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UP-DOWN DECOUPLING AND PARAXIAL WAVE EQUATION ESTIMATES

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Abstract. We provide estimates for the error incurred when a wave field produced by a directional source localized within a given plane is approximated by decoupled evolution equations describing the portions of the field moving upward and downward. The evolution equations are either pseudodifferential or of Schrödinger type (i.e. the so-called paraxial approximation) although in the latter case we must work in boundary normal coordinates. Along the way to these results we also study some microlocal, or directional, energy estimates for such wave fields which bound the energy of the wave at specific locations propagating in given directions.

1. Introduction. Directional decomposition can be rigorously developed using techniques from microlocal analysis. In [27] Taylor analyzed the reflection of singularities of solutions to systems of differential equations, for which he introduced directional decomposition normal to the reflecting boundary. In [25] Stolk introduced microlocal attenuation to accommodate the singularity which would develop in the symbols of the operators describing the decomposition and propagation associated with directions orthogonal to the direction of decomposition. Here, we control this development by the introduction of parabolic scaling as in the dyadic parabolic decomposition of phase space in the initial condition or source. We may relate this to a particular form of controlled illumination ([21, 30]). To accommodate initial conditions oriented in directions different from the decomposition direction, we propose the use of shearlets ([16]). Except for in the results for the paraxial approximation we admit the formation of caustics. In [5] de Hoop and in [25] Stolk consider the subprincipal symbols of the mentioned operators to ensure that the geometrical amplitudes of the solution of the wave equation are recovered.

The paraxial approximation was introduced in the analysis of wave phenomena in [18] by Leontovich and Fock. Their field of application was the propagation of electromagnetic waves near the surface of a convex conducting body [10]. Higher order approximations were considered by Bremmer [3] in the asymptotic evaluation of the integrals that appear in the theory of diffraction of waves by an aperture in a screen. Since then, the paraxial approximation has been applied to various wave problems, amongst which are the downwward continuation of acoustic waves in seismic prospecting [4], underwater acoustics [19, 26], and scattering in random media [15].

The properties of waves in the paraxial approximation have been studied by Joly [14] and Bamberger *et al.* [1]. These were further analyzed in the case of constant coefficients by De Hoop and De Hoop [6]. Higher-order approximations obtained through an interpolation of the symbol at particular points in the contangent space were considered by Halpern and Trefethen [13]. The relation between the solutions of the Helmholtz equation and the Schrödinger equation, through the paraxial approximation, for the propagation of sound in an acoustic wave guide was analyzed by Polyanskii [20] and De Santo [9].

More recently, the paraxial approximation has been revisited for the further development of fast propagation algorithms with a view to understanding and improving their accuracy. We mention two results. Sava & Fomel [23] consider the wave equation in geodesic coordinates either initiated by a point source or a plane wave, and then introduce the paraxial approximation, which is reminiscent of the construction of Gaussian beams. Op't Root and Stolk [22] consider a symmetric quantization of the symbol of the underlying pseudodifferential operator, to avoid the presence and evaluation of lower order terms, to obtain accurate leading-order geometrical amplitudes of the solutions. We also mention another algorithm, proposed by Benamou, Collino and Runborg [2], for the computation of a wave field near a given point at a fixed frequency via microlocal methods.

Also, recently, waves in random media in certain scaling regimes have been described in terms of Itô-Schrödinger diffusion models, in which the paraxial approximation appears, by Garnier and Sølna (see for example [12, 11]) and de Hoop, Garnier, Holman, and Sølna [7]. In these papers, the deterministic reference medium is constant; the results presented in this paper will aid in generalizing these results to heterogeneous reference media.

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The main contribution of this work (Theorem 4.1) is a set of estimates in energy norm for the approximation of a wave field by a pair of decoupled hyperbolic evolution equations. One study of the numerical solution of such equations via a wave packet representation, which naturally dovetails with the situation we study in this work, can be found in [29]. A careful analysis of the wave packet based algorithm can also be found in [8].

Another contribution of this work are the microlocal, or directional, energy estimates given by Theorem 3.1. We remark that this result actually holds in more general coordinate systems than explicitly considered here.

The final result (Theorem 5.1) gives estimates, also in energy norm, for the approximation of a wave field by a pair of Schrödinger type equations (this is the paraxial approximation). These equations fit well into the framework of Itô-Schrödinger diffusion models for waves in random media mentioned above, and we anticipate the use of the paraxial approximation also as preconditioner for the numerical solution of the Helmholtz equation for relatively high frequencies.

In the next section we begin by describing in more detail the problem considered in the current work.

2. Presentation of the problem. For most of the paper our model will be the acoustic wave equation in pseudodepth coordinates. By this we mean coordinates in which the Euclidean metric takes the form

$$[g^{ij}] = \begin{pmatrix} g' & 0 \\ 0 & g^{nn} \end{pmatrix} \tag{2.1}$$

where g' gives a metric in the first $n-1$ coordinates. Actually the results in section 3 will hold in arbitrary coordinates with the same proofs although we do not formulate them that way. We will always assume that we are working in pseudodepth coordinates which we label as (x, z) where $x \in \mathbb{R}^{n-1}$ are the horizontal coordinates and $z \in \mathbb{R}$ is the "pseudodepth". Further, the summation convention with primed indices will indicate a sum only from 1 to $n-1$. In a pseudodepth coordinate system the acoustic wave equation has the form

$$\Box_g u := \frac{\partial^2 u}{\partial t^2} - \frac{1}{\kappa|g|}\frac{\partial}{\partial z}\left(\frac{|g|g^{nn}}{\rho}\frac{\partial u}{\partial z}\right) - \frac{1}{\kappa|g|}\frac{\partial}{\partial x^{j'}}\left(\frac{|g|}{\rho}g^{j'k'}\frac{\partial u}{\partial x^{k'}}\right) = f \tag{2.2}$$

where ρ is the density and κ the compressibility. We consider the Cauchy problem for this equation with $u(t_0, z, x) = 0$ and $\partial_t u(t_0, z, x) = 0$ for a constant t_0. We will also write $L_{z,x}$ for the operator

$$L_{z,x} u := \frac{1}{\kappa|g|}\frac{\partial}{\partial z}\left(\frac{|g|g^{nn}}{\rho}\frac{\partial u}{\partial z}\right) + \frac{1}{\kappa|g|}\frac{\partial}{\partial x^{j'}}\left(\frac{|g|}{\rho}g^{j'k'}\frac{\partial u}{\partial x^{k'}}\right).$$

We assume that g', g^{nn}, κ and ρ are all uniformly bounded above and below so that $L_{z,x}$ is uniformly elliptic. As is usual we will refer to the function f as the source.

Under the assumptions of the previous paragraph, if $f \in L^1([t_0, T]; H^s(\mathbb{R}_z \times \mathbb{R}_x^{n-1}))$ for some $s \in \mathbb{R}$ and $t_0 < T$, then by [17, Theorem 23.2.2] the problem is well posed with a unique solution $u \in C^0([t_0, T]; H^{s+1}(\mathbb{R}_z \times \mathbb{R}_x^{n-1})) \cap C^1([t_0, T]; H^s(\mathbb{R}_z \times \mathbb{R}_x^{n-1}))$. In fact by [17, Formula 23.2.4'] we have the following energy estimate for a constant C sufficiently large

$$\|u\|_{L^2([t_0,T];H^{s+1}(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))} + \|\partial_t u\|_{L^2([t_0,T];H^s(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))} \leq C\|f\|_{L^2([t_0,T];H^s(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}. \tag{2.3}$$

We comment at this point that here and throughout this work C will denote a constant, which may change from step to step, but may always be chosen based only on $L_{z,x}$, s, t_0, T, and in some cases other parameters introduced below.

Our goal will be to show how well the solution of (2.2), when f has a particular form so that the waves propagate primarily along the z axis, may be approximated by a solution of an evolution evolution equation in z. The particular form for f will be

$$f_\lambda(t, z, x) = \delta(z)\phi(t)\chi_\lambda(t, x) \tag{2.4}$$

where $\chi_\lambda \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$ satisfies for some λ and $k > 0$

$$\operatorname{supp}(\widehat{\chi_\lambda}(\omega, \xi)) \subset \{|\xi|^2 \le k|\omega|\} \cap \{\lambda \le |\omega|\} =: B_{\lambda,k},$$

and $\phi(t) \in C_c^\infty(\mathbb{R}_t)$ is a cutoff function with support contained in the interval (t_0, T). In the previous formula we mean to define the set $B_{\lambda,k}$ as

$$B_{\lambda,k} = \{|\xi|^2 \le k|\omega|\} \cap \{\lambda \le |\omega|\}.$$

Note that $f_\lambda \in L^2(\mathbb{R}_t; H^s(\mathbb{R}_z \times \mathbb{R}_x^{n-1})$ for any $s < -1/2$. Intuitively, λ corresponds to the minimum frequency of the source and k the degree of "directionality." For any positive k the source produces directional waves moving up and down, but for smaller values of k this is more focused. We designed these hypotheses for χ_λ with curvelets in mind (for a construction of a curvelet frame see for example [24]), although we have made them more general in order to apply in more general situations. Our approach is to consider k to be fixed while we find estimates that depend on the parameter λ which is why we write f_λ rather than say $f_{\lambda,k}$.

The function ϕ represents the physical reality that we may only produce a source for a finite amount of time, and thus makes sense for applications of this theory. In fact the cutoff ϕ is important from a theoretical point of view as well since we wish to specify that there are "no incoming waves" (i.e. that the entire wave field is generated from the source) by setting Cauchy data at $t = -\infty$, but this may not be possible in general unless f decays exponentially as $t \to -\infty$.

We comment that the above setup accommodates only sources oriented perpendicular to the plane $z = 0$ which is a major restriction. In general we clearly need to have a method to handle sources oriented in other directions. We can do this if we shear the source function and allow more general coordinate systems than pseudodepths (i.e. if we allow sheared coordinates). The results of section 3 will hold in these coordinates systems, but the results of the other sections do not. In fact we anticipate that we can generalize section 4 to hold in such coordinates but the systems (4.3) and (4.4) will be more complicated. For completeness, we indicate more precisely how a sheared directional source could be constructed. Indeed, considering only dimension $n = 3$ we introduce the parabolic scaling and shearing matrices

$$M_\lambda = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{pmatrix}, \quad X_s = \begin{pmatrix} 1 & -s_1 & -s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and their product $D_{\lambda s} = X_s M_\lambda$. A sheared source is then defined by replacing χ_λ with

$$\chi_{\lambda s\,(t_0,x_0)}(t, x) = \lambda\, \psi(D_{\lambda s}^{-1}((t, x) - (t_0, x_0))),$$

where ψ is a mother shearlet whose Fourier transform has support contained for example in $\{|\omega| \in [1, 2], |\xi| < 1\}$. In this case we would have

$$\operatorname{supp}(\widehat{\chi_{\lambda s\,(t_0,x_0)}}) \subset \left\{|\omega| \in [\lambda, 2\lambda], |\xi - s\omega| < \lambda^{1/2}\right\}$$

and $\|\chi_{\lambda s\,(t_0,x_0)}\|_{L^2} = \|\psi\|_{L^2}$. The function ψ could be constructed in different ways for example by the Fourier transform of a tensor product. Naturally, $s_{1,2}$ determines the orientation while (t_0, x_0) determines a translation. These are essentially continuous shearlets ([16]), and our plan for possible numerical implementation of the decoupled evolution equations is to use these as a frame to decompose a general source, and then evolve each of these shearlets separately. For the shearlets oriented perpendicular to $z = 0$ the evolution should be done according the equations from section 4, while for the obliquely oriented packets the evolution should be done by first making a shearing change of coordinates to bring the wave front set of the source projected onto the spatial coordinates perpendicular to $z = 0$. In these coordinates the Euclidean metric would no longer have the special pseudodepth form (2.1), and the decoupling into pseudodifferential equations would be more complicated although still possible. In particular the acoustic wave equation in such coordinates would contain some mixed derivatives $\partial^2/\partial x^j \partial z$. The details of extending the results of this paper to that case are left for future work.

