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RECONSTRUCTION OF THE METRIC OF A RIEMANNIAN MANIFOLD FROM LOCAL BOUNDARY DIFFRACTION TRAVEL TIMES II. A DIX-TYPE ALGORITHM *

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Abstract. We consider a region M in $\mathbb{R}^n$ with boundary ∂M and a metric g on M conformal to the Euclidean metric. We analyze the inverse problem, originally formulated by Dix [7], of reconstructing g from boundary measurements associated with the single scattering of seismic waves in this region. In our formulation the measurements determine the shape operator in a subset of an extension of M which does not intersect M. We develop an explicit reconstruction procedure which consists of two steps: In the first step we reconstruct the directional curvatures and the metric in what are essentially Riemannian normal coordinates; in the second step we develop a conversion to Cartesian coordinates. We admit the presence of conjugate points. In dimension $n \geq 3$ both steps involve the solution of a system of ordinary differential equations. In dimension $n = 2$ the same is true for the first step, but the second step requires the solution of a Cauchy problem for an elliptic operator which is unstable in general.

Key words. geometric inverse problems, Riemannian manifold, shape operator

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1. Introduction. We consider a region, M, in $\mathbb{R}^n$ with a smooth boundary ∂M. We assume that there is a Riemannian metric, g, on M that is conformal to the Euclidean metric with conformal factor v^{-2} where $v \in C^\infty(\overline{M})$ is strictly positive. This means that $g(x) = v^{-2}\mathbf{e}$, where $\mathbf{e}$ is the Euclidean metric, or, in Cartesian coordinates $x = (x^1, \ldots, x^n)$, $g^{ij}(x) = v(x)^2\delta^{ij}$. We analyze the inverse problem, originally formulated by Dix [7], aiming at reconstructing g (or equivalently v) from boundary measurements associated with the single scattering (reflections) of seismic waves in M. Geodesics are rays following the propagation of singularities by a parametrix corresponding for the wave operator on (M, g). In the seismic context v is the wavespeed. Dix developed a procedure, with a formula, for reconstructing one-dimensional wavespeed profiles in a half space, which we generalize here to the case of higher dimensional regions with Riemannian metrics conformal to the Euclidean metric. The method is different in the cases of $n = 2$ and $n \geq 3$ and in fact we expect better results in the case $n \geq 3$.

Assuming that we know v and all of its derivatives on ∂M, we may extend v to a complete manifold, $\widetilde{M}$, with Riemannian metric $\widetilde{g}$, compactly containing M, so that $\widetilde{g}|_M = g$. We denote, for simplicity, $\widetilde{g} = g$. (Then we are also able to relate measurements on ∂M with measurements in $\widetilde{M} \setminus M$.) As described below, we measure the curvature of the intersection of certain geodesic spheres centered at "diffraction" points with an open subset of $\widetilde{M} \setminus M$. From these data we show an explicit method to determine the function v in the Cartesian coordinates $x = (x^1, \ldots, x^n)$ along geodesics of g which connect the diffractions points to the measurement region. This method can we viewed as a generalization of the work of Iversen and Tygel [12]. Since Dix, various adaptions have been considered to admit more general wavespeed functions in a half space. We mention the work of Shah [18], Hubral & Krey [11], Dubose, Jr. [8], and Mann [15].

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For the moment, we fix $(x_0, \eta_0) \in \Omega(\widetilde{M} \setminus M)$ (Ω indicates the unit sphere bundle with respect to g), and consider the geodesic γ_{x_0,η_0} with initial data, $\gamma_{x_0,\eta_0}(0) = x_0$, $\dot\gamma_{x_0,\eta_0}(0) = \eta_0$. For r in the domain of γ_{x_0,η_0}, we let $\mathcal{C}_r$ denote the set of times t such that $\gamma_{x_0,\eta_0}(r)$ is conjugate to $\gamma_{x_0,\eta_0}(t)$ along γ_{x_0,η_0}. When $t > r$ is in the domain of γ_{x_0,η_0} and $t \notin \mathcal{C}_r$ there is a small portion of the geodesic sphere of radius $t - r$ centered at $y_t = \gamma_{x_0,\eta_0}(t)$ that contains $\gamma_{x_0,\eta_0}(r)$ and is an embedded submanifold $\Sigma_{r,t}$ of $\widetilde{M}$. Indeed, in this case we can define a vector field $\nu_{r,t}$ in a neighborhood of $\gamma_{x_0,\eta_0}(r)$ by writing for $\xi \in T_{y_t}\widetilde{M}$ in a small neighborhood of $-(t-r)\dot\gamma_{x_0,\eta_0}(t)$:

$$\nu_{r,t}(\exp_{y_t}(\xi)) = \frac{1}{|\xi|_g} \left.\frac{\partial}{\partial s}\right|_{s=1} \exp_{y_t}(s\xi).$$

Geometrically, $\nu_{r,t}$ gives the outward pointing normal vector fields to a part of the geodesic spheres centered at y_t near $\gamma_{x_0,\eta_0}(r)$. The shape operator $S_{r,t} \in (T^1_1)_{\gamma_{x_0,\eta_0}(r)}\widetilde{M}$ of $\Sigma_{r,t}$ at $\gamma_{x_0,\eta_0}(r)$ is then given by

$$S_{r,t}X = \nabla_X \nu_{r,t}$$

for all $X \in T_{\gamma_{x_0,\eta_0}(r)}\widetilde{M}$, where ∇ is the Levi-Civita connection for g. For the reconstruction of v we assume that $S_{0,t}$ is known for all $t > 0$ such that $t \notin \mathcal{C}_0$. In reflection seismology one refers to $\Sigma_{r,t}$ as the (partial) front of a point diffractor located at y_t.

