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LOCAL INVERSE PROBLEMS: HÖLDER STABILITY AND ITERATIVE RECONSTRUCTION*

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Abstract. We consider a class of inverse problems defined by a nonlinear map from parameter or model functions to the data. We assume that solutions exist. The space of model functions is a Banach space which is smooth and uniformly convex; however, the data space can be an arbitrary Banach space. We study sequences of parameter functions generated by a nonlinear Landweber iteration and conditions under which these strongly converge, locally, to the solutions within an appropriate distance. We express the conditions for convergence in terms of Hölder stability of the inverse maps, which ties naturally to the analysis of inverse problems.

Key words. inverse problems, Landweber iteration, stability

AMS subject classifications. 86A15, 35R30

1. Introduction.

In this paper, we study the convergence of certain nonlinear iterative reconstruction methods for inverse problems in Banach spaces.

We consider a class of inverse problems defined by a nonlinear map from parameter or model functions to the data. The parameter functions and data are contained in certain Banach spaces, or Hilbert spaces. A stable reconstruction involves regularization techniques. We analyze these inverse problems locally, that is, in a neighborhood of their solutions if they exist. In particular, we explicitly construct sequences of parameter functions by a modified Landweber iteration. Our analysis pertains to obtaining natural conditions for the strong convergence of these sequences (locally) to the solutions in an appropriate distance associated with the relevant spaces.

Extensive research has been carried out to study convergence of the Landweber iteration [22] and its modifications. In the case of model and data Hilbert spaces, see Hanke, Neubauer & Scherzer [16]. An overview of iterative regularization methods for inverse problems in Hilbert spaces can be found, for example, in Engl, Hanke & Neubauer [15]. Schöpfer, Louis & Schuster [28] presented a nonlinear extension of the Landweber method to Banach spaces using duality mappings. We use this iterative method in the analysis presented here. Duality mappings also play a role in iterative schemes for monotone and accretive operators (see Alber [2], Chidume & Zegeye [12] and Zeidler [33, 34]). The model space needs to be smooth and uniformly convex, however, the data space can be an arbitrary Banach space. Due to the geometrical characteristics of Banach spaces other than Hilbert spaces, it is more appropriate to use Bregman distances instead of Ljapunov functionals to prove convergence (Osher *et al.* [24]). For convergence rates, see Hofmann *et al.* [18]. Schöpfer, Louis & Schuster [29] furthermore considered the solution of convex split feasibility problems in Banach spaces by cyclic projections. Under the so-called tangential cone condition, pertaining to the nonlinear map modelling the data, convergence has been established; invoking a source condition results in a convergence rate. Here, we build on the work of Kaltenbacher, Schöpfer and Schuster [19] and revisit these conditions with a view to stability properties of the inverse problem.

Our main result establishes convergence of the modified Landweber iteration if the inverse problem ensures a Hölder stability estimate. We prove this result both for the case of Hilbert spaces and Banach spaces. Moreover, we prove monotonicity of the residuals defined by the sequence induced by the iteration. We also obtain the convergence rates. Hence, the stability condition is the natural one in the framework of iterative reconstruction.

In many inverse problems one probes a medium, or an obstacle, with a particular type of

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field and measures the response. From these measurements one aims to determine the medium properties and/or (geometrical) structure. Typically, the physical phenomenon is modeled by partial differential equations and the medium properties by variable, and possibly singular, coefficients. The interaction of fields is usually restricted to a bounded domain with boundary. Experiments can be carried out on the boundary. The goal is thus to infer information on the coefficients in the interior of the domain from the associated boundary measurements. The map, solving the partial differential equations, from coefficients or parameter functions to the measurements or data is nonlinear. Its injectivity is studied in the analysis of inverse problems. As an example, we discuss Electrical Impedance Tomography, where the Dirichlet-to-Neumann map represents the data, and summarize the conditions leading to Lipschitz stability.

Traditionally, the Landweber iteration has been viewed as a fixed-point iteration. However, in general, the underlying fixed point operator is not a contraction. There is an extensive literature of iterative methods for approximating fixed points of non-expansive operators. Hanke, Neubauer & Scherzer [16] replace the condition of non-expansive to a local tangential cone condition, which guarantees a local result. In the finite-dimensional setting, in which, for example the model space is $\mathbb{R}^n$, non-convex constraint optimization problems admitting iterative solutions have been studied by Curtis *et al.* [14]. Under certain assumptions, they obtain convergence to stationary points of the associated feasibility problem. In the context of inverse problems defined by partial differential equations, this setting is motivated by discretizing the problems prior to studying the convergence (locally) of the iterations. Inequality constraints are necessary to enforce locality. The non-convexity is addressed by Hessian modifications based on inertia tests.

The paper is organized as follows. In the next section, we summarize certain geometrical aspects of Banach spaces, including (uniform) smoothness and (uniform) convexity, and their connection to duality mappings. Smoothness is naturally related to Gâteaux differentiability. We also introduce the Bregman distance. We then define the nonlinear Landweber iteration in Banach spaces. In Section 3 we introduce the basic assumptions including Hölder stability and analyze the convergence of the modified Landweber iteration in Hilbert spaces. In Section 4 we adapt these assumptions and generalize the analysis of convergence of the modified Landweber iteration to Banach spaces. We also establish the convergence rates. In Section 5 we give an example, namely, the reconstruction of conductivity in Electrical Impedance Tomography, and show that our assumptions can be satisfied.

2. Landweber iteration in Banach spaces.

Let X and Y be both Banach spaces. We consider the nonlinear operator equation

$$F(x) = y, \quad x \in \mathcal{D}(F),\ y \in Y, \tag{2.1}$$

with domain $\mathcal{D}(F) \subset X$. In applications, $F:\ \mathcal{D}(F) \to Y$ models the data. In the inverse problem one is concerned with the question whether y determines x. We assume that F is continuous, and that F is Fréchet differentiable, locally.

We couple the uniqueness and stability analysis of the inverse problem to a local solution construction based on the Landweber iteration. Throughout this paper, we assume that the data y in (2.1) is attainable, i.e., that (2.1) has a solution $x^\dagger$ (which need not be unique).