Our main aim is to show that we can approximate u_λ, the solution of (2.2) with $f = f_\lambda$, by solving evolution equations in z and that these estimates improve asymptotically as $\lambda \to \infty$ at least for z within a certain interval around 0 which will be determined by the ray geometry. The first major step towards this goal will be to establish a "microlocal energy estimate" which will show that we can microlocally cut out the portion of u_λ which cannot be decomposed into upward and downward moving components. These estimates will be proven in section 3. Next we show in section 4 how (2.2) may be decoupled into two pseudodifferential equations describing respectively the upward and downward moving portions of u_λ. The estimates obtained in section 3 are then used to obtain estimates on the error incurred by using these decoupled equations to approximate u_λ. In section 5 we show how the pseudodifferential equations found in section 4 can be replaced by Schrödinger type differential equations which may be considered as evolution equations in the z variable. This is the paraxial approximation. Using the results we have built up from the previous two sections we also give estimates that improve as $\lambda \to \infty$ for the error incurred when approximating u_λ by solutions of these differential equations when we use boundary normal coordinates, which corresponds in the context of (2.2) to coordinates in which $g^{nn} = \rho\kappa$. Finally in the conclusion we review the main results of the paper and briefly indicate the direction of future work.

3. Directional Energy estimate for acoustic wave equation. The main purpose of this section is to prove some microlocal energy estimates for (2.2) that will be needed later. First we describe the main result of the section. We will make extensive use of pseudodifferential operators (ΨDOs) and their calculus. As general references on ΨDOs we mention [17, 28].

As is usual we write $c = (\rho\kappa)^{-1/2}$ for the "local wave speed" of the equation (2.2). Let $\Phi_t(z, x; \zeta, \xi)$ denote the Hamiltonian flow for the Hamiltonian

$$H = c\,(\zeta^2 g^{nn} + g^{j'k'}\xi_{j'}\xi_{k'})^{1/2}$$

starting at $(z, x; \zeta, \xi) \in T^*(\mathbb{R}_z \times \mathbb{R}^{n-1}_x)$, and define the set

$$\Gamma_{r,\delta,\epsilon} = \{(z, x; \zeta, \xi) \in T^*(\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \,|\, |z| < \epsilon \quad \text{and whenever} \quad |\zeta|^2 + |\xi|^2 > r,\ |\zeta| > \delta|\xi|\}.$$

Theorem 3.1. *Suppose that* $\varphi \in S^0((\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \times (\mathbb{R}_z \times \mathbb{R}^{n-1}_x))$ *is such that*

$$\Gamma_{r,\delta,\epsilon} \cap \Phi_t(\operatorname{supp}(\varphi)) = \emptyset \quad \textit{for all} \quad t \in [-T, T]. \tag{3.1}$$

Then $\varphi(z, x, D_z, D_x)u_\lambda \in H^1((t_0, T) \times \mathbb{R}_z \times \mathbb{R}^{n-1}_x))$ *and*

$$\|\varphi(z, x, D_z, D_x)u_\lambda\|_{H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}^{n-1}_x)} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_z\times\mathbb{R}^{n-1}_x)}.$$

The proof of Theorem 3.1 is the subject of the rest of this section. It is no surprise that the proof uses methods from microlocal analysis based on high frequency asymptotics. Without loss of generality we will assume that $\widehat{\chi_\lambda} \in C^\infty_c$ which can be done since general $\chi_\lambda \in L^2$ may be approximated by such functions. Our first task is to create a parametrix for (2.2).

The methods applied here for the parametrix construction are standard. First, we require an approximate square root for $-L_{z,x}$. Since the principal symbol of $-L_{z,x}$ is positive except at $(\zeta, \xi) = 0$ this can be done via an asymptotic expansion (for a description of how to do this see [22]) yielding an elliptic symbol $b \in S^1((\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \times (\mathbb{R}_\zeta \times \mathbb{R}^{n-1}_\xi))$ with principal symbol $\sigma_p(b) = H$ such that

$$b(z, x, D_z, D_x) \circ b(z, x, D_z, D_x) = -L_{z,x} + a(z, x, D_z, D_x)$$

where $a \in S^{-\infty}((\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \times (\mathbb{R}_\zeta \times \mathbb{R}^{n-1}_\xi))$. The operator $\Box_g$ in (2.2) can then be factored as

$$\begin{aligned}\Box_g &= \left(\frac{\partial}{\partial t} + \mathrm{i}\, b(z, x, D_z, D_x)\right) \circ \left(\frac{\partial}{\partial t} - \mathrm{i}\, b(z, x, D_z, D_x)\right) - a(z, x, D_z, D_x) \\ &= \left(\frac{\partial}{\partial t} - \mathrm{i}\, b(z, x, D_z, D_x)\right) \circ \left(\frac{\partial}{\partial t} + \mathrm{i}\, b(z, x, D_z, D_x)\right) - a(z, x, D_z, D_x).\end{aligned} \tag{3.2}$$

Now we introduce the two approximate propagators $E_\pm(t,t')$ which are the respective solution operators for

$$\left(\frac{\partial}{\partial t} \pm \mathrm{i}\, b(z,x,D_z,D_x)\right).$$

This means that $E_\pm(t,t') : H^s(\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \to H^s(\mathbb{R}_z \times \mathbb{R}^{n-1}_x)$ for all $s \in \mathbb{R}$, $E_\pm(t,t) = \mathrm{Id}$ for all $t \in \mathbb{R}$, and

$$\left(\frac{\partial}{\partial t} \pm \mathrm{i}\, b(z,x,D_z,D_x)\right) \circ E_\pm(t,t') = 0.$$

These operators exist and have the claimed properties by [28, Proposition 7.1]. Next define

$$S_{t,t'} = \frac{E_-(t,t') + E_+(t,t')}{2}. \tag{3.3}$$

$S_{t,t'}$ is a parametrix for $\Box_g$ as shown in the next lemma.

Lemma 3.2. *The operator $S_{t,t'}$ defined by (3.3) has the following properties*

1. *For every t, t', and s*

$$S_{t,t'} : H^s(\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \to H^s(\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \quad \text{and} \quad \partial_t S_{t,t'} : H^s(\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \to H^{s-1}(\mathbb{R}_z \times \mathbb{R}^{n-1}_x).$$

2. *For every t, $S_{t,t} = \mathrm{Id}$, and $\partial_t S_{t,t'}|_{t'=t} = 0$.*
3. *For every t, t', s, and m*

$$K(t,t') := \Box_g \circ S_{t,t'} : H^s(\mathbb{R}_z \times \mathbb{R}^{n-1}_x) \to H^{s+m}(\mathbb{R}_z \times \mathbb{R}^{n-1}_x).$$

The continuity is also uniform for t and t' in any finite interval with fixed s and m.

Proof. Points (1) and (2) are simple consequences of the properties of $E_\pm$, the continuity of pseudodifferential operators (see [17, Theorem 18.1.13]), and the formula

$$\partial_t S_{t,t'} = \mathrm{i}\, b(z,x,D_z,D_x) \circ (E_-(t,t') - E_+(t,t')).$$

For (3) we have using the previous formula and the factorizations (3.2)

$$\begin{aligned}\Box_g \circ S_{t,t'} &= \frac{\Box_g}{2} \circ \Big(E_-(t,t') + E_+(t,t')\Big) \\ &= \frac{a(z,x,D_z,D_x)}{2} \circ \Big(E_+(t,t') + E_-(t,t')\Big).\end{aligned}$$

The result now follows from the continuity properties of $E_\pm$ and $a(z,x,D_x,D_z) \in \Psi^{-\infty}(\mathbb{R}_z \times \mathbb{R}^{n-1}_x)$. □

The next step in our proof of Theorem 3.1 will be to show that we can replace u_λ by an appropriately chosen parametrix applied to the source. To obtain the correct estimates we integrate (2.2) with f replaced by f_λ. This gives

$$\Box_g \int_{t_0}^{t} u_\lambda(t',z,x)\,\mathrm{d}t' = \int_{t_0}^{t} f_\lambda(t',z,x)\,\mathrm{d}t' =: F_\lambda(t,z,x). \tag{3.4}$$

We continue our program by using the parametrix $S_{t,t'}$ as an approximate propagator for (3.4). Indeed the following lemma gives a λ dependent estimate for such an approximation.

Lemma 3.3. *We have the following estimate*

$$\left\| u_\lambda(t,z,x) - \int_{t_0}^{t} S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t' \right\|_{H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}^{n-1}_x)} \leq C\lambda^{-1} \|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}^{n-1}_x)}.$$

The constant C depends on the operator $L_{z,x}$, $T - t_0$, and a C^k bound on ϕ for some finite k.

