon + \delta_{k-1} + E}{\varepsilon + \delta_{k-1}} \omega \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \geq \omega(1),$$

hence

$$\delta_{k-1} \leq \varepsilon + \delta_{k-1} \leq (\omega(1))^{-1}(\varepsilon + \delta_{k-1} + E) \omega \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right),$$

which with (4.20) gives that

$$(4.21) \quad \delta_k \leq C(\varepsilon + \delta_{k-1} + E) \omega \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right).$$

The above choice of  $r$  is possible only if

$$\left| \ln \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-1/4} < 2r_1.$$

However, if

$$\left| \ln \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-1/4} \geq 2r_1,$$

that is,

$$\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \geq e^{-(2r_1)^{-4}},$$

the fact that

$$\sup_{\substack{r \in (0, 2r_1) \\ t \in (e^{-(2r_1)^{-4}}, 1)}} t^{\beta^{2N_1}(12r_1 \ln 3)^{-2}r^2} |\ln t|^{\frac{n-2}{4}}$$

is finite shows that (4.20) still holds true, then (4.21) follows.

We iterate (4.21), starting from  $\delta_0 = 0$ , and find

$$(4.22) \quad \delta_k + \varepsilon \leq (C + n^{(n-2)/4})^k (E + \varepsilon) \omega_k \left( \frac{\varepsilon}{\varepsilon + E} \right),$$

where  $\omega_k$  is the composition of  $\omega$   $k$  times with itself. We recall that  $E = \delta_M$ , whence,

$$(4.23) \quad E + \varepsilon \leq (C + n^{(n-2)/4})^M (E + \varepsilon) \omega_M \left( \frac{\varepsilon}{\varepsilon + E} \right),$$

so that

$$(4.24) \quad E \leq \frac{1 - \omega_M^{-1}((C + n^{(n-2)/4})^{-M})}{\omega_M^{-1}((C + n^{(n-2)/4})^{-M})} \varepsilon,$$

which completes the proof for dimension  $n \geq 3$ .

The proof for  $n = 2$  follows from a careful inspection and adaptation of the above proof for  $n \geq 3$ . By Proposition 3.4, instead of Proposition 3.3, and the explicit form of  $\Gamma(x, y) = -\frac{2\pi(x_2 - y_2)}{|x - y|^2}$ , we obtain that

$$(4.25) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left( \left( \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} + |\ln r|^{-1} \right).$$

Then, by taking  $r = \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E}$  and adapting the function  $\omega(t)$  according to

$$\tilde{\omega}(t) = \begin{cases} |\ln t|^{-1}, & 0 < t < e^{-2}, \\ \frac{1}{2}, & t \geq e^{-2}, \end{cases}$$

we end up with

$$(4.26) \quad E \leq \frac{1 - \tilde{\omega}_M^{-1}((C + 2)^{-M})}{\tilde{\omega}_M^{-1}((C + 2)^{-M})} \varepsilon,$$

which completes the proof.  $\square$

**5. Exponential behavior of the Lipschitz stability constant.** In this section, we give a model example to show that the Lipschitz stability constant  $C = C(n, r_0, L, A, N)$  in Theorem 2.7 behaves exponentially with respect to the number  $N$  of the subdomains. The construction is an analogue of the construction in [10], pertaining to the inverse conductivity problem.

Let  $\Omega$  be the unit ball  $B_1(0) \subset \mathbb{R}^n$  and  $D = [-1/2, 1/2]^n$  be the cube of side 1 centered at the origin. We define the class of admissible potentials by

$$(5.1) \quad \mathcal{A} = \{q \in L^\infty(\Omega) \mid 1/2 \leq q \leq 3/2 \text{ in } \Omega \text{ and } q = 1 \text{ in } \Omega \setminus D\}$$

and denote the operator from potential  $q$  to  $\Lambda_q$  by  $F$ , which maps  $\mathcal{A}$  into  $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ . We fix a positive integer  $N$  and let  $N_1$  be the smallest integer such that  $N \leq N_1^n$ . We divide each side of the cube  $D$  into  $N_1$  equal parts of length  $h = 1/N_1$  and let  $S_{N_1}$  be the set of all open cubes of the type