2.1. Duality mappings.

Throughout this paper, X and Y are real Banach spaces with duals X^* and Y^*, respectively. Their norms are denoted by $\|\cdot\|$. We denote the space of continuous linear operators $X \to Y$ by $\mathcal{L}(X, Y)$. Let $A \in \mathcal{L}(X, Y)$; if A is Fréchet differentiable, $DA:\ X \to \mathcal{L}(X, Y)$. For $x \in X$ and $x^* \in X^*$, we write the dual pair as $\langle x, x^*\rangle = \langle x^*, x\rangle = x^*(x)$. We write A^* for the dual operator $A^* \in \mathcal{L}(Y^*, X^*)$ and $\|A\| = \|A^*\|$ for the operator norm of A. We let $1 < p, q < \infty$ be conjugate exponents, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{2.2}$$

For $p > 1$, the subdifferential mapping $J_p = \partial f_p : X \to 2^{X^*}$ of the convex functional $f_p : x \mapsto \frac{1}{p}\|x\|^p$ defined by

$$J_p(x) = \{x^* \in X^* \mid \langle x, x^* \rangle = \|x\|\,\|x^*\| \text{ and } \|x^*\| = \|x\|^{p-1}\} \tag{2.3}$$

is called the duality mapping of X with gauge function $t \mapsto t^{p-1}$. Generally, the duality mapping is set-valued. In order to let J_p be single valued, we need to introduce the notion of convexity and smoothness of Banach spaces.

One defines the convexity modulus δ_X of X by

$$\delta_X(\epsilon) = \inf_{x,\tilde{x}\in X}\{1 - \|\tfrac{1}{2}(x+\tilde{x})\| \mid \|x\| = \|\tilde{x}\| = 1 \text{ and } \|x - \tilde{x}\| \geq \epsilon\} \tag{2.4}$$

and the smoothness modulus ρ_X of X by

$$\rho_X(\tau) = \sup_{x,\tilde{x}\in X}\{\tfrac{1}{2}\left(\|(x+\tau\tilde{x}\| + \|(x-\tau\tilde{x}\| - 2\right) \mid \|x\| = \|\tilde{x}\| = 1\}. \tag{2.5}$$

Definition 2.1. *A Banach spaces X is said to be*

(a) uniformly convex if $\delta_X(\epsilon) > 0$ for any $\epsilon \in (0, 2]$,

(b) uniformly smooth if $\lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} = 0$,

(c) convex of power type p or p-convex if there exists a constant $C > 0$ such that $\delta_X(\epsilon) \geq C\epsilon^p$,

(d) smooth of power type q or q-smooth if there exists a constant $C > 0$ such that $\rho_X(\tau) \leq C\tau^q$.

Example 2.2.

(a) A Hilbert space X is 2-convex and 2-smooth and $J_2 : X \to X$ is the identity mapping.

(b) The Banach spaces L^p, $p > 1$ are uniformly convex and uniformly smooth, and

$$\delta_{L^p}(\epsilon) \simeq \begin{cases} \epsilon^2, & 1 < p < 2, \\ \epsilon^p, & 2 \leq p < \infty; \end{cases}$$

$$\rho_{L^p}(\tau) \simeq \begin{cases} \tau^p, & 1 < p < 2, \\ \tau^2, & 2 \leq p < \infty. \end{cases}$$

(c) For $X = L^r(\mathbb{R}^n)$, $r > 1$, we have

$$\begin{aligned} J_p : L^r(\mathbb{R}^n) &\to L^s(\mathbb{R}^n) \\ u(x) &\mapsto \|u\|_{L^r}^{p-r}|u(x)|^{r-2}u(x), \end{aligned}$$

where $\frac{1}{r} + \frac{1}{s} = 1$.

For a detailed introduction to the geometry of Banach spaces and the duality mapping, we refer to [13, 28]. We list the properties we need here in the following theorem.

Theorem 2.3. *The following statements hold true:*

(a) For every $x \in X$, the set $J_p(x)$ is not empty and it is convex and weakly closed in X^*.

(b) If a Banach space is uniformly convex, it is reflexive.

(c) A Banach space X is uniformly convex (resp. uniformly smooth) iff X^* is uniformly smooth (resp. uniformly convex).

(d) If a Banach space X is uniformly smooth, $J_p(x)$ is single valued for all $x \in X$.

(e) If a Banach space X is uniformly smooth and uniformly convex, $J_p(x)$ is bijective and the inverse $J_p^{-1} : X^* \to X$ is given by $J_p^{-1} = J_q^*$ with J_q^* being the duality mapping of X^* with gauge function $t \mapsto t^{q-1}$, where $1 < p, q < \infty$ are conjugate exponents.

Throughout this paper, we assume that X is p-convex and q-smooth, hence it is is uniformly smooth and uniformly convex. Furthermore, X is reflexive and its dual X^* has the same properties. Y is allowed to be an arbitrary Banach space; j_p will be a single-valued selection of the possibly set-valued duality mapping of Y with gauge function $t \mapsto t^{p-1}$, $p > 1$. Possible further restrictions on X and Y will be indicated in the respective theorems below.

2.2. Bregman distances. Due to the geometrical characteristics of Banach spaces different from those of Hilbert spaces, it is often more appropriate to use the Bregman distance instead of the conventional norm-based functionals $\|x-\tilde{x}\|^p$ or $\|J_p(x)-J_p(\tilde{x})\|^p$ for convergence analysis. This idea goes back to Bregman [9].

DEFINITION 2.4. *Let X be a uniformly smooth Banach space and $p>1$. The Bregman distance $\Delta_p(x,\cdot)$ of the convex functional $x\mapsto \frac{1}{p}\|x\|^p$ at $x\in X$ is defined as*

$$\Delta_p(x,\tilde{x}) = \frac{1}{p}\|\tilde{x}\|^p - \frac{1}{p}\|x\|^p - \langle J_p(x), \tilde{x}-x\rangle, \quad \tilde{x}\in X, \tag{2.6}$$

where J_p denotes the duality mapping of X with gauge function $t\mapsto t^{p-1}$.

In the following theorem, we summarize some facts concerning the Bregman distance and its relationship to the norm [1, 2, 10, 32].

THEOREM 2.5. *Let X be a uniformly smooth and uniformly convex Banach space. Then, for all $x,\tilde{x}\in X$, the following holds:*
(a)

$$\begin{aligned}\Delta_p(x,\tilde{x}) &= \frac{1}{p}\|\tilde{x}\|^p - \frac{1}{p}\|x\|^p - \langle J_p(x),\tilde{x}\rangle + \|x\|^p \\ &= \frac{1}{p}\|\tilde{x}\|^p + \frac{1}{q}\|x\|^p - \langle J_p(x),\tilde{x}\rangle.\end{aligned} \tag{2.7}$$

(b) $\Delta_p(x,\tilde{x})\ge 0$ *and* $\Delta_p(x,\tilde{x})=0 \Leftrightarrow x=\tilde{x}$.
(c) Δ_p *is continuous in both arguments.*
(d) The following statements are equivalent
   (i) $\lim_{n\to\infty}\|x_n-x\|=0$,
   (ii) $\lim_{n\to\infty}\Delta_p(x_n,x)=0$,
   (iii) $\lim_{n\to\infty}\|x_n\|=\|x\|$ *and* $\lim_{n\to\infty}\langle J_p(x_n),x\rangle = \langle J_p(x),x\rangle$.
(e) If X is p-convex, there exists a constant $C_p>0$ such that

$$\Delta_p(x,\tilde{x}) \ge \frac{C_p}{p}\|x-\tilde{x}\|^p. \tag{2.8}$$

(f) If X^* is q-smooth, there exists a constant $G_q>0$ such that

$$\Delta_q(x^*,\tilde{x}^*) \le \frac{G_q}{q}\|x^*-\tilde{x}^*\|^q, \tag{2.9}$$

for all $x^*,\tilde{x}^*\in X^*$.