Proof. If we define

$$v_\lambda(t,z,x) := u_\lambda(t,z,x) - \int_{t_0}^{t} S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t',$$

then it is straightforward to check using (3.4) and Lemma 3.2 part (2) that

$$\Box_g v_\lambda(t,z,x) = -\int_{t_0}^{t} K(t,t')[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t',$$

$v_\lambda(t_0,z,x)=0$, and $\partial_t v_\lambda(t_0,z,x)=0$. Therefore we can apply the energy estimate (2.3) as well as Lemma 3.2 part (3) to obtain

$$\begin{aligned}\|v_\lambda\|_{H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})} &\le C\|\Box_g v_\lambda\|_{L^2((t_0,T)\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})}\\ &\le C\|F_\lambda\|_{L^2((t_0,T);H^{-1}(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}.\end{aligned}$$

Note that since $f_\lambda = 0$ for $t \le t_0$ we may apply the Fourier inversion formula and Riemann-Lebesgue Lemma to obtain

$$\begin{aligned}\mathcal{F}_{z,x}[F_\lambda](t,\zeta,\xi) &= \frac{1}{2\pi}\lim_{a\to\infty}\int_{t_0-a}^{t}\int_{\mathbb{R}_\omega} \mathrm{e}^{\mathrm{i}\omega t'}[\widehat{\phi}(\cdot_\omega) *_\omega \widehat{\chi_\lambda}(\cdot_\omega,\xi)](\omega)\,\mathrm{d}\omega\,\mathrm{d}t'\\ &= \frac{1}{2\pi}\int_{|\omega|<1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{\mathrm{i}\omega t_0})\widehat{\phi}(\omega-\tau)\widehat{\chi_\lambda}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\omega\\ &\quad + \frac{1}{2\pi}\lim_{a\to\infty}\int_{|\omega|\ge 1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{\mathrm{i}\omega(t_0-a)})\widehat{\phi}(\omega-\tau)\widehat{\chi_\lambda}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{|\omega|<1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{\mathrm{i}\omega t_0})\widehat{\phi}(\omega-\tau)\widehat{\chi_\lambda}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\omega\\ &\quad + \frac{1}{2\pi}\int_{|\omega|\ge 1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\widehat{\phi}(\omega-\tau)\widehat{\chi_\lambda}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{|\omega|<1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{\mathrm{i}\omega t_0})\widehat{\phi}(\omega-\tau)\widehat{\chi_\lambda}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\omega\\ &\quad + \frac{1}{2\pi}\int_{|\omega|\ge 1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\widehat{\phi}(\omega-\tau)\widehat{\chi_\lambda}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\omega.\end{aligned} \tag{3.5}$$

Now using the assumptions on $\mathrm{supp}(\widehat{\chi_\lambda})$ together with Parseval's identity and the Minkowski inequality we have

$$\|F_\lambda\|_{L^2((t_0,T);H^{-1}(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))} \le C\lambda^{-1}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}. \tag{3.6}$$

Combining the previous estimates proves the lemma. □

Remark 1. *The proof of Lemma 3.3 could also give stronger estimates involving higher order derivatives with respect to z and x on the left hand side, but the H^1 estimate is all we will need in the sequel.*

We are now in a position to complete the proof of Theorem 3.1.

Proof. [Proof of Theorem 3.1] By Lemma 3.3 and the continuity properties of $\varphi(z,x,D_z,D_x)$ it is sufficient to show

$$\left\|\varphi(z,x,D_z,D_x)\int_{t_0}^{t} S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t'\right\|_{H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}.$$

To begin we prove the result only for the derivatives in z and x. That is, we prove this estimate with the norm $H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})$ replaced by $L^2((t_0,T);H^1(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))$. By the continuity properties of the pseudodifferential operators, we may bring $\varphi(z,x,D_z,D_x)$ inside the integral and thus we may complete this portion of the proof by finding an estimate of

$$\left\|\varphi(z,x,D_z,D_x)S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\right\|_{L^2((t_0,T);H^1(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}.$$

We begin with the calculation

$$\begin{aligned}\varphi(z,x,D_z,D_x)S_{t,t'} = \frac{1}{2}\Big(&E_-(t,t')[E_-(t',t)\varphi(z,x,D_z,D_x)E_-(t,t')]\\ &+E_+(t,t')[E_+(t',t)\varphi(z,x,D_z,D_x)E_+(t,t')]\Big).\end{aligned}$$

To bound this quantity we use Egorov's theorem [28, Theorem 8.1] which shows that

$$[E_\pm(t',t)\varphi(z,x,D_z,D_x)E_\pm(t,t')] = h^\pm_{t-t'}(z,x,D_z,D_x) + r^\pm_{t-t'}(z,x,D_z,D_x)$$

where $h^\pm_{t-t'}\in S^0((\mathbb{R}_z\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_\zeta\times\mathbb{R}_\xi^{n-1}))$ has support contained in $\Phi_{\mp(t-t')}(\mathrm{supp}(\varphi))$ and $r^\pm_{t-t'}(z,x,D_z,D_x)\in\Psi^{-2}(\mathbb{R}_z\times\mathbb{R}_x^{n-1})$. In order to make use of this decomposition, let $\kappa\in S^0((\mathbb{R}_z\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_\zeta\times\mathbb{R}_\xi^{n-1}))$ denote a microlocal cutoff function that is equal to 1 on the set

$$\{|\zeta|^2+|\xi|^2>2r\}\cap\{|\zeta|>2\delta|\xi|\}$$

and with support contained in the set

$$\{|\zeta|^2+|\xi|^2>r\}\cap\{|\zeta|>\delta|\xi|\}.$$

Also let $Z\in C_c^\infty(\mathbb{R}_z)$ be a cutoff equal to 1 on $[-\epsilon/2,\epsilon/2]$ and with support in $[-\epsilon,\epsilon]$. By the hypotheses

$$h^\pm_{t-t'}(z,x,D_z,D_x) = h^\pm_{t-t'}(z,x,D_z,D_x)(1-Z\kappa)(z,D_z,D_x)+\widetilde{r}_{t-t'}(z,x,D_z,D_x)$$

where $\widetilde{r}_{t-t'}(z,x,D_z,D_x)\in\Psi^{-\infty}(\mathbb{R}_z\times\mathbb{R}_x^{n-1})$ for all t and $t'\in[t_0,T]$ with $t'\le t$. Using this identity and the uniform continuity of $h^\pm_{t-t'}(z,x,D_z,D_x)$ with respect to t and t' we have

$$\begin{aligned}\left\|h^\pm_{t-t'}(z,x,D_z,D_x)F_\lambda\right\|_{L^2((t_0,T);H^1(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}\le C\Big(&\|(1-Z\kappa)(z,D_z,D_x)F_\lambda\|_{L^2((t_0,T);H^1(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}\\ &+\|F_\lambda\|_{L^2((t_0,T);H^{-1}(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}\Big).\end{aligned}$$

The second term on the right hand side of the previous estimate has already been bounded by (3.6), and so all we need is a bound on the first term. To accomplish this we first apply a version of (3.5) without the Fourier transform in z and x to obtain

$$\begin{aligned}(1-Z\kappa)(z,D_z,D_x)F_\lambda =&\frac{1}{2\pi}\int_{|\omega|<1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{\mathrm{i}\omega t_0})\widehat{\phi}(\omega-\tau)(1-\kappa)(D_z,D_x)[\delta_z\mathcal{F}_t[\chi_\lambda](\tau,\cdot_x)](z,x)\,\mathrm{d}\tau\,\mathrm{d}\omega\\ &+\frac{1}{2\pi}\int_{|\omega|\ge1/\lambda}\int_{\mathbb{R}_\tau}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\widehat{\phi}(\omega-\tau)(1-\kappa)(D_z,D_x)[\delta_z\mathcal{F}_t[\chi_\lambda](\tau,\cdot_x)](z,x)\,\mathrm{d}\tau\,\mathrm{d}\omega.\end{aligned}$$

Next, using the properties of κ and the fact that $\mathrm{supp}(\widehat{\chi}_\lambda(\tau,\xi))\subset\{|\xi|<k^{1/2}|\tau|^{1/2}\}$, we have

$$\begin{aligned}\left\|(1-\kappa)(D_z,D_x)[\delta_z\mathcal{F}_t[\chi_\lambda](\tau,\cdot_x)](z,x)\right\|^2_{H^1(\mathbb{R}_z\times\mathbb{R}_x)}&\le\iint_{\{|\zeta|^2+|\xi|^2\le2r\}\cup\{|\zeta|\le2\delta|\xi|\}}\left|\widehat{\chi}_\lambda(\tau,\xi)\right|^2(1+|\zeta|^2+|\xi|^2)\mathrm{d}\zeta\mathrm{d}\xi\\ &\le C|\tau|^{3/2}\int\left|\widehat{\chi}_\lambda(\tau,\xi)\right|^2\mathrm{d}\xi.\end{aligned}$$

Now we can apply this estimate and the previous identity to obtain

$$\|(1-Z\kappa)(z,D_z,D_x)F_\lambda\|_{L^2((t_0,T);H^1(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}\le\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_z\times\mathbb{R}_x^{n-1})}$$

which completes the first part of the proof.

It now remains to estimate the derivative with respect to t. The proof here is much the same, but we begin with the calculation

$$\begin{aligned}\partial_t\Big(\varphi(z,x,D_z,D_x)\int_{t_0}^{t} S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t'\Big)\\ &=\varphi(z,x,D_z,D_x)[F_\lambda(t,\cdot_z,\cdot_x)](z,x)+\varphi(z,x,D_z,D_x)\int_{t_0}^{t}\partial_t S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t'\\ &=\varphi(z,x,D_z,D_x)[F_\lambda(t,\cdot_z,\cdot_x)](z,x)\\ &\quad+\frac{\mathrm{i}}{2}\int_{t_0}^{t}[\varphi(z,x,D_z,D_x),b(z,x,D_z,D_x)](E_-(t,t')-E_+(t,t'))[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t'\\ &\quad+\frac{\mathrm{i}}{2}b(z,x,D_z,D_x)\int_{t_0}^{t}\varphi(z,x,D_z,D_x)(E_-(t,t')-E_+(t,t'))[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\,\mathrm{d}t'.\end{aligned}$$

We now discuss how to estimate each of the three terms on the right hand side separately. The first term has precisely the same form as $h^{\pm}_{t-t'}(z,x,D_z,D_x)F_\lambda$ (in fact it is the case $t=t'$) which has already been estimated above in a stronger norm than is required here. For the second term note that using the calculus of pseudodifferential operators

$$[\varphi(z,x,D_z,D_x),b(z,x,D_z,D_x)]=\widetilde{\varphi}(z,x,D_z,D_x)+r(z,x,D_z,D_x)$$

where $\widetilde{\varphi}\in S^0((\mathbb{R}_z\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))$ with $\mathrm{supp}(\widetilde{\varphi})\subset\mathrm{supp}(\varphi)$ and $r\in S^{-1}((\mathbb{R}_z\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))$. Arguing as before the second term can thus be bounded in $L^2((t_0,T)\times\mathbb{R}_z\times\mathrm{R}_x^{n-1})$. Finally, using the continuity properties of $b(z,x,D_z,D_x)\in\Psi^1(\mathbb{R}_z\times\mathbb{R}_x^{n-1})$ the $L^2((t_0,T)\times\mathbb{R}_z\times\mathrm{R}_x^{n-1})$ norm of the last term is bounded by

$$C\,\|\varphi(z,x,D_z,D_x)(E_-(t,t')-E_+(t,t'))[F_\lambda(t',\cdot_z,\cdot_x)](z,x)\|_{L^2((t_0,T);H^1(\mathbb{R}_z\times\mathrm{R}_x^{n-1}))}.$$

This last quantity is in turn bounded in the same way as $\varphi(z,x,D_z,D_x)S_{t,t'}[F_\lambda(t',\cdot_z,\cdot_x)](z,x)$. This completes the proof. □

To finish this section we prove a few more estimates that will also be required in the rest of the paper. They provide bounds on the negative Sobolev norms of u_λ with respect to t. We recall that for negative s and an open finite time interval I_t the space $H^s(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))$ is defined as the dual of $H_0^{-s}(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))$ with the natural norm

$$\|v\|_{H_0^s(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))}=\sup_{\|u\|_{H_0^{-s}(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))}=1}\langle v,u\rangle.$$

When restricted to distributions with support compactly contained in I_t, the norm in $H^s(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))$ is equivalent to the norm in $H^s(\mathbb{R}_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))$ which may be defined in the usual way through the Fourier transform. Our reason for spelling out the relationship between these spaces is to be able to use both estimates coming from the theory of ΨDO's and energy estimates. To establish the needed bounds we note that when $v\in L^2(\mathbb{R}_t\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})$ and $\widetilde{\phi}\in C^\infty(\mathbb{R}_t)$ has support contained in I_t then for $u\in H_0^{-s}(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))$

$$\langle\widetilde{\phi}v,u\rangle=\int_{I_t}\int_{\mathbb{R}_z}\int_{\mathbb{R}_x^{n-1}}\widetilde{\phi}\,\overline{v}\,u\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t.$$

If we define an operator $E[v]=\int_{t_0}^{t}v(t',z,x)\,\mathrm{d}t'$ then $\partial_t^kE^k[v]=v$, and so if $k\leq -s$ then

$$|\langle\widetilde{\phi}v,u\rangle|=\left|(-1)^k\int_{I_t}\int_{\mathbb{R}_z}\int_{\mathbb{R}_x^{n-1}}\overline{E^k[v]}\,\partial_t^k(\widetilde{\phi}u)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\right|\leq\|E^k[v]\|_{L^2(I_t\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})}\|\widetilde{\phi}u\|_{H_0^s(I_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))}.$$

Therefore $\|\widetilde{\phi}v\|_{H^s(\mathbb{R}_t;L^2(\mathbb{R}_z\times\mathbb{R}_x))}\leq C\|E^k[v]\|_{L^2(I_t\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})}$. Using this estimate together with some of the arguments from earlier in this section we may now prove the following lemma.