We now introduce a set of coordinates in which we will perform our initial calculations. We begin with picking a large $t_0 > 0$ in the domain of γ_{x_0,η_0} such that $t_0 \notin \mathcal{C}_0$. Next, let us take arbitrary coordinates $\widehat{x} = (\widehat{x}^1, \ldots, \widehat{x}^{n-1})$ on Σ_{0,t_0} such that $\widehat{x} = 0$ defines x_0. We let $\gamma_{\widehat{x}}$ be the geodesic with a special choice of initial data: $\dot\gamma_{\widehat{x}}(0) = -\nu_{0,t_0}(\widehat{x})$. Then we define coordinates, $(\widehat{x}, r)$, on some set W by the inverse of the map

$$(\widehat{x}, r) \mapsto \gamma_{\widehat{x}}(r);$$

sometimes we use the notation $\widehat{x}^n = r$. The $(\widehat{x}, r)$ coordinates, basically, are Riemannian normal coordinates centered at y_{t_0}, but parametrized in a particular way: $\widehat{x}$ can be thought of as a parametrization of part of the geodesic sphere of radius t_0 in $T_{y_{t_0}}\widetilde{M}$, and then r corresponds to the radial variable in $T_{y_{t_0}}\widetilde{M}$. We note that the domain W of these coordinates includes $\gamma_{x_0,\eta_0}([0,t_0)) \setminus \{\gamma_{x_0,\eta_0}(r) : r \in \mathcal{C}_{t_0}\}$. Also, we note that along γ_{x_0,η_0} the coordinate vectors are Jacobi fields, and are defined even at the conjugate points. Finally, we introduce frames $\{F_j(\widehat{x}, r)\}_{j=1}^n$ defined by parallel translation along $\gamma_{\widehat{x}}$ such that

$$F_j(\widehat{x}, 0) = \left.\frac{\partial}{\partial \widehat{x}^j}\right|_{(\widehat{x},0)},$$

and also write $\{f^j(\widehat{x}, r)\}_{j=1}^n$ for the corresponding dual frame; that is[1]

$$\langle f^j(\widehat{x}, r), F_k(\widehat{x}, r)\rangle = \delta^j_k.$$

Thus,

$$F_n(\widehat{x}, 0) = \dot\gamma_{\widehat{x}}(0)$$

points towards the interior of M.

In the sequel will also consider the shape operators $S_{r,t}$ when x_0 is replaced in the above construction by another point in Σ_{0,t_0} represented in the coordinates by $\widehat{x}$. We thus have for each

[1]Here, $\langle ., . \rangle$ denotes the usual pairing of $T_{\gamma_{\widehat{x}}(r)}\widetilde{M}$ and $T^*_{\gamma_{\widehat{x}}(r)}\widetilde{M}$.

$\widehat{x}$ and $0 \leq r < t \leq t_0$ such that $\gamma_{\widehat{x}}(r)$ and $\gamma_{\widehat{x}}(t)$ are not conjugate along $\gamma_{\widehat{x}}$ a tensor $S_{r,t}(\widehat{x}) \in (T_1^1)_{\gamma_{\widehat{x}}(r)}\widetilde{M}$. We represent $S_{r,t}(\widehat{x})$ using the frames constructed above as

$$S_{r,t}(\widehat{x}) = \mathbf{s}_j^k(\widehat{x}, r, t) f^j(\widehat{x}, r) \otimes F_k(\widehat{x}, r).$$

We will also we the notation $\mathbf{s}_j^k(r,t) = \mathbf{s}_j^k(0,r,t)$. Note that immediately from the definition we have $\mathbf{s}_j^n(\widehat{x}, r, t) = \mathbf{s}_n^j(\widehat{x}, r, t) = 0$ for all j and because of this in what follows when we write $\mathbf{s}(\widehat{x}, r, t)$ (respectively $\mathbf{s}(r,t)$) without indices we will actually be referring to the $(n-1) \times (n-1)$ matrix $\mathbf{s}_j^k(\widehat{x}, r, t)$ (respectively $\mathbf{s}_j^k(r,t)$) with $j,\, k = 1, \ldots, n-1$. The data for our recovery are the matrix elements $\mathbf{s}_j^k(\widehat{x}, 0, t)$, and their first three derivatives with respect to t, for $0 < t < t_0$ and $\gamma_{\widehat{x}}(t)$ not conjugate to $\gamma_{\widehat{x}}(0)$ along $\gamma_{\widehat{x}}$.

In a companion paper, we obtained the following result: We can uniquely determine the Riemannian metric in a neighborhood of $\gamma_{x_0,\eta_0}([0,t_0))$ in Riemannian normal coordinates having the origin at the point y_{t_0}. Here, we cast this result into an algorithm, and construct a conversion from the mentioned coordinates to Cartesian coordinates, which is the main contribution of this paper. Essentially, we generalize the time-to-depth conversion in Dix' original method to multi-dimensional manifolds with Riemannian metrics conformal to the Euclidean metric. The construction comprises solving a system of $n + 3n^2 + n^3$ nonlinear differential equations. The discretization of this system is directly related to the available "density" of scatterers.

2. Preliminaries. We summarize the basic differential equations from Riemannian geometry that we will use. We mention some general references to Riemannian geometry [9, 16, 17]. In this work we will use the conventions from [16] for the curvature tensor and related quantities in local coordinates.

2.1. Geodesics. We evaluate the geodesics by solving

$$\frac{\mathrm{d}^2\gamma^i}{\mathrm{d}\,t^2} + \Gamma^i_{kl}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\mathrm{d}\gamma^l}{\mathrm{d}t} = 0, \tag{2.1}$$

where

$$\Gamma^i_{kl} = \frac{1}{2} g^{pi} \left(\frac{\partial g_{kp}}{\partial x^l} + \frac{\partial g_{lp}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^p} \right)$$

are the Christoffel symbols. It is also possible to find the solutions of (2.1) using the Hamiltonian flow for the Hamiltonian $H(x,p) = \frac{1}{2} p_j p_k g^{jk}(x)$. Although we will not use the Hamiltonian formulation here, we note that it gives the system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\gamma^i = g^{ij} p_j \\ \frac{\mathrm{d}}{\mathrm{d}t} p_i = -\frac{1}{2} p_j p_k \frac{\partial g^{jk}}{\partial x^i} \end{cases}$$

for the geodesics. From this, we see that, in terms of seismic ray tracing, the geodesics may be identified with generalized image rays.