$$D' = (-1/2 + (j'_1 - 1)h, -1/2 + j'_1 h) \times \cdots \times (-1/2 + (j'_n - 1)h, -1/2 + j'_n h),$$

where  $j'_1, \dots, j'_n$  are integers belonging to  $\{1, \dots, N_1\}$ . We order such cubes as follows. For any two different cubes  $D'$  and  $D''$  belonging to  $S_{N_1}$ , we say that  $D' \prec D''$  if and only if there exists an  $i_0 \in \{1, \dots, n\}$  such that  $j'_i = j''_i$  for any  $i < i_0$  and  $j'_{i_0} < j''_{i_0}$ . We define

$$\mathcal{A}_N = \{q \in L^\infty(\Omega) \mid q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x) + \chi_{D_0}(x), \quad q_j \in [1/2, 3/2]\}.$$

Our aim is to estimate from below the Lipschitz constant  $C(N)$  in terms of  $N$ . A simple computation shows polynomial behavior of the lower bound estimate of  $C(N)$ . To obtain the exponential estimate, we then need to employ a topological argument.

Consider a subset  $\tilde{\mathcal{A}}_N \subset \mathcal{A}_N$  defined by

$$\tilde{\mathcal{A}}_N = \left\{ q \in L^\infty(\Omega) \mid q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x) + \chi_{D_0}(x), \quad q_j \in \left\{ \frac{1}{2}, 1, \frac{3}{2} \right\} \right\}.$$

It is easy to check that  $\tilde{\mathcal{A}}_N$  is a  $1/2$ -net of  $\mathcal{A}_N$  with  $3^N$  elements and, for any two different  $q_1, q_2 \in \tilde{\mathcal{A}}_N$ , we have  $\|q_1 - q_2\|_{L^\infty(\Omega)} = 1/2$ . Based on Mandache's result [9, Lemma 3], there exist a constant  $K$ , which only depends on dimension  $n$ , such that for every  $\varepsilon \in (0, e^{-1})$ , there is an  $\varepsilon$ -net  $Y$  for  $F(\mathcal{A})$  with at most  $e^{K(-\ln \varepsilon)^{2n-1}}$  elements. For  $\varepsilon \in (0, e^{-1})$  and  $N \in \mathbb{N}$  let

$$Q(\varepsilon, N) = e^{K(-\ln \varepsilon)^{2n-1}}.$$

Note that

$$3^N > e^{K(-\ln \varepsilon)^{2n-1}}$$

if

$$\varepsilon > e^{-K_1 N^{1/(2n-1)}} = \varepsilon_0(N)$$

where  $K_1 = (K^{-1} \ln 3)^{1/(2n-1)}$ . There exists  $N_0$  such that for  $N \geq N_0$  we have that  $\varepsilon < e^{-1}$ . Thus, for  $N \geq N_0$ , if we take  $\varepsilon = \varepsilon_0$  we have  $3^N > Q(\varepsilon, N)$ . Then, there exist two different  $q_1, q_2 \in \tilde{\mathcal{A}}_N$  such that  $\|q_1 - q_2\|_{L^\infty(\Omega)} = 1/2$  with their images under  $F$  in the same ball of radius  $\varepsilon$  centered at a point of  $Y$ , that is,

$$\frac{1}{2} = \|q_1 - q_2\|_{L^\infty(\Omega)} \leq C_N \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \leq 2C_N \varepsilon_0(N)$$

from which we get

$$C(N) \geq \frac{1}{4} e^{K_1 N^{1/(2n-1)}}.$$

**Acknowledgements.** This paper was initialized at a Special semester on Inverse Problems and Applications at MSRI, Berkeley, in the Fall of 2010. The work of E. Beretta was partially supported by MIUR grant PRIN 20089PWTPS003. The research of M. de Hoop and L. Qiu was supported in part by National Science Foundation grant CMG DMS-1025318, and in part by the members of the Geo-Mathematical Imaging Group at Purdue University.

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