LEMMA 3.4. *If $s < 0$ is an integer, $(t_0, T) \subset I_t \subset \mathbb{R}_t$ is a finite interval and $\widetilde{\phi} \in C_c^\infty(I_t)$, then*

$$\|\widetilde{\phi} u_\lambda\|_{H^s(\mathbb{R}_t; L^2(\mathbb{R}_z \times \mathbb{R}_x))} \leq C\lambda^s \|\chi_\lambda\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})}.$$

Proof. By the paragraph above the lemma it is sufficient to show

$$\|E^k[u_\lambda]\|_{L^2(I_t \times \mathbb{R}_z \times \mathbb{R}_x^{n-1})} \leq C\lambda^s \|\chi_\lambda\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})}.$$

We can do this essentially by following the same proof as Lemma 3.3. Indeed,

$$\Box_g E^k[u_\lambda] = E^k[f],$$

and so the energy estimate (2.3) implies

$$\|E^k[u_\lambda]\|_{L^2(I_t \times \mathbb{R}_z \times \mathbb{R}_x^{n-1})} \leq C\|E^k[f]\|_{L^2(I_t; H^{-1}(\mathbb{R}_z \times \mathbb{R}_x^{n-1}))}.$$

Now the argument is completed in the same manner as in the proof of Lemma 3.3. Note that the proof of that lemma actually completes the proof of this lemma in the case $s = -1$ fully, and in fact that is all that we need although we have stated the lemma in more generality. □

4. Decoupling into pseudodifferential equations. In this section we first review how (2.2) may be decoupled into two hyperbolic evolution equations involving first order pseudodifferential operators. This decoupling is certainly not new and has been studied recently for example in [22, 25]. The contribution in this section is to establish true estimates on the error incurred by using these decoupled formulae to approximate the wave field u_λ given by (2.2) with source f_λ. The directional nature of the source, given by (2.4), is what allows us to do this.

Since we are only interested in approximating u_λ in the finite time interval $[t_0, T]$ we begin by replacing u_λ by $\widetilde{u}_\lambda = \widetilde{\phi}(t) u_\lambda$ where $\widetilde{\phi} \in C_c^\infty(\mathbb{R})$ is a cutoff function equal to one on an open set containing $[t_0, T]$, and with support in a slightly larger interval. Then $\widetilde{u}_\lambda$ satisfies

$$\Box_g \widetilde{u}_\lambda = -2(\partial_t \widetilde{\phi})\partial_t u_\lambda - (\partial_t^2 \widetilde{\phi}) u_\lambda + f_\lambda. \tag{4.1}$$

Our next step is to decompose the wave field $\widetilde{u}_\lambda$ into one component in which we have a microlocal directional decomposition, and another which we will control using the estimates of section 3. Indeed, for some $0 < \theta_1 < 1$ and $\widetilde{k} = 0$, 1, or 2 let $\varphi_1^{\widetilde{k}}(s) \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function that is equal to one for $|s| < \theta_1^{2^{\widetilde{k}}}$ and equal to zero for $|s| > \theta_1^{2^{\widetilde{k}-1}}$. Also, choose another function $\psi_1(s) \in C_c^\infty(\mathbb{R})$ equal to zero on the set $\{|s| < 1/2\}$, and equal to one on $\{|s| > 1\}$. We will also use the notation $\psi_1^{\widetilde{k}}(s) = \psi_1(\widetilde{k}s)$. With these functions in hand we define

$$\varphi^{\widetilde{k}}(z, x, \omega, \zeta, \xi) = \psi_1^{1/\widetilde{k}}(\omega)\, \varphi_1^{\widetilde{k}}\left(\frac{c^2 g^{j'k'} \xi_{j'} \xi_{k'}}{\omega^2}\right).$$

and for each $\widetilde{k}$ introduce the decomposition

$$\widetilde{u}_\lambda = \varphi^{\widetilde{k}}(z, x, D_t, D_x)\widetilde{u}_\lambda + \left(1 - \varphi^{\widetilde{k}}(z, x, D_t, D_x)\right)\widetilde{u}_\lambda =: \widetilde{u}_1^{\widetilde{k},\lambda} + \widetilde{u}_2^{\widetilde{k},\lambda}.$$

The motivation for this decomposition is that when there are no turning rays we may microlocally estimate $\widetilde{u}_2^{\widetilde{k},\lambda}$ (see Lemma 4.2), while as we show below $\widetilde{u}_1^{1,\lambda}$ may be decoupled into portions which are moving in the positive and negative z directions for which we have a pair of evolution equations in z (see (4.6) and (4.7)).

Now we apply $\varphi^1(z, x, D_t, D_x)$ to (4.1) to obtain

$$\Box_g \varphi^1(z, x, D_t, D_x)\widetilde{u}_\lambda = \left[\Box_g, \varphi^1(z, x, D_t, D_x)\right]\widetilde{u}_\lambda - \varphi^1(z, x, D_t, D_x)(2(\partial_t \widetilde{\phi})\partial_t u_\lambda + (\partial_t^2 \widetilde{\phi}) u_\lambda - f_\lambda).$$

By analyzing the support of the symbols of the operators involved and applying the calculus of ΨDOs we see that we may insert $1 - \varphi^2(z, x, D_t, D_x)$ before the commutator if we are willing to add some lower order terms. Indeed, this leads to

$$\Box_g \widetilde{u}_1^{1,\lambda} = \left[\Box_g, \varphi^1(z, x, D_t, D_x)\right] \widetilde{u}_2^{2,\lambda} + R_1 \widetilde{u}_\lambda - \varphi^1(z, x, D_t, D_x)(2(\partial_t \widetilde{\phi})\partial_t u_\lambda + (\partial_t^2 \widetilde{\phi}) u_\lambda - f_\lambda) \tag{4.2}$$

where $R_1 \in \Psi^{-\infty}(\mathbb{R}_t \times \mathbb{R}_z \times \mathbb{R}_x^{n-1})$. Next we will proceed to show how (4.2) may be decoupled into upward and downward moving components. We essentially follow methods presented in [22], although our situation is a little more general.

To begin the decoupling we multiply by $\kappa|g|$ and define

$$\mathbf{u}_\lambda = \left(\widetilde{u}_1^{1,\lambda}, \frac{|g|g^{nn}}{\rho}\partial_z \widetilde{u}_1^{1,\lambda}\right), \quad \widetilde{f}_\lambda^1 = \kappa|g|\varphi^1(z, x, D_t, D_x) f_\lambda, \quad \text{and} \quad R_2 = -\kappa|g|\varphi^1(z, x, D_t, D_x)(2\partial\widetilde{\phi}\,\partial_t + \partial_t^2 \widetilde{\phi})$$

to rewrite (4.2) as the system

$$\partial_z \mathbf{u}_\lambda = \begin{pmatrix} 0 & \frac{\rho}{|g|g^{nn}} \\ \kappa|g|\partial_t^2 - L_x & 0 \end{pmatrix} \mathbf{u}_\lambda - \begin{pmatrix} 0 \\ \kappa|g| \left[\Box_g, \varphi^1(z, x, D_t, D_x)\right] \widetilde{u}_2^{2,\lambda} + R_1 \widetilde{u}_\lambda + R_2 u_\lambda + \widetilde{f}_\lambda^1 \end{pmatrix}$$

where

$$L_x = \frac{\partial}{\partial x^{j'}} \frac{|g|}{\rho} g^{j'k'} \frac{\partial}{\partial x^{k'}}.$$

We note here that $\kappa|g|\partial_t^2 - L_x$ is formally symmetric, which will be important later. Indeed, in the following constructions we take measures to preserve this symmetry. In particular, we adopt the notation of [22] for the symmetric quantization of a symbol $b \in \Psi^\infty(\mathbb{R}_t \times \mathbb{R}_x)$

$$Op_{\mathcal{S}}(b) = \frac{1}{2}\left(b(t, x, D_t, D_x) + b(t, x, D_t, D_x)^*\right).$$

Next we have the decomposition

$$\begin{aligned} -\kappa|g|\partial_t^2 + L_x &= Op_{\mathcal{S}}\left(\psi_1\,(2\omega)^4\ \varphi_1^0\left(\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^4 \sigma\left(-\kappa|g|\partial_t^2 + L_x\right)\right) \\ &\quad + Op_{\mathcal{S}}\left(\left(1 - \psi_1\,(2\omega)^4\ \varphi_1^0\left(\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^4\right)\sigma\left(-\kappa|g|\partial_t^2 + L_x\right)\right) \\ &=: \mathcal{A}_1 + \mathcal{A}_2. \end{aligned}$$

The system above can now be written

$$\partial_z \mathbf{u}_\lambda = \begin{pmatrix} 0 & \frac{\rho}{|g|g^{nn}} \\ -\mathcal{A}_1 & 0 \end{pmatrix} \mathbf{u}_\lambda - \begin{pmatrix} 0 \\ \mathcal{A}_2 \widetilde{u}_1^{1,\lambda} + \kappa|g| \left[\Box_g, \varphi^1(z, x, D_t, D_x)\right] \widetilde{u}_2^{2,\lambda} + R_1 \widetilde{u}_\lambda + R_2 u_\lambda + \widetilde{f}_\lambda^1 \end{pmatrix}. \tag{4.3}$$