In our case, assuming isotropy, we have

$$\Gamma^l_{qm} = -\left(\delta^l_q \delta^k_m + \delta^l_m \delta^k_q - \delta_{qm}\delta^{kl}\right)\frac{\partial f}{\partial x^k}, \quad f = \log(v).$$

To make the notation more concise below we introduce the shorthand

$$\Theta^{lk}_{qm} = \delta^l_q \delta^k_m + \delta^l_m \delta^k_q - \delta_{qm}\delta^{kl}.$$

2.2. A frame, parallel transport. As mentioned above, $F_j(\widehat{x}, r)$ denotes the parallel translation of $\partial/\partial\widehat{x}^j$ along $\gamma_{\widehat{x}}$ from 0 to r. Thus, for every $\widehat{x}$ and $r > 0$, $\{F_1(\widehat{x}, r), \ldots, F_n(\widehat{x}, r)\}$ forms a basis for $T_{\gamma_{\widehat{x}}(r)}\widetilde{M}$. The invariant formula for parallel translation is

$$\nabla_{\dot{\gamma}_{\widehat{x}}(r)} F_k(\widehat{x}, r) = 0.$$

If we introduce matrices which give the frames $F_j(\widehat{x}, r)$ in Cartesian coordinates

$$F_j(\widehat{x}, r) = F_j^k(\widehat{x}, r)\, \left.\frac{\partial}{\partial x^k}\right|_{\gamma_{\widehat{x}}(r)}$$

then the invariant formula implies that these satisfy

$$\frac{\partial}{\partial r} F_j^l + \dot{\gamma}_{\widehat{x}}^k(r)\, \Gamma_{km}^l F_j^m = 0 \quad \text{or} \quad \frac{\partial}{\partial r} F_j^l(\widehat{x}, r) + F_n^k(\widehat{x}, r)\, \Gamma_{km}^l(\gamma_{\widehat{x}}(r)) F_j^m(\widehat{x}, r) = 0, \tag{2.2}$$

and since the coordinates $\widehat{x}$ on Σ_{0,t_0} are known, we also know the initial conditions, $F_p^l(\widehat{x}, 0)$.

2.3. Curvature. The Riemannian curvature tensor in any coordinate system is given as a $(1, 3)$ tensor by

$$R_{jkl}^p = \frac{\partial}{\partial x^j} \Gamma_{kl}^p - \frac{\partial}{\partial x^k} \Gamma_{jl}^p + \Gamma_{kl}^i \Gamma_{ji}^p - \Gamma_{jl}^i \Gamma_{ki}^p,$$

or as a $(0, 4)$ tensor as

$$g_{ip} R_{jkl}^i = R_{jklp}.$$

The Ricci curvature tensor is given by

$$Ric_{ij} = R_{kij}^k$$

and the scalar curvature is $scal = g^{ij} Ric_{ij}$. In the Cartesian coordinates the Riemmanian and Ricci curvature tensors are given respectively by the following formulae in terms of f:

$$R = e^{-2f} \left(\delta \odot \left(-\text{Hess}(f) - \nabla f \cdot \nabla f + \frac{1}{2} |\nabla f|^2 \delta \right) \right)$$

where $\odot$ is the Kulkarni-Nomizu product (in coordinates, see [16] for the invariant formula), Hess(f) is the Hessian matrix of second derivatives, ∇f is the (Euclidean) gradient, and $|\nabla f|$ is the Euclidean norm of ∇f;

$$Ric = (n-2)\, (\text{Hess}(f) + \nabla f \cdot \nabla f) + \delta_{ij} \left(\Delta f + (2-n)\, |\nabla f|^2 \right). \tag{2.3}$$

Also, when the dimension is $n = 2$ the scalar curvature satisfies the so-called scalar curvature equation

$$\Delta_g f = -2\, scal \tag{2.4}$$

where Δ_g is the Laplace-Beltrami operator corresponding to g given in any coordinate system $\{y^j\}_{j=1}^2$ by

$$\Delta_g = g(y)^{-1/2} \frac{\partial}{\partial y^j} g(y)^{1/2} g^{jk}(y) \frac{\partial}{\partial y^j}.$$

Here $g = \det([g_{jk}]_{j,k=1}^2)$ and $[g^{jk}]_{j,k=1}^2$ is the inverse matrix of $[g_{jk}]_{j,k=1}^2$. From these formulae we see a fundamental difference between the two dimensional case and the case of three or more dimensions.

In two dimensions the Ricci curvature and scalar curvature essentially give the Laplacian of f, and so to find f from these curvature tensors would require the solution of an elliptic equation. On the other hand, in three or more dimensions the Ricci curvature tensor depends on all the second partial derivatives of f, and in general we can recover a formula Hess(f) in terms of the Ricci curvature and ∇f. Indeed, if we define

$$G = Ric - (n-2)\left(\nabla f \cdot \nabla f - |\nabla f|^2 \delta\right) \tag{2.5}$$

then from (2.3) we may calculate

$$\mathrm{Hess}(f) = \frac{1}{n-2}\left(G - \frac{1}{2(n-1)}\mathrm{tr}\,(G)\,\delta\right). \tag{2.6}$$

This is possible in three dimensions, but not in two dimensions, and is the reason we must consider the two cases separately.