REMARK 2.6. *The Bregman distance Δ_p is similar to a metric, but does not satisfy the triangle inequality nor symmetry. In a Hilbert space, $\Delta_2(x,\tilde{x})=\frac{1}{2}\|x-\tilde{x}\|^2$.*

2.3. Modified Landweber iteration. In this subsection, we introduce an iterative method for minimizing the functional

$$\Phi(x) = \frac{1}{p}\|F(x)-y\|^p. \tag{2.10}$$

For regularization, Tikhonov proposed to minimize the functional

$$\Phi(x) = \phi(F(x),y) + \beta\mathcal{R}(x) \tag{2.11}$$

assuming that ϕ is a functional measuring the error between $F(x)$ and y, $\beta>0$ and $\mathcal{R}$ is a non-negative functional, see for instance [23]. In this paper, we only consider the case with $\phi(F(x),y) =$

$\frac{1}{p}\|F(x)-y\|^p$ and $\mathcal{R}(x) = \frac{1}{p}f_p(x-x_0) = \frac{1}{p}\|x-x_0\|^p$. The iterates are generated by iterative calculating

$$\begin{aligned} x_{k+1} = &\quad \arg\min \Phi^{(k)}(x_k), \\ \Phi^{(k)}(x_k) = &\quad \tfrac{1}{p}\|F(x_k)-y\|^p + \tfrac{\beta_k}{\mu p}\|x_k-x_0\|^p, \quad k=0,1,\dots \end{aligned}$$

with the steepest descent flow given by

$$\partial\Phi^{(k)}(x_k) = DF(x_k)^* j_p(F(x_k)-y) + \frac{\beta_k}{\mu}J_p(x_k-x_0). \tag{2.12}$$

To be more precisely, we study the iterative method in Banach spaces,

$$\begin{aligned} J_p(x_{k+1}) = &\quad J_p(x_k) - \mu DF(x_k)^* j_p(F(x_k)-y) - \beta_k J_p(x_k-x_0), \\ x_{k+1} = &\quad J_q^*(J_p(x_{k+1})), \end{aligned} \tag{2.13}$$

where $J_p : X \to X^*$, $J_q^* : X^* \to X$ and $j_p : Y \to Y^*$ denote duality mappings in corresponding spaces. When X and Y are Hilbert spaces and $p = 2$, this reduces to the modified Landweber iteration in Hilbert spaces [26], which consists with the quadratic Tikhonov regularization in Hilbert space. We specify μ and the β_k below. Equation (2.13) defines a sequence (x_k).

If $F(x^\dagger) = y$, the so-called tangential cone condition [19],

$$\|F(x) - F(\tilde{x}) - DF(x)(x-\tilde{x})\| \le c_{\mathrm{TC}}\ \|F(x) - F(\tilde{x})\| \quad \forall x, \tilde{x} \in \mathcal{B}_\rho^\Delta(x^\dagger), \tag{2.14}$$

for some $0 < c_{\mathrm{TC}} < 1$, is critical to obtain convergence of (x_k) to $x^\dagger$ [17, 18, 19]; $\mathcal{B}_\rho^\Delta(x^\dagger) = \{x \in X \mid \Delta_p(x, x^\dagger) \le \rho\} \subset \mathcal{D}(F)$. A source condition controls the convergence rate. Here, we study convergence and convergence rates in relation to a single, alternative condition replacing the tangential cone and source conditions, namely, Hölder type stability,

$$\Delta_p(x,\tilde{x}) \le C_F^p \|F(x) - F(\tilde{x})\|^p \quad \forall x, \tilde{x} \in \mathcal{B}_\rho^\Delta(x^\dagger).$$

This condition implies the tangential cone condition, and, hence, convergence is guaranteed; however, it also implies a certain convergence rate.

3. Convergence rate – Hilbert spaces. In this section, we assume that X and Y are Hilbert spaces. Then the mappings J_p, j_p and J_q^* are all identity mappings. Let $\mathcal{B}_\rho(x_0)$ denote a closed ball centered at x_0 with radius ρ, such that $\mathcal{B} = \mathcal{B}_{\rho'}(x_0) \subset \mathcal{D}(F)$, $\rho' > \rho$. As before, let $x^\dagger$ generate the data y, that is

$$F(x^\dagger) = y. \tag{3.1}$$

We assume that $x^\dagger \in \mathcal{B}_\rho(x_0)$.

Assumption 3.1.

(a) The Fréchet derivative, DF, of F is Lipschitz continuous locally on $\mathcal{B}$.

(b) F is weakly sequentially closed, i.e.,

$$\left.\begin{array}{l} x_n \rightharpoonup x, \\ F(x_n) \to y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x \in \mathcal{D}(F), \\ F(x) = y. \end{array}\right.$$

(c) The inversion has the uniform Hölder type stability, i.e., there exists a constant, $C_F > 0$, such that

$$\|x - \tilde{x}\| \le C_F \|F(x) - F(\tilde{x})\|^{\frac{1+\varepsilon}{2}} \quad \forall x, \tilde{x} \in \mathcal{B} \tag{3.2}$$

for some $\varepsilon \in (0,1]$

The modified Landweber iteration in Hilbert spaces [26] is given by

$$x_{k+1} = x_k - \mu DF(x_k)^*(F(x_k) - y) - \beta_k(x_k - x_0), \tag{3.3}$$

cf. (2.13). In the remainder of this section, we discuss the convergence criterion and convergence rate for the modified Landweber iteration (3.3), with $\beta_k \in [0, \frac{1}{4}]$ and (β_k) converges to 0.