Note that we have defined $\mathcal{A}_2$ and $\widetilde{u}_1^{1,\lambda}$ so that $\mathcal{A}_2 \widetilde{u}_1^{1,\lambda}$ is equivalent to a pseudodifferential operator in $\Psi^{-\infty}(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$, depending on the parameter z, applied to $\widetilde{u}_\lambda$. In this section when we say that a pseudodifferential operator is in $\Psi^m(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$ for some m we will generally be allowing for dependence of the operator on the parameter z. Also, we have intentionally defined $\mathcal{A}_1$ so that the fourth root of its principal symbol is again a symbol, and can be inverted on the set $\left\{\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2} < \theta_1\right\} \cap \left\{|\omega| > \frac{1}{2}\right\}$. Taking this into consideration we define the symbol

$$b(z, x, \omega, \xi) := \sqrt{\frac{\omega}{c}}\ (g^{nn})^{1/4}\ \psi_1\,(2\omega)\ \varphi_1^0\left(\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)\left(1 - \frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^{1/4}$$

and use the notation b^{-1} for a symbol of the microlocal inverse of $Op_{\mathcal{S}}(b)$ on the set given above. Also we will write

$$a(z,x,\omega,\xi) := \left(1 - \frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^{1/2}.$$

Now it is possible, using methods described for example in [22], to find a 2×2 matrix Λ of formally symmetric ΨDOs with principal symbols given by

$$\sigma_p(\Lambda) = \frac{1}{2}\begin{pmatrix} b & -\mathrm{i}\, b^{-1} \\ b & \mathrm{i}\, b^{-1}\end{pmatrix}$$

such that $\mathbf{v} = \Lambda \mathbf{u}$ satisfies the system

$$\partial_z \mathbf{v} = \begin{pmatrix} \mathrm{i}\mathcal{A}^{1/2} & 0 \\ 0 & -\mathrm{i}\mathcal{A}^{1/2}\end{pmatrix}\mathbf{v} - \Lambda\begin{pmatrix} 0 \\ \mathcal{A}_2 \widetilde{u}_1^{\,1,\lambda} + \kappa|g|\left[\Box_g, \varphi^1(z,x,D_t,D_x)\right]\widetilde{u}_2^{\,2,\lambda} + R_1\widetilde{u}_\lambda + R_2 u_\lambda + \widetilde{f}_\lambda^{\,1}\end{pmatrix} + \mathcal{R}\mathbf{u} \tag{4.4}$$

where $\mathcal{A}^{1/2} \in \Psi^1(\mathbb{R}_t \times \mathbb{R}^{n-1}_x)$ has principal and subprincipal symbols given by

$$\begin{aligned}\psi_1\,(2\omega)^2\,\varphi_1^0\left(\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^2 &\left(\frac{\omega}{c}\,(g^{nn})^{-1/2}\,a + \frac{1}{2a}\frac{\partial}{\partial x^l}\left(\frac{c}{(g^{nn})^{1/2}}g^{jl}\right) i\,\frac{\xi_j}{\omega}\right. \\ &\left. + \frac{i}{4a^3}\frac{c}{(g^{nn})^{1/2}}g^{p'l'}\frac{\partial}{\partial x^{l'}}\left(c^2 g^{j'k'}\right)\frac{\xi_{p'}\xi_{j'}\xi_{k'}}{\omega^3}\right) + r(z,x,\omega,\xi).\end{aligned} \tag{4.5}$$

Here $r \in S^0$ has support contained in the set $\left\{\theta_1 \le \frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2} \le \theta_1^{1/2}\right\} \cup \left\{\frac14 \le |\omega| \le \frac12\right\}$. The formula (4.5) can be calculated by first noting that the principal and subprincipal symbol of $\mathcal{A}^{1/2}$ should be the same as the principal and subprincipal symbols of

$$\frac{1}{2}\left(Op_{\mathcal{S}}\left(b^{-1}\right)\mathcal{A}_1\,Op_{\mathcal{S}}\left(b^{-1}\right) + Op_{\mathcal{S}}(b)\,\frac{\rho}{|g|g^{nn}}\,Op_{\mathcal{S}}(b)\right).$$

The principal symbol of the operator in this equation is straightforward to calculate, and for the subprincipal symbol we note that the operator is formally symmetric and the subprincipal symbol is pure imaginary which determines the subprincipal symbol by a (relatively) simple formula.

The operator $\mathcal{R}$ in the remainder term in (4.4) can be taken to be a matrix of pseudodifferential operators in the following respective spaces

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_{11} \in \Psi^{-3/2}(\mathbb{R}_t \times \mathbb{R}^{n-1}_x) & \mathcal{R}_{12} \in \Psi^{-5/2}(\mathbb{R}_t \times \mathbb{R}^{n-1}_x) \\ \mathcal{R}_{21} \in \Psi^{-3/2}(\mathbb{R}_t \times \mathbb{R}^{n-1}_x) & \mathcal{R}_{22} \in \Psi^{-5/2}(\mathbb{R}_t \times \mathbb{R}^{n-1}_x)\end{pmatrix}$$

all of which depend on z uniformly in finite intervals. In fact the decoupling can be done to higher order, but this is all we require.

To approximate the wave field u_λ we replace the previous equation by the following two completely decoupled equations to approximate the components of $\mathbf{v}$. Indeed for $z \ge 0$, let v^{up} be defined by

$$\left(\partial_z + \mathrm{i}\mathcal{A}^{1/2}\right)v^{up} = 0, \quad \text{and} \quad v^{up}(0) = -\frac{\mathrm{i}}{2}b^{-1}(0,x,D_t,D_x)\kappa|g|\varphi^1(0,x,D_t,D_x)\phi\chi_\lambda \tag{4.6}$$

and $v^{up} = 0$ for $z < 0$. Similarly, for $z \le 0$ we define z^{do} by

$$\left(\partial_z - \mathrm{i}\mathcal{A}^{1/2}\right)v^{do} = 0, \quad \text{and} \quad v^{do}(0) = \frac{\mathrm{i}}{2}b^{-1}(0,x,D_t,D_x)\kappa|g|\varphi^1(0,x,D_t,D_x)\phi\chi_\lambda, \tag{4.7}$$

and set $v^{do} = 0$ for $z > 0$. The final approximation for u_λ is then given by

$$u_a = b^{-1}(z,x,D_t,D_x)v^{up} + b^{-1}(z,x,D_t,D_x)v^{do}. \tag{4.8}$$

The main result of this section is the following theorem.

THEOREM 4.1. *Suppose that, for given Z and $\delta, \epsilon > 0$, no null bicharacteristic for $\Box_g$ beginning in the set*

$$\left\{|z| < \epsilon, |\zeta| > \delta|\xi|, t \in \operatorname{supp}(\widetilde{\phi})\right\}$$

enters the set

$$\left\{z \in I_Z, t \in \operatorname{supp}(\widetilde{\phi}), \frac{g^{j'k'}\xi_{j'}\xi_{k'}}{g^{nn}\zeta^2 + g^{j'k'}\xi_{j'}\xi_{k'}} > \Theta\right\}$$

where $I_Z = (-Z, Z)$ and Θ is a positive constant less than one that depends on θ_1. Then provided that

$$Z' = Z - \frac{\|c\|_{C^0}}{(1-\theta_1^{1/2})^{1/2}}(\epsilon' + T - t_0) > 0$$

for some small $\epsilon' > 0$ we have the following estimates

$$\|u_\lambda\|_{L^2((t_0,T)\times I_{Z'}\times\mathbb{R}_x^{n-1})} \leq C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})} \tag{4.9}$$

and

$$\|u_\lambda - u_a\|_{L^2(I_{Z'};H^1((t_0,T)\times\mathbb{R}_x^{n-1}))} \leq C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}. \tag{4.10}$$

The hypothesis about the ray geometry simply requires that there are no turning rays which are close to perpendicular to the plane $\{z = 0\}$. This encapsulates the intuitive notion that the wavefronts must be moving only in the positive and negative z directions for the directional decoupling to be possible. The constant Θ also goes to 1 as θ_1 goes to 1. Thus by adjusting the constant θ_1 we may allow rays as close as we like to turning. We obtain the estimates over $I_{Z'}$ rather than I_Z since it is possible that energy could propagate out of the the region corresponding to I_Z, and then turn and reenter the region within the time interval (t_0, T) and therefore we must assume there are no turning rays on a larger depth interval then that on which we have our estimates. This value of Z' could actually be improved by applying finite speed of propagation to the original equation (2.2).

The remainder of the section is dedicated to the proof of Theorem 4.1 which we break into several steps starting with the following lemma that provides estimates for $\widetilde{u}_2^{\widetilde{k},\lambda}$.

LEMMA 4.2. *Suppose the hypotheses of Theorem 4.1 are satisfied. Then for $\widetilde{k} = 1$ or 2*

$$\|\widetilde{u}_2^{\widetilde{k},\lambda}\|_{H^1((t_0,T)_t\times I_Z\times\mathbb{R}_x^{n-1})} \leq C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}.$$

Proof. The first thing to note is that we may easily simplify by cutting out low frequencies to estimating

$$\|\psi_1^{1/\widetilde{k}}(D_t/4)\widetilde{u}_2^{\widetilde{k},\lambda}\|_{H^1((t_0,T)_t\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})}.$$

This simplification is possible working from the original equation (2.2), estimating $\psi_1^{1/\widetilde{k}}(D_t/4)f_\lambda$, and then applying the energy estimates (2.3) and the continuity of ΨDOs.

Now let

$$\widetilde{\varphi}^{\widetilde{k}} = \psi_1^{\widetilde{k}}\left(\frac{\omega^2 - c^2g^{nn}\zeta^2 - c^2g^{j'k'}\xi_{j'}\xi_{k'}}{1 + \omega^2 + c^2g^{nn}\zeta^2 + c^2g^{j'k'}\xi_{j'}\xi_{k'}}\right).$$

Using standard methods from microlocal analysis we may find an operator $\Box_g^{-1} \in \Psi^{-2}(\mathbb{R}_t\times\mathbb{R}_z\times\mathbb{R}_x^{n-1})$ such that

$$\Box_g^{-1}\Box_g = \widetilde{\varphi}^{\widetilde{k}}(z, x, D_t, D_z, D_x) + R$$

where $R \in \Psi^{-\infty}(\mathbb{R}_t \times \mathbb{R}_z \times \mathbb{R}^{n-1}_x)$. Thus applying $\square_g^{-1}$ to (4.1) we have

$$\widetilde{\varphi}^{\widetilde{k}}(z, x, D_t, D_z, D_x)\widetilde{u}_\lambda = \square_g^{-1}\left(-2(\partial_t\widetilde{\phi})\partial_t u_\lambda - (\partial_t^2\widetilde{\phi})u_\lambda + f_\lambda\right) - R\widetilde{u}_\lambda.$$

Since the support of $\partial_t\widetilde{\phi}$ is disjoint from (t_0, T) this last formula leads to an estimate

$$\|\widetilde{\varphi}^{\widetilde{k}}(z, x, D_t, D_z, D_x)\widetilde{u}_\lambda\|_{H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}^{n-1}_x)} \leq C(\|u_\lambda\|_{H^{-1}((t_0,T)\times\mathbb{R}_z\times\mathbb{R}^{n-1}_x)} + \|f_\lambda\|_{H^{-1}((t_0,T)\times\mathbb{R}_z\times\mathbb{R}^{n-1}_x)}).$$