We will also write the Riemannian curvature on the geodesic $\gamma_{\widehat{x}}$ in the frame obtained by parallel transport as

$$\widehat{R}^p_{jkl}(\widehat{x}, r) F_p(\widehat{x}, r) = R_{\gamma_{\widehat{x}}(r)}(F_j(\widehat{x}, r), F_k(\widehat{x}, r)) F_l(\widehat{x}, r),$$

or

$$\widehat{R}^p_{jkl}(\widehat{x}, r) = \langle f^p(\widehat{x}, r), R_{\gamma_{\widehat{x}}(r)}(F_j(\widehat{x}, r), F_k(\widehat{x}, r)) F_l(\widehat{x}, r)\rangle.$$

Recalling that $F_n(\widehat{x}, r) = \dot{\gamma}_{\widehat{x}}(r)$, we also write

$$\mathbf{r}^p_j(\widehat{x}, r) = \widehat{R}^p_{jnn}(\widehat{x}, r),$$

or

$$\mathbf{r}^p_j(r) := \mathbf{r}^p_j(0, r) = \left\langle f^p(0, r), R_{\gamma_{\widehat{x}}(r)}(F_j(0, r), \dot{\gamma}_{x_0,\eta_0}(r)) \dot{\gamma}_{x_0,\eta_0}(r)\right\rangle, \tag{2.7}$$

for the directional curvature operator which we reconstruct in the first step of our procedure. Note that as with $\mathbf{s}$, for any j $\mathbf{r}^n_j(r) = \mathbf{r}^j_n(r) = 0$, and so when we write $\mathbf{r}$ without indices we will actually be referring to the corresponding $(n-1) \times (n-1)$ matrix.

We continue in the next sections to describe the actual reconstruction algorithm.

3. Reconstruction procedure – Step 1: Determination of the metric in $(\widehat{x}, r)$ coordinates. The reconstruction procedure consists of two steps. In the first step we consider only the single geodesic γ_{x_0,η_0} and reconstruct $\mathbf{r}^p_j(r)$, and then the metric g as a function of r for $\widehat{x} = 0$. In the second step, we determine v in W by also varying $\widehat{x}$.

Following the geometric analysis of the companion paper we now describe the first step of the procedure which is itself broken up into a number of substeps below.

1. Let $V^j = V^j(r, t)$, $j = 0, \dots, 3$ represent $(n-1)\times(n-1)$ matrices. We solve the autonomous system of ordinary differential equations for $V(r, t) = (V^j(r, t))^3_{j=0}$,

$$\begin{aligned}
\frac{\partial}{\partial r} V^0 &= -I + \frac{1}{2} V^0 (\mathcal{T} V^3) V^0, \\
\frac{\partial}{\partial r} V^1 &= \frac{1}{2}(V^1(\mathcal{T} V^3) V^0 + V^0(\mathcal{T} V^3) V^1), \\
\frac{\partial}{\partial r} V^2 &= \frac{1}{2}(V^2(\mathcal{T} V^3) V^0 + V^0(\mathcal{T} V^3) V^2 + 2V^1(\mathcal{T} V^3) V^1), \\
\frac{\partial}{\partial r} V^3 &= \frac{1}{2}(V^3(\mathcal{T} V^3) V^0 + V^0(\mathcal{T} V^3) V^3 + 3V^2(\mathcal{T} V^3) V^1 + 3V^1(\mathcal{T} V^3) V^2),
\end{aligned} \tag{3.1}$$

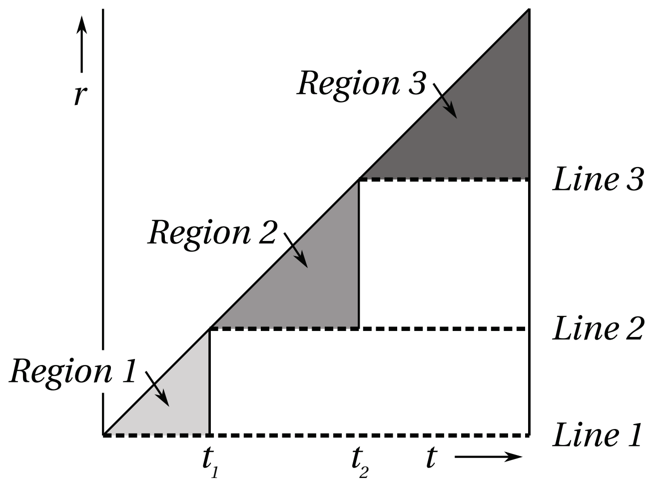

FIG. 1. *A depiction of how the first step of the algorithm proceeds. The original data give* $\mathbf{s}$ *on line 1. Then substep 1 recovers* $\{V^j\}_{j=1}^3$ *in region 1 by solving (3.1). Next substep 2 gives* $\mathbf{r}(r)$ *for* $0 \leq r \leq t_1$ *by (3.3), and then substep 3 gives* $\mathbf{s}$ *on line 2. Returning to substep 1 again gives* $\{V^j\}_{j=1}^3$ *in region 2. Then as before substep 2 gives* $\mathbf{r}(r)$ *for* $t_1 \leq r \leq t_2$*, and substep 3 gives* $\mathbf{s}$ *on line 3. Continuing in this way we reconstruct* $\mathbf{r}$ *as far along* γ_{x_0,η_0} *as we like.*

in which

$$(\mathcal{T}V^3)(r) = V^3(r,r),$$

for $0 \leq r \leq t \leq t_0$. This system is supplemented with initial data,

$$V^j(0,t) = \{\partial_t^j(\mathbf{s}(0,t))^{-1}\}_{j=0}^3. \tag{3.2}$$

We may use a Runge-Kutta method to solve the system numerically for $0 \leq r \leq t \leq t_1$ for some time t_1. The system will not generally have a solution all the way up to t_0; in this case, we must divide the interval $[0, t_0]$ into several subintervals, and reconstruct on each of these in turn as described below in substep 3.