THEOREM 3.2. *Assume there exists a solution $x^\dagger$ to (3.1) and that Assumption 3.1 holds. We further assume that*

$$\|DF(x)\| \le \hat{L} \quad \forall x \in \mathcal{B} \tag{3.4}$$

and that

$$\|DF(x) - DF(\tilde{x})\| \le L\|x - \tilde{x}\| \quad \forall x, \tilde{x} \in \mathcal{B}. \tag{3.5}$$

Let $\beta_k \in [0, \frac{1}{4}]$, $\quad \beta_k \to 0$ as $k \to \infty$ and the positive stepsize, μ, be such that

$$\begin{aligned} \mu \le& \quad \frac{1}{2\hat{L}^2}, \\ \mu(1 - \mu\hat{L}^2) <& \quad C_F^{\frac{4}{1+\varepsilon}} \end{aligned} \tag{3.6}$$

Let

$$\rho = (L\hat{L}^\varepsilon C_F^2)^{-2/\varepsilon}.$$

If

$$\|x_0 - x^\dagger\|^2 \le \frac{2}{3}\rho, \tag{3.7}$$

then the iterates satisfy

$$\|x_k - x^\dagger\|^2 \le \rho, \quad k = 1, 2, \ldots \tag{3.8}$$

and $x_k \to x^\dagger$ as $k \to \infty$. Moreover, let

$$c = \mu(1 - \mu\hat{L}^2)C_F^{-\frac{4}{1+\varepsilon}}. \tag{3.9}$$

The convergence rate is given by

$$\|x_k - x^\dagger\|^2 \le \frac{2\rho}{3}((1 - c)^k + (1 + \mu\hat{L}^2)\sum_{j=0}^{k-1}(1 - c)^j\beta_{k-1-j}), \tag{3.10}$$

if $\varepsilon = 1$. For $\varepsilon \in (0, 1)$, if

$$\beta_k \le \frac{3\tilde{C}}{4(1 + \mu\hat{L}^2)\rho}(k + 2)^{-\frac{1+\varepsilon}{1-\varepsilon}-1}, \tag{3.11}$$

with

$$\tilde{C} = \max\left(\frac{2}{3}\rho,\ 2\left(\frac{3 + \varepsilon}{2c(1 - \varepsilon)}\right)^{\frac{1+\varepsilon}{1-\varepsilon}}\right) \tag{3.12}$$

the convergence rate is given by

$$\|x_k - x^\dagger\|^2 \le \tilde{C}(k + 2)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \quad k = 0, 1, \ldots. \tag{3.13}$$

Proof. From (3.3), we obtain the sequence of residuals,

$$x_{k+1} - x^\dagger = (1-\beta_k)(x_k - x^\dagger) - \beta_k(x^\dagger - x_0) - \mu DF(x_k)^*(F(x_k) - y). \tag{3.14}$$

We have

$$\begin{aligned} \|x_{k+1} - x^\dagger\|^2 \leq \quad & (1-\beta_k)^2\|x_k - x^\dagger\|^2 + \beta_k^2\|x^\dagger - x_0\|^2 + \mu^2\hat{L}^2\|F(x_k) - y\|^2 \\ - \quad & 2\beta_k(1-\beta_k)\langle x_k - x^\dagger, x^\dagger - x_0\rangle \\ - \quad & 2\mu(1-\beta_k)\langle DF(x_k)(x_k - x^\dagger), F(x_k) - y\rangle \\ + \quad & 2\mu\beta_k\langle DF(x_k)(x^\dagger - x_0), F(x_k) - y\rangle. \end{aligned} \tag{3.15}$$

The fourth term is estimated as

$$-2\beta_k(1-\beta_k)\langle x_k - x^\dagger, x^\dagger - x_0\rangle \leq \beta_k(1-\beta_k)(\|x_k - x^\dagger\|^2 + \|x^\dagger - x_0\|^2). \tag{3.16}$$

Using the Hölder type stability (3.2), we estimate the fifth term as

$$\begin{aligned} & -2\mu(1-\beta_k)\langle DF(x_k)(x_k - x^\dagger), F(x_k) - y\rangle \\ = \quad & -2\mu(1-\beta_k)(\|F(x_k) - y\|^2 - \langle F(x_k) - y - DF(x_k)(x_k - x^\dagger), F(x_k) - y\rangle) \\ \leq \quad & -2\mu(1-\beta_k)\|F(x_k) - y\|^2 + \mu(1-\beta_k)L\|x_k - x^\dagger\|^2\|F(x_k) - y\| \\ \leq \quad & -2\mu(1-\beta_k)\|F(x_k) - y\|^2 + \mu(1-\beta_k)LC_F^2\|F(x_k) - y\|^{2+\varepsilon}. \end{aligned} \tag{3.17}$$

The sixth term satisfies the estimate

$$2\mu\beta_k\langle DF(x_k)(x^\dagger - x_0), F(x_k) - y\rangle \leq \mu\beta_k(\hat{L}^2\|x^\dagger - x_0\|^2 + \|F(x_k) - y\|^2). \tag{3.18}$$

Using the notation

$$\gamma_k = \|x_k - x^\dagger\|^2,$$

combining (3.16), (3.17) and (3.18), we find that

$$\begin{aligned} \gamma_{k+1} \leq \quad & (1-\beta_k)\gamma_k + (1+\mu\hat{L}^2)\beta_k\gamma_0 \\ + \quad & (-\mu + 2\mu\beta_k + \mu^2\hat{L}^2)\|F(x_k) - y\|^2 \\ & -\mu(1-\beta_k)\|F(x_k) - y\|^2 + \mu LC_F^2(1-\beta_k)\|F(x_k) - y\|^{2+\varepsilon}. \end{aligned} \tag{3.19}$$

We claim that the sequence (γ_k) is bounded,

$$\gamma_{k+1} \leq \rho, \quad k = 0, 1, \ldots \tag{3.20}$$

which we prove by induction. Assume that

$$\gamma_k \leq \rho \tag{3.21}$$

holds. With this assumption, the mean value inequality yielding

$$\|F(x_k) - y\| \leq \sup_{t\in[0,1]} \|DF((1-t)x_k + tx^\dagger)\|\,\|x_k - x^\dagger\|, \tag{3.22}$$

and (3.4), it follows that

$$\|F(x_k) - y\|^\varepsilon \leq (\hat{L}\rho^{1/2})^\varepsilon = (LC_F^2)^{-1}.$$

Therefore,

$$-\mu(1-\beta_k)\|F(x_k)-y\|^2 + \mu L C_F^2(1-\beta_k)\|F(x_k)-y\|^{2+\varepsilon} \leq 0.$$

Dropping this non-positive term in (3.19), we obtain

$$\begin{aligned}\gamma_{k+1} \leq & \quad (1-\beta_k)\gamma_k + (1+\mu\hat{L}^2)\beta_k\gamma_0 \\ + & \quad \mu(2\beta_k + \mu\hat{L}^2 - 1)\|F(x_k)-y\|^2.\end{aligned} \tag{3.23}$$

Now, we choose a stepsize $\mu > 0$ which satisfies (3.6). Noting that with $\beta_k \leq \frac{1}{4}$ the factor $2\beta_k + \mu\hat{L}^2 - 1 \leq 0$, we drop the last term in (3.23) and, with (3.7) and (3.21), obtain that

$$\begin{aligned}\gamma_{k+1} \leq & \quad (1-\beta_k)\gamma_k + (1+\mu\hat{L}^2)\beta_k\gamma_0 \\ \leq & \quad (1-\beta_k)\gamma_k + \tfrac{3}{2}\beta_k\gamma_0 \\ \leq & \quad \rho,\end{aligned} \tag{3.24}$$

whence the claim holds true.