Applying Lemma 3.4 and making a straight forward estimate of the norm of f_λ then leads to

$$\|\widetilde{\varphi}^{\widetilde{k}}(z, x, D_t, D_z, D_x)\widetilde{u}_\lambda\|_{H^1((t_0,T)\times\mathbb{R}_z\times\mathbb{R}^{n-1}_x)} \leq C\lambda^{-1}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}^{n-1}_x)}.$$

Thus the proof reduces to estimating

$$\psi_1^{1/\widetilde{k}}(2D_t)\,(1 - \varphi^{\widetilde{k}}(z, x, D_t, D_x))\,(1 - \widetilde{\varphi}^{\widetilde{k}}(z, x, D_t, D_z, D_x))\,\widetilde{u}_\lambda.$$

Carefully analyzing the support of the three ΨDOs in the previous formula we find that modulo a ΨDO of arbitrarily low order the composition has symbol with support contained in a set of the form

$$\left\{\frac{g^{j'k'}\xi_{j'}\xi_{k'}}{g^{nn}\zeta^2 + g^{j'k'}\xi_{j'}\xi_{k'}} \geq \Theta\right\}$$

for a constant $\Theta < 1$ depending on θ_1. Therefore we may precompose the three operators by an operator in $\Psi^0(\mathbb{R}_z \times \mathbb{R}^{n-1}_x)$ with symbol having support contained in a slightly larger set than the one above, and apply Theorem 3.1 using the hypothesis about the ray geometry to complete the proof. □

For the remainder of the section we will write v^1 and v^2 for the two components of $\mathbf{v}$, and introduce the notation $E^{up}(z, z')$ and $E^{do}(z, z')$ for the respective solution operators of the evolution equations in (4.6) and (4.7). The next result concerns properties of v^1 and v^2.

Lemma 4.3. *Suppose the hypotheses of Theorem 4.1 are satisfied. Then for $z \in I_{Z'}$*

$$v^1(\cdot_t, z, \cdot_x), v^2(\cdot_t, z, \cdot_x) = O_{\|\cdot\|_{H^{-1/2}}}\left(\lambda^{-1/4}\|\chi_\lambda\|_{L^2}\right).$$

Also, v^1 and v^2 satisfy the following jump conditions at $z = 0$:

$$v^1(t, 0^-, x) = -\frac{\mathrm{i}}{2}b^{-1}(0, x, D_t, D_x)\,[\kappa|g|\phi\chi_\lambda] + O_{\|\cdot\|_{H^{1/2}}}\left(\lambda^{-1/4}\|\chi_\lambda\|_{L^2}\right),\ \|v^1(t, 0^+, x)\|_{H^{1/2}} \leq \lambda^{-1/4}\|\chi_\lambda\|_{L^2},$$

$$v^2(t, 0^+, x) = \frac{\mathrm{i}}{2}b^{-1}(0, x, D_t, D_x)\,[\kappa|g|\phi\chi_\lambda] + O_{\|\cdot\|_{H^{1/2}}}\left(\lambda^{-1/4}\|\chi_\lambda\|_{L^2}\right),\ \|v^2(t, 0^-, x)\|_{H^{1/2}} \leq \lambda^{-1/4}\|\chi_\lambda\|_{L^2}.$$

The notation $O_{\|\cdot\|}$ refers to a function depending on λ whose $\|\cdot\|$ norm may be bounded by the given quantity as $\lambda \to \infty$. Also $H^{\pm 1/2} = H^{\pm 1/2}((t_0, T) \times \mathbb{R}^{n-1}_x)$ while $L^2 = L^2(\mathbb{R}_t \times \mathbb{R}^{n-1}_x)$.

Proof. To prove the lemma for v^2 we note that for $z_1 \leq -Z$ and $z \in I_{Z'}$ we have from (4.4)

$$\begin{aligned} v^2(t, z, x) = &\int_{z_1-1}^{z_1} E^{up}(z, z'')v^2(\cdot_t, z'', \cdot_x)\mathrm{d}z'' \\ &- \int_{z_1-1}^{z_1}\int_{z''}^{z} E^{up}(z, z')\Big(\Lambda_{22}\left(\mathcal{A}_2\widetilde{u}_1^{1,\lambda} + \kappa|g|\left[\square_g, \varphi^1(z, x, D_t, D_x)\right]\widetilde{u}_2^{2,\lambda} + R_1\widetilde{u}_\lambda + R_2 u_\lambda\right) \\ &- \mathcal{R}_{21}\widetilde{u}_1^{1,\lambda} - \mathcal{R}_{22}\frac{|g|g^{nn}}{\rho}\partial_z\widetilde{u}_1^{1,\lambda}\Big)\mathrm{d}z'\mathrm{d}z'' + H(z)E^{up}(z, 0)\Lambda_{22}\kappa|g|\varphi^1(z, x, D_t, D_x)\phi\chi_\lambda. \end{aligned} \tag{4.11}$$

Here $H(z)$ is the Heaviside function.

We will estimate each term in (4.11) separately beginning with

$$\int_{z_1-1}^{z_1}\int_{z''}^{z} E^{up}(z,z')\Lambda_{22}\mathcal{A}_2\widetilde{u}_1^{1,\lambda}\,\mathrm{d}z'\mathrm{d}z''.$$

As mentioned above, $\mathcal{A}_2 u_1^{1,\lambda}$ is equivalent to an operator in $\Psi^{-\infty}(\mathbb{R}_t\times\mathbb{R}_x^{n-1})$ applied to u_λ. Therefore the $H^{\pm 1/2}$ norm of the above term can be estimated by $\|\widetilde{u}_\lambda\|_{H^s((\widetilde{t}_0,\widetilde{T});L^2(\mathbb{R}_z\times\mathbb{R}_x^{n-1})}$ for any $s<0$ and an interval $(\widetilde{t}_0,\widetilde{T})$ which is slightly larger than (t_0,T). Thus by Lemma 3.4 this term is bounded by $C\lambda^s\|\chi_\lambda\|_{L^2}$ which is certainly sufficient for the proof. The terms including $R_1\widetilde{u}_\lambda$ and $\mathcal{R}_{21}\widetilde{u}_1^{1,\lambda}$ are bounded in essentially the same way

Next we move to the term

$$\int_{z_1-1}^{z_1}\int_{z''}^{z} E^{up}(z,z')\mathcal{R}_{22}\frac{|g|g^{nn}}{\rho}\partial_z\widetilde{u}_1^{1,\lambda}\,\mathrm{d}z'\mathrm{d}z''.$$

Using the continuity of $\mathcal{R}_{22}\frac{|g|}{\rho}g^{nn}\in\Psi^{-5/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1})$ this may be estimated in $H^{\pm1/2}$ by

$$C\left\|\partial_z\widetilde{u}_1^{1,\lambda}\right\|_{L^2(I_Z;H^{-5/2\pm1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))}.$$

To complete this estimate we decompose $\widetilde{u}_1^{1,\lambda}$ into two components similar to how $\widetilde{u}_2^{\widetilde{k},\lambda}$ was decomposed in the proof of Lemma 4.2. Indeed, on the component away from the characteristic set of $\Box_g$ we can apply a parametrix, and near the characteristic set the z derivative may be estimated by a t derivative and so the last expression may be bounded by

$$C\left\|\widetilde{u}_1^{1,\lambda}\right\|_{L^2(I_Z;H^{-3/2\pm1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))}.$$

Now Lemma 3.4 can be used to complete the estimate of this term.

Now we deal with the term

$$\int_{z_1-1}^{z_1}\int_{z''}^{z}\Lambda_{22}\kappa|g|\left[\Box_g,\varphi^1(z,x,D_t,D_x)\right]\widetilde{u}_2^{2,\lambda}\,\mathrm{d}z'\mathrm{d}z''.$$

By the continuity of ΨDOs the $H^{\pm1/2}$ norm of this term is bounded by

$$\left\|\left[\Box_g,\varphi^1(z,x,D_t,D_x)\right]\widetilde{u}_2^{2,\lambda}\right\|_{L^2(I_Z;H^{-1/2\pm1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))}\le C\left\|\widetilde{u}_2^{2,\lambda}\right\|_{H^1(I_Z\times\mathbb{R}_t\times\mathbb{R}_x^{n-1})}.$$

This quantity is in turn bounded by Lemma 4.2.

Next is the term

$$H(z)E^{up}(z,0)\Lambda_{22}\kappa|g|\varphi^1(z,x,D_t,D_x)\phi\chi_\lambda$$

which is where the difference between the $H^{\pm1/2}$ cases lies. Indeed, since the principal symbol of Λ_{22} is $\frac{\mathrm{i}}{2}b^{-1}$, elementary estimates of the norm of χ_λ making use of the continuity of the relevant ΨDOs shows that in the $H^{1/2}$ case we must subtract precisely the terms appearing in the jump condition formula in the theorem, and in the $H^{-1/2}$ case the terms can simply be estimated as required.

We finally deal with the terms

$$\text{(4.12)}\qquad \int_{z_1-1}^{z_1}E^{up}(z,z'')v^2(\cdot_t,z'',\cdot_x)\mathrm{d}z'' \quad\text{and}\quad \int_{z_1-1}^{z_1}\int_{z''}^{z}E^{up}(z,z')\Lambda_{22}R_2u_\lambda\mathrm{d}z'\mathrm{d}z''.$$

The idea here is to localize in time to an interval slightly larger than (t_0,T), and then show using Egorov's theorem that modulo lower order terms these terms only depend on u_λ for $t<t_0$ where it is identically equal to zero. To do this, let $\widetilde{\phi}_2\in C_c^\infty(\mathbb{R}_t)$ have support contained in the set $\{\widetilde{\phi}=1\}$ and equal to one on $[t_0,T]$, and define $t_m=\sup(\mathrm{supp}(\widetilde{\phi}_2))$. If we carefully analyze the t component of the Hamiltonian flow of $\sigma_p\left(-\mathcal{A}^{1/2}\right)$ then we see that by Egorov's theorem

$$\widetilde{\phi}_{z,z''}:=E^{up}(z'',z)\,\widetilde{\phi}_2\,E^{up}(z,z'')=A(t,x,D_t,D_x)+r(z,z'',t,x,D_t,D_x)$$

where for fixed z and z'', $r(z, z'', t, x, D_t, D_x) \in \Psi^{-2}(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$ and

$$\operatorname{supp}(A) \subset \left\{ t \leq \frac{(1-\theta_1^{1/2})^{1/2}}{\|c\|_{C^0}}(z''-z) + t_m \right\}.$$

Choosing $\widetilde{\phi}_2$ such that t_m is close enough to T we have

$$z_1 \leq -Z = -Z' + \frac{\|c\|_{C^0}}{(1-\theta_1^{1/2})^{1/2}}(t_0 - T - \epsilon) \leq z + \frac{\|c\|_{C^0}}{(1-\theta_1^{1/2})^{1/2}}(t_0 - t_m)$$

which implies

$$\operatorname{supp}(A) \subset \{t \leq t_0\}.$$

Now, if we multiply (4.12) by $\widetilde{\phi}_2$, which does not change the $H^{\pm 1/2}$ norm, then we see that two terms in (4.12) may be bounded respectively by

$$C\|[\widetilde{\phi}_{z,z''} v^2](t, z'', x)\|_{L^2((z_1-1,z_1)_{z''}; H^{\pm 1/2}(\mathbb{R}_t \times \mathbb{R}_x^{n-1}))}$$

and

$$C\|[\widetilde{\phi}_{z,z''} \Lambda_{22} R_2 u_\lambda](t, z'', x)\|_{L^2((z_1-1,z_1)_{z''}; H^{\pm 1/2}(\mathbb{R}_t \times \mathbb{R}_x^{n-1}))}.$$

Estimates for both of these terms can now be found by applying the calculus of ΨDOs and analyzing the support of the result applied to u_λ which is identically equal to zero for $t \leq t_0$. We also apply Lemma 3.4 here. This completes the proof for v^2, and for v^1 the proof proceeds in the same manner with only a few changes of sign and E^{up} replaced by E^{do}. □

Then Lemma 4.3 can be used to bound the first two terms and Lemma 3.4 to bound the third one.
The next lemma brings us closer to the proof of Theorem 4.1.