2. We extract the directional curvature operator,

$$\mathbf{r}(r) = \frac{1}{2}(\mathcal{T}V)(r). \tag{3.3}$$

Note that this matrix $\mathbf{r}(r)$ is $(n-1) \times (n-1)$, but recall that from this we can recover the full directional curvature operator $\mathbf{r}_p^j(r)$ since the nth row and column of $\mathbf{r}_p^j(r)$ are both equal to zero.

3. In general the first two steps only reconstruct $\mathbf{r}_k^j(r)$ for $0 \leq r \leq t_1$ where $t_1 < t_0$ since we may not able to solve (3.1) all the way up to t_0. Here, we describe how to find $\mathbf{r}_k^j(r)$ for r in the entire interval $[0, t_0]$. The idea is to replace 0 by t_1, and then use knowledge of $\mathbf{s}_j^k(t_1, t)$ for $t > t_1$. Indeed, let us fix $t > t_1$. Now we find the coordinates of the Jacobi fields along γ_{x_0,η_0} that agree with $\{F_j(0,0)\}_{j=1}^n$ at x_0 and vanish at $\gamma_{x_0,\eta_0}(t)$ with respect to the parallel frame. These can be found by solving the system

$$\frac{\partial}{\partial r}\begin{pmatrix} \mathbf{j}_k^j(r,t) \\ \dot{\mathbf{j}}_k^j(r,t) \end{pmatrix} = \begin{pmatrix} 0 & \delta_p^j \\ -\mathbf{r}_p^j(r) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{j}_k^p(r,t) \\ \dot{\mathbf{j}}_k^p(r,t) \end{pmatrix}, \tag{3.4}$$

with initial conditions

$$\begin{pmatrix} \mathbf{j}_k^j(0,t) \\ \dot{\mathbf{j}}_k^j(0,t) \end{pmatrix} = \begin{pmatrix} \delta_k^j \\ -\mathbf{s}_k^j(0,t) \end{pmatrix}. \tag{3.5}$$

We can then recover $\mathbf{s}(t_1, t)$ by the equation

$$\frac{\partial \mathbf{j}}{\partial r}(t_1, t)\,(\mathbf{j}^{-1})(t_1, t) = -\mathbf{s}(t_1, t).$$

Now we may return to substep 1. and solve (3.1) with (3.2) replaced by

$$V^j(t_1, t) = \{\partial_t^j(\mathbf{s}(t_1, t))^{-1}\}_{j=0}^3.$$

This then allows us to use (3.3) to recover $\mathbf{r}(r)$ for r up another time t_2 say with $t_1 < t_2 \leq t_0$. If t_2 is less than t_0 we repeat this same procedure again with t_1 replaced by t_2, and so on. By results in the companion paper there is a lower bound on the size of each step we take (i.e. a lower bound on $t_i - t_{i-1}$), and so by induction we eventually recover $\mathbf{r}_k^j(r)$ for r on the entire interval $[0, t_0]$. For a visual depiction of how the first three substeps proceed see Figure 1.

4. We now obtain the metric in the $(\widehat{x}, r)$ coordinates along γ_{x_0,η_0}, which we write as $\widehat{g}_{jk}(0, r)$, by the formula

$$\widehat{g}_{jk}(0, r) = \mathbf{j}_j^p(r, t_0)\mathbf{j}_k^q(r, t_0)\,\mathring{g}_{pq}, \tag{3.6}$$

where $\mathring{g}_{pq} = g(F_p(0,0), F_q(0,0))$ which is known. By adjusting the choice of x_0 we also find the metric with respect to the $(\widehat{x}, r)$ coordinates where they are defined (i.e. on all of W).

4. Reconstruction procedure – Step 2: Transformation of coordinates. By the procedure described in the previous section we can reconstruct the metric $\widehat{g}_{jk}(\widehat{x}, r)$ in the coordinates $(\widehat{x}, r)$ everywhere in the domain W of those coordinates. Thus we can also reconstruct the Ricci curvature tensor in these coordinates and also the scalar curvature, which we will use below. In this section, we show how to determine the velocity function $v(\gamma_{\widehat{x}}(r))$ and the geodesics $\gamma_{\widehat{x}}(r)$ in the Cartesian coordinates from this information.

As observed above, the coordinates vectors in the $(\widehat{x}, r)$ coordinates are Jacobi fields along $\gamma_{\widehat{x}}$; in particular, if

$$\left.\frac{\partial}{\partial \widehat{x}^k}\right|_{\gamma_{\widehat{x}}(r)} = \mathbf{j}_k^l(\widehat{x}, r)F_l(\widehat{x}, r) = \mathbf{j}_k^l(\widehat{x}, r)F_l^p(\widehat{x}, r)\,\left.\frac{\partial}{\partial x^p}\right|_{\gamma_{\widehat{x}}(r)},$$

then the matrix $\mathbf{j}_k^l(\widehat{x}, r)$ satisfies

$$\frac{\partial}{\partial r}\begin{pmatrix} \mathbf{j}_k^l(\widehat{x}, r) \\ \dot{\mathbf{j}}_k^l(\widehat{x}, r) \end{pmatrix} = \begin{pmatrix} 0 & \delta_p^l \\ -\mathbf{r}_p^l(\widehat{x}, r) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{j}_k^p(\widehat{x}, r) \\ \dot{\mathbf{j}}_k^p(\widehat{x}, r) \end{pmatrix}, \tag{4.1}$$

supplemented with the initial data

$$\left.\begin{pmatrix} \mathbf{j}_k^l(\widehat{x}, 0) \\ \dot{\mathbf{j}}_k^l(\widehat{x}, 0) \end{pmatrix}\right|_{r=0} = \begin{pmatrix} \delta_k^l \\ -\mathbf{s}_k^l(\widehat{x}, 0, t_0) \end{pmatrix}.$$