To prove convergence, we reorder the terms in (3.23),

$$\begin{aligned}\gamma_{k+1} \leq & \quad \gamma_k - \mu(1-\mu\hat{L}^2)\|F(x_k)-y\|^2 + (1+\mu\hat{L}^2)\beta_k\gamma_0 \\ - & \quad \beta_k\gamma_k + 2\mu\beta_k\|F(x_k)-y\|^2.\end{aligned} \tag{3.25}$$

We note that with (3.22), $-\beta_k\gamma_k + 2\mu\beta_k\|F(x_k)-y\|^2 \leq \beta_k(2\mu\hat{L}^2-1)\gamma_k$, whence this term by the choice of μ is non-positive. Hence, we drop it in the above inequality. Then, by the Hölder type stability (3.2), we have that

$$\begin{aligned}\gamma_{k+1} \leq & \quad \gamma_k - \mu(1-\mu\hat{L}^2)\|F(x_k)-y\|^2 + (1+\mu\hat{L}^2)\beta_k\gamma_0 \\ \leq & \quad \gamma_k - \underbrace{\mu(1-\mu\hat{L}^2)C_F^{-\frac{4}{1+\varepsilon}}}_{=c}\gamma_k^{\frac{2}{1+\varepsilon}} + (1+\mu\hat{L}^2)\beta_k\gamma_0.\end{aligned} \tag{3.26}$$

By letting k go to infinity on both sides of the above inequality, we conclude that

$$\gamma_k \to 0 \text{ as } k \to \infty.$$

In the rest of the proof, we obtain the convergence rate. Note that, with the choice of μ (3.6),

$$0 < c < 1. \tag{3.27}$$

With $\varepsilon = 1$, We have

$$\begin{aligned}\gamma_{k+1} \leq & \quad ((1-c)^{k+1} + (1+\mu\hat{L}^2)\textstyle\sum_{j=0}^k (1-c)^j\beta_{k-j})\gamma_0 \\ \leq & \quad \tfrac{2\rho}{3}((1-c)^{k+1} + (1+\mu\hat{L}^2)\textstyle\sum_{j=0}^k (1-c)^j\beta_{k-j}),\end{aligned} \tag{3.28}$$

which expresses the convergence rate (3.10). We note that

$$\sum_{j=0}^{k+1}(1-c)^j\beta_{k+1-j} - (1-c)\sum_{j=0}^{k}(1-c)^j\beta_{k-j} = \beta_{k+1}.$$

Using (3.27), this implies that

$$\lim_{k\to\infty}\sum_{j=0}^{k}(1-c)^j\beta_{k-j} = 0.$$

For the convergence rate with $\varepsilon \in (0,1)$, we claim that

$$\gamma_{k+1} \leq \tilde{C}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \quad k=0,1,\ldots \tag{3.29}$$

which we prove by induction. Assume that

$$\gamma_k \leq \tilde{C}(k+1)^{-\frac{1+\varepsilon}{1-\varepsilon}} \tag{3.30}$$

holds. If

$$\gamma_k \leq \frac{\tilde{C}}{2}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}},$$

by (3.26) and (3.11),

$$\gamma_{k+1} \leq \gamma_k(1-c\gamma_k^{\frac{1-\varepsilon}{1+\varepsilon}}) + (1+\mu\hat{L}^2)\beta_k\gamma_0 \leq \tilde{C}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}}; \tag{3.31}$$

otherwise

$$\frac{\tilde{C}}{2}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}} < \gamma_k \leq \tilde{C}(k+1)^{-\frac{1+\varepsilon}{1-\varepsilon}}. \tag{3.32}$$

Then, we obtain that

$$\begin{aligned} \gamma_{k+1} \leq \quad & \gamma_k(1-c\gamma_k^{\frac{1-\varepsilon}{1+\varepsilon}}) + (1+\mu\hat{L}^2)\beta_k\gamma_0 \\ \leq \quad & \tilde{C}(k+1)^{-\frac{1+\varepsilon}{1-\varepsilon}}\left(1-c\left(\tfrac{\tilde{C}}{2}\right)^{\frac{1-\varepsilon}{1+\varepsilon}}(k+2)^{-1}\right) + \tfrac{\tilde{C}}{2}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}-1}. \end{aligned} \tag{3.33}$$

Note that (3.12) implies

$$c\left(\frac{\tilde{C}}{2}\right)^{\frac{1-\varepsilon}{1+\varepsilon}} \geq (k+2)\left(1-\left(\frac{k+1}{k+2}\right)^{\frac{1+\varepsilon}{1-\varepsilon}}\right) + \frac{1}{2}\left(\frac{k+1}{k+2}\right)^{\frac{1+\varepsilon}{1-\varepsilon}}, \quad k=0,1,\ldots. \tag{3.34}$$

Plugging (3.34) into (3.33), we get

$$\gamma_{k+1} \leq \tilde{C}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \tag{3.35}$$

whence the claim holds true. ☐

The convergence is sublinear if $0 < \varepsilon < 1$ and speed up with $\varepsilon \to 1$ refers to the fact that it switches to a linear convergence.

REMARK 3.3. *The convergence radius condition (3.7) can be replaced by the condition on the data,*

$$\|F(x_0)-y\|^{\frac{1+\varepsilon}{2}} \leq \frac{2}{\sqrt{6}}\left(\frac{1}{2L\hat{L}^{\varepsilon}C_F^{2+\varepsilon}}\right)^{1/\varepsilon}. \tag{3.36}$$

4. Convergence rate – Banach spaces. In this section, we discuss the convergence and convergence rate of the modified Landweber iteration (2.13) in Banach spaces. Let $\mathcal{B}_\rho(x_0)$ denote a closed ball centered at x_0 with radius ρ, and $\mathcal{B} = \mathcal{B}_\rho^\Delta(x^\dagger)$ denote a ball with respect to the Bregman distance centered at some solution $x^\dagger$. We assume that $\mathcal{B}_\rho^\Delta(x^\dagger) \subset \mathcal{D}(F)$.