Lemma 4.4. *Suppose the hypotheses of Theorem 4.1 are satisfied. Then we have the following estimates*

$$\|\mathbf{v}\|_{L^2(I_{Z'}; H^{-1/2}(\mathbb{R}_t \times \mathbb{R}_x^{n-1}))} \leq C\lambda^{-1/4} \|\chi_\lambda\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})}$$

and

$$\left\| \mathbf{v} - \begin{pmatrix} v^{do} \\ v^{up} \end{pmatrix} \right\|_{L^2(I_{Z'}; H^{1/2}(\mathbb{R}_t \times \mathbb{R}_x^{n-1}))} \leq C\lambda^{-1/4} \|\chi_\lambda\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^{n-1})}.$$

Proof. As in the proof of Lemma 4.3, the estimates for the two components are obtained in almost identical ways, and so we only consider the case of $v^2 - v^{up}$. From (4.4) and (4.6) we have on $\{z \neq 0\}$ the following equation

$$\begin{aligned} \left(\partial_z + \mathrm{i}\mathcal{A}^{1/2}\right)(v^2 - v^{up}) = & -\Lambda_{22}\left(\mathcal{A}_2 \widetilde{u}_1^{1,\lambda} + \kappa|g|\left[\Box_g, \varphi^1(z, x, D_t, D_x)\right]\widetilde{u}_2^{2,\lambda} + R_1 \widetilde{u}_\lambda + R_2 u_\lambda\right) \\ & - \mathcal{R}_{21} \widetilde{u}_1^{1,\lambda} - \mathcal{R}_{22} \frac{|g| g^{nn}}{\rho} \partial_z \widetilde{u}_1^{1,\lambda} \end{aligned} \tag{4.13}$$

The proof now follows by applying energy estimates to this equation for $z > 0$ using the results of Lemma 4.3 for the required bounds of the initial conditions at $z = 0$. The terms coming from the source in the energy estimate are bounded in the same way as in the proof of Lemma 4.3. □

Now we can use Lemmas 4.2 and 4.4 to prove (4.9) and (4.10). First Lemma 4.2 shows that it is sufficient to estimate respectively $\widetilde{u}_1^{1,\lambda}$ and $\widetilde{u}_1^{1,\lambda} - u_a$. Applying a paramterix V for Λ to $\mathbf{v} = \Lambda \mathbf{u}$ gives

$$\widetilde{u}_1^{1,\lambda} = V_{11} v^1 + V_{12} v^2 - \widetilde{R}_1 \widetilde{u}_1^{1,\lambda} - \widetilde{R}_2 \partial_z \widetilde{u}_1^{1,\lambda}.$$

where $V_{11}, V_{12} \in \Psi^{-1/2}(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$ and $\widetilde{R}_1, \widetilde{R}_2 \in \Psi^{-\infty}(\mathbb{R}_t \times \mathbb{R}_x^{n-1})$ all depend uniformly on $z \in I_{Z'}$. Therefore, using the same techniques as in the proof of Lemma 4.2 to estimate the remainder terms

$$\|\widetilde{u}_1^{1,\lambda}\|_{L^2((t_0,T)\times I_{Z'}\times\mathbb{R}_x^{n-1})} \leq C\left(\|v^1\|_{L^2(I_{Z'};H^{-1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))} + \|v^2\|_{L^2(I_{Z'};H^{-1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))} + \|\widetilde{u}\|_{H^{-1}((\widetilde{t_0},\widetilde{T});L^2(\mathbb{R}_z\times\mathbb{R}_x^{n-1}))}\right).$$

Then Lemma 4.4 can be used to bound the first two terms and Lemma 3.4 to bound the third one. This proves (4.9). For the second part we have

$$\widetilde{u}_1^{1,\lambda} - u_a = \mathcal{A}^{-1/4}(v^1 - v^{do}) + \mathcal{A}^{-1/4}(v^2 - v^{up}) + V_{11}^s v^1 + V_{12}^s v^2 - \widetilde{R}_1\widetilde{u}_1^{1,\lambda} - \widetilde{R}_2\partial_z\widetilde{u}_1^{1,\lambda}$$

where V_{11}^s and $V_{12}^s \in \Psi^{-3/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1})$. The result now follows as in the previous case using also Lemma (4.4) to bound the initial terms including the differences $v^1 - v^{do}$ and $v^2 - v^{up}$. This completes the proof of Theorem 4.1.

5. Paraxial equations. Our main intention in this section is to show that if we make a particular choice of coordinates we can replace the evolution equations (4.6) and (4.7) by another pair of differential equations which we will introduce shortly. The requirement for the coordinates is that $g^{nn} = c^{-2}$. This can be accomplished by using boundary normal coordinates with respect to $z = 0$ for the metric $c^{-2}\mathbf{e}$ where $\mathbf{e}$ is the Euclidean metric. Of course if there are caustics then this is not possible globally, but the results still hold up to the value of z at which caustics form. In light of this we will assume that $g^{nn} = c^{-2}$ for this section.

First we must define a family of microlocal cutoffs depending on λ, k, and $1 \leq \alpha \leq 2$ with symbols $\tau_{\lambda,k}^\alpha \in S^0_{(3-\alpha)/2,0}((\mathbb{R}_t\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))$ that will play a role in the initial conditions for these equations and some of the analysis. The symbols are defined by

$$\tau_{\lambda,k}^\alpha(\omega,\xi) = \varphi_1^1\left(\frac{\theta_1}{k}\frac{|\xi|^\alpha}{|\omega|}\right)\psi_1^1\left(\frac{1}{\lambda}\omega\right)$$

and we will write

$$B_{\lambda/2,k}^\alpha := \left\{\frac{|\xi|^\alpha}{|\omega|} \leq k\right\}\bigcap\left\{|\omega| \geq \frac{\lambda}{2}\right\}$$

for a set which contains the support of $\tau_{\lambda,k}^\alpha$. Then (4.6) and (4.7) may be replaced by

$$\left(\partial_z + \mathrm{i}\,\omega + \mathrm{i}\frac{1}{2\omega}\frac{\partial}{\partial x^{j'}}c^2 g^{j'k'}\frac{\partial}{\partial x^{k'}}\right)v_\omega^{up} = 0, \tag{5.1}$$

$$v_\omega^{up}(0,\cdot) = -\frac{\mathrm{i}}{2}\mathcal{F}_t\left[\tau_{\lambda,k}^2(D_t,D_x)b^{-1}(0,x,D_t,D_x)\kappa|g|\varphi^1(0,x,D_t,D_x)\phi\chi_\lambda\right], \tag{5.2}$$

and

$$\left(\partial_z - \mathrm{i}\,\omega - \mathrm{i}\frac{1}{2\omega}\frac{\partial}{\partial x^{j'}}c^2 g^{j'k'}\frac{\partial}{\partial x^{k'}}\right)v_\omega^{do} = 0, \tag{5.3}$$

$$v_\omega^{do}(0,\cdot) = \frac{\mathrm{i}}{2}\mathcal{F}_t\left[\tau_{\lambda,k}^2(D_t,D_x)b^{-1}(0,x,D_t,D_x)\kappa|g|\varphi^1(0,x,D_t,D_x)\phi\chi_\lambda\right]. \tag{5.4}$$

For notational convenience we write

$$v_0^{do} := \frac{\mathrm{i}}{2}\tau_{\lambda,k}^2(D_t,D_x)b^{-1}(0,x,D_t,D_x)\kappa|g|\varphi^1(0,x,D_t,D_x)\phi\chi_\lambda =: -v_0^{up}$$

and

$$L_\omega = \omega + \frac{1}{2\omega}\frac{\partial}{\partial x^{j'}}c^2 g^{j'k'}\frac{\partial}{\partial x^{k'}}.$$

We comment that the equations (5.2) and (5.4) for the initial conditions appear quite complicated, but in practice are actually fairly simple. If the material parameters and g are constant on $z = 0$ then at sufficiently high frequencies in fact v_0^{do} simplifies to $\frac{\mathrm{i}}{2} b^{-1}(0, x, D_t, D_x)\phi\chi_\lambda$.

The main result of this section is the following theorem.

THEOREM 5.1. *For every ω there are unique solutions of (5.1) and (5.3) with the given initial conditions. There is a constant $Z > 0$ such that these solutions may be used to approximate v^{up} and v^{do} in the sense*

$$\left\| v^{up} - \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega\cdot_t} v_\omega^{up}(z,\cdot_x)\,\mathrm{d}\omega \right\|_{C^0([0,Z];H^{1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})} \tag{5.5}$$

and

$$\left\| v^{do} - \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega\cdot_t} v_\omega^{do}(z,\cdot_x)\,\mathrm{d}\omega \right\|_{C^0([-Z,0];H^{1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}. \tag{5.6}$$

REMARK 2. *By combining Theorem 5.1 with a slight extension of the arguments at the end of Section 4 we can further obtain estimates*

$$\left\| u - \mathcal{A}^{-1/4}\frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega\cdot_t} v_\omega^{up}(z,\cdot_x)\,\mathrm{d}\omega \right\|_{L^2((0,Z);H^1((t_0,T)\times\mathbb{R}_x^{n-1}))} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})} \tag{5.7}$$

and

$$\left\| u - \mathcal{A}^{-1/4}\frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega\cdot_t} v_\omega^{do}(z,\cdot_x)\,\mathrm{d}\omega \right\|_{L^2((-Z,0);H^1((t_0,T)\times\mathbb{R}_x^{n-1}))} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})} \tag{5.8}$$

which relate the solutions of the paraxial equations to the actual wave field. Here Z should be the lesser of the Z in Theorem 5.1 and the Z' in Theorem 4.1

REMARK 3. *In fact it is possible to get a better rate of decay than $\lambda^{-1/4}$ in the estimates (5.5) and (5.6) without seriously modifying the proof. However, in light of Theorem 4.1 we cannot improve, or at least not with our methods, the decay rates in (5.7) and (5.8) which is why we only state and prove Theorem 5.1 with decay rate $\lambda^{-1/4}$.*

Proof. We first comment that the existence of solutions to (5.1) and (5.3) in $L^2((0,Z);H^1(\mathbb{R}_x^{n-1}))$ with initial data at $z = 0$ in $L^2(\mathbb{R}_x^{n-1})$ can be shown by Galerkin's method, and uniqueness follows from (for v a solution of (5.1) or (5.3))

$$\frac{\mathrm{d}}{\mathrm{d}z}\|v\|^2_{L^2(\mathbb{R}_x^{n-1})} = \pm\mathrm{Re}\left(\mathrm{i}\int\left(\omega|v|^2 + \frac{c^2}{2\omega}g^{j'k'}\frac{\partial v}{\partial x^{j'}}\frac{\partial\overline{v}}{\partial x^{k'}}\right)\mathrm{d}x\right) = 0.$$

We write $S^{up}(z,z')$ and $S^{do}(z,z')$ for the respective solution operators of (5.1) and (5.3) and the last formula shows that the $L^2(\mathbb{R}_x^{n-1})$ norm is preserved by these solution operators.