In fact, here we are simply adding the dependence on $\widehat{x}$ to the same quantities already considered in the previous section, and suppressing the dependence on t_0 which is now considered to be fixed. Since parallel translation preserves the metric, we also have the relation

$$v(\gamma_{\widehat{x}}(0))^{-2}F_j^l(\widehat{x}, 0)F_k^q(\widehat{x}, 0)\delta_{lq} = v(\gamma_{\widehat{x}}(r))^{-2}F_j^l(\widehat{x}, r)F_k^q(\widehat{x}, r)\delta_{lq}.$$

By taking determinants of the matrices on each side and then the natural log, we obtain the following formula

$$f(\gamma_{\widehat{x}}(r)) = \frac{1}{n}\log\left(v(\gamma_{\widehat{x}}(0))^n\left|\frac{\det(F_j^l(\widehat{x}, r))}{\det(F_j^l(\widehat{x}, 0))}\right|\right). \tag{4.2}$$

Also, combining some of the previous formulas we have

$$\frac{\partial}{\partial \widehat{x}^k} f(\gamma_{\widehat{x}}(r)) = \mathbf{j}^p_k(\widehat{x}, r) F^l_p(\widehat{x}, r) \frac{\partial f}{\partial x^l}(\gamma_{\widehat{x}}(r)). \tag{4.3}$$

Away from conjugate points along $\gamma_{\widehat{x}}$, we may invert to obtain

$$\frac{\partial f}{\partial x^l}(\gamma_{\widehat{x}}(r)) = (F^{-1})^p_l(\widehat{x}, r)(\mathbf{j}^{-1})^j_p(\widehat{x}, r) \frac{\partial}{\partial \widehat{x}^j} f(\gamma_{\widehat{x}}(r)) \tag{4.4}$$

which may further be combined with (4.2) to show

$$\frac{\partial f}{\partial x^l}(\gamma_{\widehat{x}}(r)) = \frac{1}{n}(F^{-1})^p_l(\widehat{x}, r)(\mathbf{j}^{-1})^j_p(\widehat{x}, r) \left(\frac{\partial \sigma}{\partial \widehat{x}^j}(\widehat{x}) + (F^{-1})^c_b(\widehat{x}, r) \frac{\partial F^b_c}{\partial \widehat{x}^j}(\widehat{x}, r) \right) \tag{4.5}$$

where we use the notation

$$\sigma(\widehat{x}) := \log \left(\frac{v(\gamma_{\widehat{x}}(0))^n}{\left|\det(F^l_j(\widehat{x}, 0))\right|} \right).$$

A "stepping" recovery procedure in the spirit of Dix' original method may now be presented. We introduce a step size h in r, and for $\alpha \in \mathbb{N}$ we label $r_\alpha = \alpha h$. If we know $f(\gamma_{\widehat{x}}(r_{\alpha-1}))$, $F^l_j(\widehat{x}, r_{\alpha-1})$, and $\mathbf{j}^l_k(\widehat{x}, r_{\alpha-1})$, then we approximate the same quantities at r_α by the following strategy. First we numerically estimate the derivatives

$$\frac{\partial}{\partial \widehat{x}^k} f(\gamma_{\widehat{x}}(r_{\alpha-1})).$$

Then we use (4.4) to estimate the derivatives

$$\frac{\partial f}{\partial x^k}(\gamma_{\widehat{x}}(r_{\alpha-1})).$$

We then have an estimate of $\Gamma^l_{km}(\gamma_{\widehat{x}}(r_{\alpha-1}))$. Next, we use this estimate to perform a forward Euler step in (2.2) and get an approximation of $F^l_j(\widehat{x}, r_\alpha)$. Then, finally, we use (4.2) to obtain the approximation for $f(\gamma_{\widehat{x}}(r_\alpha))$. We note that $\mathbf{j}^l_k(\widehat{x}, r_\alpha)$ may be obtained through a completely independent calculation using (4.1). Because $F^l_n(\widehat{x}, r) = \dot{\gamma}^l_{\widehat{x}}(r)$, we can then approximate $\gamma^l_{\widehat{x}}(r_\alpha)$ as well. This essentially completes the generalization of Dix' method.

A closed system of ordinary differential equations for $n \geq 3$. The above technique is not very satisfying, and in fact we seek a single system of ordinary differential equaitons that could be solved using any numerical scheme to give all the desired quantities. This is possible, although the method we present here works in 3 or higher dimensions only. The reason for this limitation, as explained above already, comes from our use of formulas (2.5) and (2.6) to express the Hessian of f in terms of the Ricci curvature and the first order derivatives of f in Cartesian coordinates. Since we actually know the Ricci curvature in $(\widehat{x}, r)$ coordinates, which we will label as $\widehat{Ric}_{pq}$, we also need the formula for the tensorial change

$$Ric_{ij}(\gamma_{\widehat{x}}(r)) = \widehat{Ric}_{pq}(\widehat{x}, r)(\mathbf{j}^{-1})^p_l(\widehat{x}, r)(F^{-1})^l_i(\widehat{x}, r)(\mathbf{j}^{-1})^q_m(\widehat{x}, r)(F^{-1})^m_j(\widehat{x}, r). \tag{4.6}$$

Now we can describe how to get the closed system of ordinary differential equations.

We differentiate (2.2) and use (4.3) to get

$$\frac{\partial}{\partial r} \frac{\partial F^l_j}{\partial \widehat{x}^a} = \frac{\partial F^q_n}{\partial \widehat{x}^a} F^m_j \Theta^{lk}_{qm} \frac{\partial f}{\partial x^k} + F^q_n \frac{\partial F^m_j}{\partial \widehat{x}^a} \Theta^{lk}_{qm} \frac{\partial f}{\partial x^k} + F^q_n F^m_j \Theta^{lk}_{qm} \mathbf{j}^p_a F^c_p \frac{\partial^2 f}{\partial x^k \partial x^c}. \tag{4.7}$$

where we have suppressed the dependence on $(\widehat{x}, r)$. We may use (2.5), (2.6), (4.5), and (4.6) to express the right-hand side of (4.7) only in terms of $\mathbf{j}$, F, derivatives of F, and the Ricci curvature

$\widehat{Ric}_{pq}$. Combining all the previous equations we now have a system of ordinary differential equations that may be solved. In the next paragraph, we summarize the entire method for convenience.