ASSUMPTION 4.1.
(a) The Fréchet derative, DF, of F is locally Lipschitz continuous.
(b) F is weakly sequentially closed, i.e.,

$$\left.\begin{array}{l} x_n \rightharpoonup x, \\ F(x_n) \to y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x \in \mathcal{D}(F), \\ F(x) = y. \end{array}\right.$$

(c) *The inversion has the uniform Hölder type stability, i.e., there exists a constant* $C_F > 0$ *such that*

$$\Delta_p(x, \tilde{x}) \leq C_F^p \|F(x) - F(\tilde{x})\|^{\frac{1+\varepsilon}{2}p} \quad \forall x, \tilde{x} \in \mathcal{B}. \tag{4.1}$$

Remark 4.2. *Note that the nonemptyness of the interior (with respect to norm) of* $\mathcal{D}(F)$ *is sufficient for* $\mathcal{B} \subset \mathcal{D}(F)$.

Remark 4.3. *With the assumption that* X *is p-convex,* (4.1) *with* (2.8) *implies the regular notion of Hölder stability in norm.*

Remark 4.4. *Under the Lipschitz type stability assumption, i.e.,* (4.1) *with* $\varepsilon = 1$, *we have that*

$$\begin{aligned} \langle J_p(x^\dagger), x - x^\dagger \rangle \leq & \ \|x^\dagger\|^{p-1} \|x - x^\dagger\| \\ \leq & \ C \Delta_p(x, x^\dagger)^{1/p} \\ \leq & \ C C_F \|F(x) - F(x^\dagger)\|, \quad \forall x \in \mathcal{B} \end{aligned}$$

for some constant $C > 0$. *It has been show in [27] that this implies the source-wise condition*

$$J_p(x^\dagger) = DF(x^\dagger)^* \omega$$

with some ω *satisfying* $\|\omega\| \leq 1$.

Theorem 4.5. *Assume there exists a solution* $x^\dagger$ *to (3.1) and that Assumption 4.1 holds. We further assume that*

$$\|DF(x)\| \leq \hat{L} \quad \forall x \in \mathcal{B} \tag{4.2}$$

and that

$$\|DF(x) - DF(\tilde{x})\| \leq L \|x - \tilde{x}\| \quad \forall x, \tilde{x} \in \mathcal{B}. \tag{4.3}$$

Let the positive stepsize, μ, *be such that*

$$\begin{aligned} \mu^{q-1} \leq & \ \frac{q}{2^q G_q \hat{L}^q}, \\ \mu \left(1 - \frac{2^q G_q \hat{L}^q}{q} \mu^{q-1} \right) < & \ 2 C_F^{\frac{2p}{1+\varepsilon}} \end{aligned} \tag{4.4}$$

and let (β_k) *satisfy*

$$\frac{p}{C_p} (1 + 2^{-1/p})^{p-1} \left((1 + 2^{-1/p}) \frac{2^{q-1} G_q}{q} \sum_{m=0}^{k} \beta_m^q + \sum_{m=0}^{k} \beta_m \right) < \frac{1}{2} \tag{4.5}$$

Let

$$\rho = \hat{L}^{-p} (L C_F^2)^{-\frac{p}{\varepsilon}} \left(\frac{C_p}{p} \right)^{1 + \frac{2}{\varepsilon}}.$$

If

$$\Delta_p(x_0, x^\dagger) \leq \frac{\rho}{2}, \tag{4.6}$$

then the iterates satisfy

$$\Delta_p(x_k, x^\dagger) \leq \rho, \quad k = 1, 2, \ldots \tag{4.7}$$

and $\Delta_p(x_k, x^\dagger) \to 0$ *as* $k \to \infty$. *Moreover, let*

$$c = C_F^{-\frac{2p}{1+\varepsilon}} \left(\frac{1}{2}\mu - \frac{2^{q-1}G_q}{q}\mu^q \hat{L}^q \right) \tag{4.8}$$

and

$$\alpha_k = \frac{p}{C_p}(1+2^{-1/p})^{p-1} \left((1+2^{-1/p}) \frac{2^{q-1}G_q}{q}\beta_k^q + \beta_k \right). \tag{4.9}$$

The convergence rate is given by

$$\Delta_p(x_k, x^\dagger) \le \rho \left(\frac{1}{2}(1-c)^k + \sum_{j=0}^{k-1}(1-c)^j \alpha_{k-1-j} \right), \tag{4.10}$$

if $\varepsilon = 1$. *For* $\varepsilon \in (0,1)$, *if*

$$\alpha_k \le \frac{\tilde{C}}{2\rho}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}-1} \tag{4.11}$$

with

$$\tilde{C} = \max\left(\frac{\rho}{2},\ 2\left(\frac{3+\varepsilon}{2c(1-\varepsilon)} \right)^{\frac{1+\varepsilon}{1-\varepsilon}} \right), \tag{4.12}$$

the convergence rate is given by

$$\Delta_p(x_k, x^\dagger) \le \tilde{C}(k+2)^{-\frac{1+\varepsilon}{1-\varepsilon}}, \quad k = 0, 1, \ldots. \tag{4.13}$$

Proof. Using (2.7) and (2.3), we obtain, for the sequence of residues,

$$\begin{aligned} \Delta_p(x_{k+1}, x^\dagger) = {}& \Delta_p(x_k, x^\dagger) + \frac{1}{q}\left(\|x_{k+1}\|^p - \|x_k\|^p\right) - \langle J_p(x_{k+1}) - J_p(x_k), x^\dagger\rangle \\ = {}& \Delta_p(x_k, x^\dagger) + \frac{1}{q}\left(\|J_p(x_{k+1})\|^q - \|J_p(x_k)\|^q\right) - \langle J_p(x_{k+1}) - J_p(x_k), x^\dagger\rangle. \end{aligned} \tag{4.14}$$

Applying (2.7) and (f) of Theorem 2.5 with $x^* = J_p(x_{k+1})$ and $\tilde{x}^* = J_p(x_k)$, we get

$$\frac{1}{q}\left(\|J_p(x_{k+1})\|^q - \|J_p(x_k)\|^q\right) \le \frac{G_q}{q}\|J_p(x_{k+1}) - J_p(x_k)\|^q + \langle J_p(x_{k+1}) - J_p(x_k), x_k\rangle. \tag{4.15}$$