To begin the rest of the proof of Theorem 5.1 we first note that we may replace $v^{up}(0)$ and $v^{do}(0)$ in (4.6) and (4.7) by v_0^{up} and v_0^{do} and in so doing incur only an error that may be bounded in the $H^{1/2}$ norm by $C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}$. Indeed, considering only v^{up}, we have

$$v^{up}(0) - v_0^{up} = -\frac{\mathrm{i}}{2}\left(1-\tau^\alpha_{\lambda,k}(D_t,D_x)\right)b^{-1}(0,x,D_t,D_x)\kappa|g|\varphi^1(0,x,D_t,D_x)\phi\chi_\lambda.$$

Analyzing the support of the symbols of the ΨDOs which form the composition on the right hand side of this equation and then using the continuity of the resulting ΨDO this shows

$$\|v^{up}(0) - v_0^{up}\|_{H^{1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1})} \le C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}$$

and using an energy estimate this can be extended to the difference between v^{up} and the solution of (4.6) with $v^{up}(0)$ replaced by v_0^{up}.

The principal and subprincipal symbols of $\mathcal{A}^{1/2}$ are give by (4.5). Expanding the respective powers of a in each of the terms about $c^2 g^{j'k'}\xi_{j'}\xi_{k'}/\omega^2 = 0$ we get

$$\mathcal{A}^{1/2} = \psi_1\,(2\omega)^2\;\varphi_1^0\left(\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^2\left(\omega - c^2\frac{g^{j'k'}\xi_{j'}\xi_{k'}}{2\omega} + \frac{1}{2\omega}\frac{\partial}{\partial x^{j'}}\left(c^2 g^{j'k'}\right) i\xi_{k'}\right) + \widetilde{r} + order(-1)$$

where

$$\begin{aligned}\widetilde{r} = &\left(c^4\left(\frac{(g^{j'k'}\xi_{j'}\xi_{k'})^2}{\omega^3}\right)\int_0^1 \frac{s-1}{4}\left(1 - c^2\frac{g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}s\right)^{-3/2}\mathrm{d}s\right.\\ &+ \frac{\mathrm{i}}{4}c^2 g^{pq}\frac{\partial}{\partial x^l}\left(c^2 g^{jl}\right)\frac{\xi_p\xi_q\xi_j}{\omega^3}\int_0^1\left(1 - c^2\frac{g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}s\right)^{-3/2}\mathrm{d}s\\ &\left.+ \frac{\mathrm{i}}{4a^3}c^2 g^{p'l'}\frac{\partial}{\partial x^{l'}}\left(c^2 g^{j'k'}\right)\frac{\xi_{p'}\xi_{j'}\xi_{k'}}{\omega^3}\right)\psi_1\,(2\omega)^2\;\varphi_1^0\left(\frac{c^2 g^{j'k'}\xi_{j'}\xi_{k'}}{\omega^2}\right)^2.\end{aligned} \tag{5.9}$$

As complicated as this expression may appear, in fact it is not difficult to see that it is a symbol of order 1 (note that the first term in parentheses is homogenous of order 1 in (ξ,ω) while the second and third are homogeneous of order 0). The reason for considering $\widetilde{r}$ is that, roughly speaking, when we restrict to a set of the form $B_{k,\lambda}$ for some k and λ, $\widetilde{r}$ is actually a symbol of order -1.

Motivated by the expansion in the previous paragraph let us consider the following where for the moment we will focus only on (5.5). For $1 \le \alpha < 2$ we note that $\tau^\alpha_{\lambda/2,2k}(D_t,D_x)v_0^{up} = v_0^{up}$, and so

$$\begin{aligned}&(\partial_z + \mathrm{i}\,L_\omega)(S^{up}(z,0)\mathcal{F}_t - \mathcal{F}_t\,E^{up}(z,0))\,v_0^{up}\\ &\quad= \mathrm{i}\left(\mathcal{F}_t\,\mathcal{A}^{1/2}\,\mathcal{F}_t^{-1} - L_\omega\right)\mathcal{F}_t\,E^{up}(z,0)\,\tau^\alpha_{\lambda/2,2k}(D_t,D_x)v_0^{up}\\ &\quad= \mathrm{i}\left(\mathcal{F}_t\,\mathcal{A}^{1/2}\,\mathcal{F}_t^{-1} - L_\omega\right)\mathcal{F}_t\,[E^{up}(z,0)\,\tau^\alpha_{\lambda/2,2k}(D_t,D_x)\,E^{up}(0,z)]\,E^{up}(z,0)\,v_0^{up}\end{aligned} \tag{5.10}$$

Now let Φ_z denote the Hamiltonian flow of Hamiltonian $\sigma_p\left(-\mathcal{A}^{1/2}\right)$. By a slight refinement of Egorov's theorem the operator

$$\mathcal{B} = [E^{up}(z,0)\,\tau^\alpha_{\lambda/2,2k}(D_t,D_x)\,E^{up}(0,z)] \in \Psi^0_{(3-\alpha)/2,(\alpha-1)/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1})$$

with symbol given by

$$b_0 + b_1$$

where $b_0(z,x,t,\xi,\omega) \in S^0_{(3-\alpha)/2,(\alpha-1)/2}((\mathbb{R}_t\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))$ has support contained in the set

$$\bigcup_{z'\le z}\Phi_{z'}\left(B^\alpha_{\lambda/4,2k}\right)$$

and $b_1 \in S^{-3}_{(3-\alpha)/2,(\alpha-1)/2}((\mathbb{R}_t\times\mathbb{R}_x^{n-1})\times(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))$.

Lemma 5.2. *For any k, and λ sufficiently large there is $Z>0$, k', and λ' such that for $z\in[0,Z]$*

$$\pi_{(\xi,\omega)}\circ\Phi_z((\mathbb{R}_t\times\mathbb{R}_x^{n-1})\times B^\alpha_{\lambda/4,2k}) \subset B^\alpha_{\lambda'/4,2k'}.$$

Proof. Suppose that $(t,x;\omega,\xi) \in (\mathbb{R}_t\times\mathbb{R}_x^{n-1})\times B^\alpha_{\lambda/4,2k}$ with $\lambda \ge 2$. Note that since $\sigma(-\mathcal{A}^{1/2})$ is homogeneous of degree 1 in (ω,ξ) for $|\omega| \ge 1/2$ and does not depend on t it is sufficient to show that when $|\xi_0| \le 2k|\omega|^{1-\alpha}$ then the ξ component, which we write as ξ_z, of $\Phi_z(t,x;\xi_0,1)$ satisfies $|\xi_z| \le 2k'|\omega|^{1-\alpha}$. In fact for z in a fixed finite interval and ω large enough (how large depends on the interval for z) then by analyzing the Hamiltonian flow we can show that $|\xi_z| \le C|\xi_0|$, which combined with the above comments is enough to complete the proof. □

Applying this lemma we can show that for $16/13 \leq \alpha < 2$

$$\mathrm{i}\left(\mathcal{A}^{1/2} - \mathcal{F}_t^{-1} L_\omega \mathcal{F}_t\right) \mathcal{B} \in \Psi^{1/4}_{(3-\alpha)/2,(\alpha-1)/2}.$$

In fact this result can be improved by restricting α further which is the reason for remark 3, but this is all that is required to obtain the $\lambda^{-1/4}$ rate of decay in the estimate (5.5). Therefore, noting that

$$\mathcal{F}_t^{-1}(S^{up}(z,0)\mathcal{F}_t - E^{up}(z,0))\, v_0^{up} = v^{up} - \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega t} v_\omega^{up}(z,x)\,\mathrm{d}\omega$$

and using (5.10), the fact that S^{up} preserves the L^2 norm, and the continuity of E^{up} we have for $z \in [0, Z]$

$$\left\| v^{up} - \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\omega\cdot_t} v_\omega^{up}(z,\cdot_x)\,\mathrm{d}\omega \right\|_{H^{1/2}(\mathbb{R}_t\times\mathbb{R}_x^{n-1}))} \leq C\|v_0^{up}\|_{H^{1/4}(\mathbb{R}_t\times\mathbb{R}_x^{n-1})} \leq C\lambda^{-1/4}\|\chi_\lambda\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^{n-1})}.$$

This completes the proof for the case of v^{up}, and the proof for v^{do} proceeds in the same manner. □

6. Conclusion. We summarize the main results of the paper, and indicate directions of future work. The first result is Theorem 3.1 which gives a microlocal energy estimate for solutions of the acoustic wave equation with a directional source localized in a plane. The result may appear somewhat limited because it is stated so that the directional orientation of the source must be perpendicular to the plane containing the source, however the result actually holds in arbitrary coordinate systems, and so by changing coordinates to orient the direction of the source perpendicular to the plane containing the source the result still holds in other cases.

The second result is Theorem 4.1 which under the hypothesis that there are no turning rays gives an H^1 estimate of the difference between the solutions of two decoupled hyperbolic evolution equations, combined in a certain way, and a wave field produced by a directional source once again localized in a plane. In this case the restriction that the source be oriented perpendicular to plane really does hold, and our proof does not allow us to simply avoid the problem of obliquely oriented sources by a change of coordinates. However this should still be possible, but the system (4.3) will be more complicated and in particular contain entries on the diagonal. This type of system may still be diagonalized, but we reserve finding estimates equivalent to those in Theorem 4.1 in that case for future work. Here is where we plan to use shearlets to represent orientations other than perpendicular to the plane, which is necessary if we wish to use solutions of the decoupled equations as preconditioners for other solvers. This will also be important for incorporating a scattering surface into our models.

The final result is Theorem 5.1 which shows how the wave field in the same situation as in Theorem 4.1 can also be approximated by solutions of decoupled paraxial equations (i.e. (5.5) and (5.6)). This provides a connection with some previous work on wave propagation in random media [8], and we plan to use Theorem 5.1 to generalize those results to the case when there are random fluctuations of a smoothly varying background medium.

We finally comment that we plan to extend these results to the elastic case.

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