As claimed above we have now produced a system of ordinary differential equations that may be solved to obtain v and $\gamma_{\widehat{x}}$ in Cartesian coordinates. The system is nonlinear and contains $n+3n^2+n^3$ equations. It may be written as

$$\frac{\partial}{\partial r}\begin{pmatrix} \gamma^l \\ \mathbf{j}^l_k \\ \dot{\mathbf{j}}^l_k \\ F^l_k \\ \frac{\partial F^l_k}{\partial \widehat{x}^p} \end{pmatrix} = \begin{pmatrix} W^l_\gamma(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F,\frac{\partial F}{\partial \widehat{x}}) \\ W^l_{\mathbf{j};k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F,\frac{\partial F}{\partial \widehat{x}}) \\ W^l_{\dot{\mathbf{j}};k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F,\frac{\partial F}{\partial \widehat{x}}) \\ W^l_{F;k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F,\frac{\partial F}{\partial \widehat{x}}) \\ W^l_{\frac{\partial F}{\partial \widehat{x}};kp}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F,\frac{\partial F}{\partial \widehat{x}}) \end{pmatrix}. \tag{4.8}$$

We describe how each of the "W" functions on the right-hand side is to be evaluated. W^l_γ and $W^l_{\mathbf{j};k}$ are the simplest. Recalling that $F_n(\widehat{x},r) = \dot{\gamma}_{\widehat{x}}(r)$ and using (4.1) they are given by

$$W^l_\gamma = F^l_n \quad \text{and} \quad W^l_{\mathbf{j};k} = \dot{\mathbf{j}}^l_k.$$

Next, again according to (4.1), $W^l_{\dot{\mathbf{j}};k}$ is given by

$$W^l_{\dot{\mathbf{j}};k} = -\mathbf{r}^l_j \mathbf{j}^j_k.$$

Since $\widehat{x}^n = r$, $W^l_{F;k}$ is given by

$$W^l_{F;k} = \frac{\partial F^l_k}{\partial \widehat{x}^n}.$$

Finally, $W^l_{\frac{\partial F}{\partial \widehat{x}};kp}$ is given by (4.7) where we calculate $\frac{\partial f}{\partial x^j}$ using (4.5), and calculate $\frac{\partial^2 f}{\partial x^i \partial x^j}$ in several steps using the values of $\frac{\partial f}{\partial x^j}$ already calculated, (4.6), (2.5), and then (2.6). We may now use a Runge-Kutta method to solve the system numerically for r up to any conjugate point. Note that at the conjugate points the matrix $\mathbf{j}^{-1}$ will blow up, and we will not be able to continue.

We note that we do not explicitly solve for v in this system, but after we have found F^l_k, then f and therefore v may be calculated from (4.2).

The presence of conjugate points. It will be possible to solve the system (4.8) up to conjugate points of y_{t_0} along γ_{x_0,η_0}. Thus we can recover everything if there is no conjugate point. However, if there is a conjugate point, then we must follow a more sophisticated strategy. Note that once we are able to calculate the matrix $\mathbf{j}(\widehat{x},r)$ after Step 1, then we can identify all of the conjugate points. If we pick an alternate value of t_0, t_0' say, such that y_{t_0} and $y_{t_0'}$ do not have any of the same conjugate points, and also consider the system (4.8) with t_0 replaced by t_0', then we can find v and γ in Cartesian coordinates along the entire geodesic γ_{x_0,η_0} by switching back and forth between the systems corresponding to t_0 and t_0'.

5. Two-dimensional case. The method of the previous section, step 2 in the reconstruction procedure, will not work in two dimensions. The basic reason for this is that we cannot determine all of the second partial derivatives of f from the curvature of g, but rather can only obtain the Laplacian of f as shown in (2.4). Therefore in the two-dimensional case we are forced to recover f by solving (2.4). We discuss this in more detail below, but first we will also revisit step 1 in the two-dimensional case where the formulae can be simplified.

The main simplification comes from the fact that in the two dimensional case the trace of $S_{r,t}(\widehat{x})$ contains all of the same information as $S_{r,t}(\widehat{x})$ itself, and so it is actually easier to consider this trace. Indeed, let us define

$$\alpha(\widehat{x},r,t) = \operatorname{tr}\left(S_{r,t}(\widehat{x})\right).$$

In this case we have the following simple method of calculating $\alpha(\widehat{x}, r, s)$ away from conjugate points. If $\gamma_{\widehat{x}}(r)$ is not conjugate to $\gamma_{\widehat{x}}(t)$ along $\gamma_{\widehat{x}}$, then there is a distance function defined for $(\widehat{z}, s)$ in a neighborhood of $(\widehat{x}, r)$ by

$$d_{\widehat{x},r,t}(\widehat{z}, s) = \left|\exp^{-1}_{\gamma_{\widehat{x}}(t)}(\gamma_{\widehat{z}}(s))\right|_g .$$

In the seismic context this is nothing other than the local travel time along rays close to $\gamma_{\widehat{x}}$ from $\gamma_{\widehat{x}}(t)$ to points near $\gamma_{\widehat{x}}(r)$. By [16, p.46] we have

$$\alpha(\widehat{x}, r, t) = \Delta_g d_{\widehat{x},r,t}.$$

We continue to review step 1 in the two dimensional case.