Substituting (2.13) and using this inequality in (4.14) yields

$$\begin{aligned} \Delta_p(x_{k+1}, x^\dagger) - \Delta_p(x_k, x^\dagger) \le {}& \frac{G_q}{q}\|\mu DF(x_k)^* j_p(F(x_k) - y) + \beta_k J_p(x_k - x_0)\|^q \\ & - \langle \mu DF(x_k)^* j_p(F(x_k) - y), x_k - x^\dagger\rangle - \beta_k \langle J_p(x_k - x_0), x_k - x^\dagger\rangle. \end{aligned} \tag{4.16}$$

We estimate each term in (4.16) separately. By Jensen's inequality, the first term satisfies the estimate

$$\begin{aligned} & \frac{G_q}{q}\|\mu DF(x_k)^* j_p(F(x_k) - y) - \beta_k J_p(x_k - x_0)\|^q \\ & \quad \le \frac{2^{q-1}G_q}{q}\left(\|\mu DF(x_k)^* j_p(F(x_k) - y)\|^q + \|\beta_k J_p(x_k - x_0)\|^q\right) \\ & \quad \le \frac{2^{q-1}G_q}{q}\left(\mu^q \hat{L}^q \|(F(x_k) - y)\|^p + \beta_k^q \|x_k - x_0\|^p\right). \end{aligned} \tag{4.17}$$

For the second term, using (2.8) and stability (d) of Assumption 4.1, we have

$$
\begin{aligned}
&-\langle \mu DF(x_k)^* j_p(F(x_k)-y), x_k - x^\dagger\rangle \\
&\quad= -\mu\langle j_p(F(x_k)-y), DF(x_k)(x_k-x^\dagger)\rangle \\
&\quad= -\mu(\langle j_p(F(x_k)-y), F(x_k)-y\rangle \\
&\qquad\qquad -\langle j_p(F(x_k)-y), F(x_k)-y-DF(x_k)(x_k-x^\dagger)\rangle) \\
&\quad= -\mu\|F(x_k)-y\|^p + \mu\langle j_p(F(x_k)-y), F(x_k)-y-DF(x_k)(x_k-x^\dagger)\rangle \\
&\quad\le -\mu\|F(x_k)-y\|^p + \frac{\mu}{2}L\|(F(x_k)-y)\|^{p-1}\|x_k-x^\dagger\|^2 \\
&\quad\le -\mu\|F(x_k)-y\|^p + \frac{\mu}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\|F(x_k)-y\|^{p+\varepsilon}.
\end{aligned}
\tag{4.18}
$$

For the third term, by the definition of duality mapping,

$$
|-\beta_k\langle J_p(x_k-x_0), x_k-x^\dagger\rangle| \le \beta_k\|x_k-x_0\|^{p-1}\|x_k-x^\dagger\|. \tag{4.19}
$$

Combining these estimates and using the notation

$$
\gamma_k = \Delta_p(x_k, x^\dagger),
$$

we obtain

$$
\begin{aligned}
\gamma_{k+1}-\gamma_k \le\ & \left(\frac{2^{q-1}G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu\right)\|F(x_k)-y\|^p \\
-\ & \frac{1}{2}\mu\|F(x_k)-y\|^p + \frac{\mu}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\|F(x_k)-y\|^{p+\varepsilon} \\
+\ & \frac{2^{q-1}G_q}{q}\beta_k^q\|x_k-x_0\|^p + \beta_k\|x_k-x_0\|^{p-1}\|x_k-x^\dagger\|.
\end{aligned}
\tag{4.20}
$$

We claim that

$$
\gamma_{k+1} = \Delta_p(x_{k+1}, x^\dagger) \le \rho, \tag{4.21}
$$

which we prove by induction. Assume that

$$
\Delta_p(x_m, x^\dagger) \le \rho \tag{4.22}
$$

holds for $m = 0, 1, \ldots, k$. With the mean value inequality, it follows that

$$
\|F(x_m)-y\|^\varepsilon \le \hat{L}^\varepsilon\left(\frac{p}{C_p}\rho\right)^{\frac{\varepsilon}{p}} = \frac{1}{LC_F^2(p/C_p)^{2/p}}, \quad m = 0, 1, 2, \ldots, k. \tag{4.23}
$$

Therefore,

$$
-\tfrac{1}{2}\mu\|F(x_m)-y\|^p + \tfrac{1}{2}\mu LC_F^2(p/C_p)^{2/p}\|F(x_m)-y\|^{p+\varepsilon} \le 0, \quad m = 0, 1, 2, \ldots, k. \tag{4.24}
$$

Dropping this non-positive term, we obtain

$$
\begin{aligned}
\gamma_{k+1}-\gamma_k \le\ & \left(\frac{2^{q-1}G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu\right)\|F(x_k)-y\|^p \\
+\ & \frac{2^{q-1}G_q}{q}\beta_k^q\|x_k-x_0\|^p + \beta_k\|x_k-x_0\|^{p-1}\|x_k-x^\dagger\|.
\end{aligned}
\tag{4.25}
$$

Then, by (4.4), we drop the non-positive term $(\frac{2^{q-1}G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu)\|F(x_k)-y\|^p$ and, with the aid of (2.8), (4.5), (4.6) and (4.22), obtain that

$$\begin{aligned} \gamma_{k+1}-\gamma_0 &\leq \frac{2^{q-1}G_q}{q}\sum_{m=0}^{k}\beta_m^q\|x_m-x_0\|^p + \sum_{m=0}^{k}\beta_m\|x_m-x_0\|^{p-1}\|x_m-x^\dagger\| \\ &\leq \rho\frac{p}{C_p}(1+2^{-1/p})^{p-1}\left((1+2^{-1/p})\frac{2^{q-1}G_q}{q}\sum_{m=0}^{k}\beta_m^q + \sum_{m=0}^{k}\beta_m\right) \\ &\leq \frac{\rho}{2}. \end{aligned} \tag{4.26}$$

Hence

$$\Delta_p(x_{k+1},x^\dagger) \leq \rho, \tag{4.27}$$

which establishes the claim.

Now, we return to (4.25) and use (4.22) to obtain that

$$\begin{aligned} \gamma_{k+1} \leq\ & \gamma_k + \left(\frac{2^{q-1}G_q}{q}\mu^q\hat{L}^q - \frac{1}{2}\mu\right)\|F(x_k)-y\|^p \\ +\ & \rho\frac{p}{C_p}(1+2^{-1/p})^{p-1}\left((1+2^{-1/p})\frac{2^{q-1}G_q}{q}\beta_k^q + \beta_k\right). \end{aligned} \tag{4.28}$$

Then, by the Hölder type stability (4.1), we have that

$$\gamma_{k+1} \leq \gamma_k - c\gamma_k^{\frac{2}{1+\varepsilon}} + \rho\alpha_k \tag{4.29}$$

Note that, by the conditions on μ and β_k, we have $0<c<1$ and $\lim_{k\to\infty}\alpha_k = 0$. By letting k go to infinity on both sides of the above inequality, we conclude that

$$\gamma_k \to 0 \text{ as } k\to\infty.$$

The convergence rate (4.6) and (4.13) could be deduced from (4.29) by using the same argument in the proof of the Hilbert space case.