5.1. Step 1 redux: The two dimensional case. We note that the $\{V^j\}_{j=0}^3$ which appear in equation (3.1) are actually all scalars and the only nonzero component of $\mathbf{r}_j^k$ is $\mathbf{r}_1^1$ which is what we recover in (3.3). Note also that in the two dimensional case, $\mathbf{r}_1^1$ contains the same information as the sectional curvature (if F_1 is chosen to have unit length with respect to g then in fact $\mathbf{r}_1^1$ is the sectional curvature).

The other simplification occurs in substep 4. In the two dimensional case the metric must have the form

$$\widehat{g}_{jk} = \varphi(\widehat{x}, r)^2 d\widehat{x}^2 + dr^2,$$

Now by [16, p.46], $\varphi(\widehat{x}, r)$ satisfies the equation

$$\frac{\partial^2}{\partial r^2}\varphi(\widehat{x}, r) + \mathbf{r}_1^1(\widehat{x}, r)\varphi(\widehat{x}, r) = 0 \tag{5.1}$$

where $\mathbf{r}_1^1(\widehat{x}, r)$ is already known. Since the metric is also known in a neighborhood of Σ_{0,t_0} we may simply solve this equation with the known initial data $\varphi(\widehat{x}, 0)$ and $\partial_r\varphi(\widehat{x}, 0)$ in order to find the metric in the $(\widehat{x}, r)$ coordinates. We see that it is not necessary in this case to compute the matrix $\mathbf{j}$ corresponding to the Jacobi fields.

Now we continue to show how step 2 may be accomplished in the two dimensional case. The method makes use of the scalar curvature rather than the Ricci curvature.

5.2. Step 2 redux: The two dimensional case via the scalar curvature equation. In step 2 we must take a slightly different strategy for the two dimensional case. The difference is that we cannot express the second derivatives of f which appear in (4.7) in terms of $\mathbf{j}$, F, derivatives of F, and the Ricci curvature. Instead we use a method inspired by the treatment from [13, Section 4.5.6] of a different problem. After we recover the metric g in the coordinates $(\widehat{x}, r)$ we use the scalar curvature equation (2.4) to directly solve for f in these coordinates. Indeed, we assume that

$$v|_{\Sigma_{0,t_0}} \text{ and } \nu_{0,t_0}(v)|_{\Sigma_{0,t_0}} \tag{5.2}$$

are known, and so in fact we have Cauchy data for f on Σ_{0,t_0}. Thus f satisfies a Cauchy problem for the elliptic operator Δ_g (see (2.4)) expressed in $(\widehat{x}, r)$ coordinates which has a unique solution by the unique continuation principal (for a modern review of Cauchy problems for elliptic operators see [2]). Thus we can reconstruct f on the connected component of W (recall W is the domain of the coordinates $(\widehat{x}, r)$) containing Σ_{0,t_0} in the coordinates $(\widehat{x}, r)$. We note, however, that this reconstruction is generally unstable (once again see [2] for a detailed review of the stability of this type of problem).

Now once we have recovered f in $(\widehat{x}, r)$ coordinates the system (4.8) can be replaced by a significantly simpler system. Indeed, if we combine the equation (4.1) for the Jacobi field matrix with (2.2) and (4.4), then we have a closed system of ordinary differential equations which may be solved just like (4.8) for the higher dimensional case. For convenience we write this system down explicitly. The system is

$$\frac{\partial}{\partial r}\begin{pmatrix}\gamma^l\\ \mathbf{j}^l_k\\ \dot{\mathbf{j}}^l_k\\ F^l_k\end{pmatrix}=\begin{pmatrix}W^l_\gamma(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)\\ W^l_{\mathbf{j};k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)\\ W^l_{\dot{\mathbf{j}};k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)\\ W^l_{F;k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)\end{pmatrix}. \tag{5.3}$$

Here $W^l_\gamma(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)$, $W^l_{\mathbf{j};k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)$, and $W^l_{\dot{\mathbf{j}};k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)$ are given by the same formulae shown below (4.8). The last entry on the right hand side is given, according to (2.2) and (4.4), by

$$W^l_{F;k}(r,\widehat{x},\mathbf{j},\dot{\mathbf{j}},F)=\Theta^{lj}_{pq}F^p_nF^q_k(F^{-1})^i_j(\mathbf{j}^{-1})^a_i\frac{\partial}{\partial\widehat{x}^a}f(\gamma_{\widehat{x}}(r)).$$

Solving these equations we can recover f in Cartesian coordinates on the connected component of W containing Σ_{0,t_0} using the map $(\widehat{x},r)\mapsto\gamma_{\widehat{x}}(r)$. In the case that W is not connected (i.e. there are conjugate points) we must do the recovery in steps as described at the end of Section 4.

6. Conclusion. We generalized the method of Dix for reconstructing a depth varying velocity in a half space, where depth is the Cartesian coordinate normal to the boundary, to a procedure for reconstructing a metric conformal to the Euclidean metric on a region of $\mathbb{R}^n$ from expansions of diffraction travel times generated by scatterers in the region and measured on its boundary. Our procedure consists of two steps: In the first step, we reconstruct the directional curvature operator along geodesics as well as the metric in Riemannian normal coordinates. Riemannian normal coordinates can be thought of as "time" coordinates as they appear in so-called seismic time migration. We note that the directional curvature operator did not appear in the method of Dix because of the class of velocity models he considered. In the second step, the velocity and the geodesics on which the velocity is reconstructed are obtained through a transformation to Cartesian coordinates; this can be thought of as a generalization of the "time-to-depth" conversion in the framework of Dix' original formulation. In dimension three or more both steps are essentially formulated in terms of solving a closed system of nonlinear ordinary differential equations, for example, by application of the Runge-Kutta method. In dimension two the second step requires the solution of a Cauchy problem for an elliptic operator which may suffer from stability issues. Through the associated discretization, we accommodate the case of a finite number of scatterers in the manifold. We admit the formation of caustics.

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