□

Remark 4.6. *The convergence radius condition (4.6) can be replaced by the condition on data*

$$\|F(x_0)-y\|^{\frac{1+\varepsilon}{2}p} \leq \rho C_F^{-p}. \tag{4.30}$$

Remark 4.7. *Concerning the conditions (4.4) and (4.5), for given $p, q, C_p, G_q, \hat{L}, C_F$ and $\varepsilon = 1$, we may choose*

$$\mu = \min(C_F^p, \left(\frac{q}{2^{q+1}G_q\hat{L}^q}\right)^{\frac{1}{q-1}}) \tag{4.31}$$

and

$$\beta_k = (C\zeta(1+s))^{-1}k^{-(1+s)}, \tag{4.32}$$

for any $s>0$, where

$$C = 2^p\frac{p}{C_p}\left(\frac{2^{q-1}G_q}{q}+1\right) \tag{4.33}$$

and $\zeta(s)$ is the Riemann zeta function.

REMARK 4.8. *In Hilbert spaces, after rescaling the operator F such that $\max(L, \hat{L}) < \frac{1}{4}$, Scherzer [26] obtain the convergence with $\beta_k < \frac{1}{2}$ and $\sum \beta_k < C$ for some constant C. In Banach space, Hein & Kazimierski [17] suggest a posteriori choice $\beta_k \sim \left(\frac{R_k}{q2^{q-1}G_q\|J_p(x_k)\|^q+C}\right)^{p-1}$, where one chooses $R_0 \geq \Delta_p(x_0, x^\dagger)$ and R_k is given by*

$$R_{k+1} = (1 - \frac{\beta_k}{q})R_k + (2^{q-1}G_q\|J_p(x_k)\|^q + C)\beta_k^q.$$

5. Example: Electrical Impedance Tomography. In this section, we discuss an example of the electrical impedance tomography (EIT) problem. The current formulation of this problem is essentially due to Calderón [11]. For the uniqueness results, we refer to Kohn & Vogelius [20, 21], Sylvester & Uhlmann [30] and Astala & Päivärinta [7] and a recent review [31] by Uhlmann. For the stability issue, we refer to Alessandrini [3, 4, 5]. Especially, Alessandrini & Vesella [6] and Beretta & Francini [8] show that one has a Lipschitz type stability estimate if the conductivity is piecewise constant with jumps on a finite number of subdomains for real and complex conductivities, respectively.

5.1. Forward operator. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. Given a potential $f \in H^{1/2}(\partial\Omega)$ on the boundary, the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet-to-Neumann map, or voltage-to-current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\bigg|_{\partial\Omega},$$

where ν denotes the unit outer normal vector to $\partial\Omega$.

The forward operator F is defined by

$$\begin{aligned} F: \quad X \subset L^\infty(\Omega) &\to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)), \\ \gamma &\mapsto \Lambda_\gamma. \end{aligned} \tag{5.1}$$

The Fréchet derivative DF of F at $\gamma = \gamma_0$ is given by

$$\begin{aligned} DF(\gamma_0): \quad X \subset L^\infty(\Omega) &\to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \\ \delta\gamma &\mapsto DF(\gamma_0)(\delta\gamma), \end{aligned} \tag{5.2}$$

and $DF(\gamma_0)(\delta\gamma)$ is given by

$$\langle DF(\gamma_0)(\delta\gamma)\, f, g\rangle = \int_\Omega \delta\gamma \nabla u \cdot \nabla v \mathrm{d}x, \quad f, g \in H^{1/2}(\partial\Omega) \tag{5.3}$$

where

$$\begin{cases} \nabla \cdot (\gamma_0 \nabla u) = \nabla \cdot (\gamma_0 \nabla v) = 0, & \text{in } \Omega, \\ u = f, \quad v = g & \text{on } \partial\Omega. \end{cases}$$

Note that $L^\infty(\Omega)$ is neither smooth nor convex. Furthermore, to get the Hölder type stability, the preimage space need to be narrowed. We will specify the proper space X in subsection 5.3.

REMARK 5.1. *For $n = 2$, Astala and Päivärinta proved that Λ_γ can uniquely determine γ under the assumption $\gamma \in L^\infty(\Omega)$. For $n \geq 3$, Päivärinta, Panchenko and Uhlmann [25] proved the uniqueness under the assumption $\gamma \in W^{3/2,\infty}(\Omega)$. It is a open problem to understand whether, in dimension $n \geq 3$, uniqueness holds in general for $\gamma \in L^\infty(\Omega)$.*

5.2. Lipschitz stability. Roughly speaking, to achieve the Lipschitz type stability, we need assume that the domain Ω could be divided into known subdomains $\{D_k\}$ of $C^{1,\alpha}$ class and the conductivity γ is in a finite dimensional space. The precise mathematical description is the following.

For every $x \in \mathbb{R}^n$, let us set $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ for $n \geq 2$. With $B_R(x)$, $B'_R(x')$ and $Q_R(x)$ we denote respectively the open ball in $\mathbb{R}^n$ centered at x of radius R, the ball in $\mathbb{R}^{n-1}$ centered at x' of radius R and the cylinder $B'_R(x') \times (x_n - R, x_n + R)$. For simplicity, $B_R(0)$, $B'_R(0)$ and $Q_R(0)$ are denoted by B_R, B'_R and Q_R.

Definition 5.2. *Let Ω be a bounded domain in $\mathbb{R}^n$. We say that $\partial\Omega$ is of Lipschitz class with constants $r_0, L > 0$ if, for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates such that $P = 0$ and*

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \phi(x')\}$$

where ϕ is a Lipschitz continuous function on B'_{r_0} with $\phi(0) = 0$ and

$$\|\phi\|_{C^{0,1}(B'_{r_0})} \leq L r_0.$$

Definition 5.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Given $\alpha \in (0,1)$, we say that $\partial\Omega$ is of $C^{1,\alpha}$ class with constants $r_0, L > 0$ if, for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates such that $P = 0$ and*

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \phi(x')\}$$

where ϕ is a $C^{1,\alpha}$ function on B'_{r_0} with $\phi(0) = |\nabla\phi(0)| = 0$ and

$$\|\phi\|_{C^{1,\alpha}(B'_{r_0})} \leq L r_0.$$

Assumption 5.4. *$\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying*

$$|\Omega| \leq A|B_{r_0}|.$$

Here and in the sequel $|\Omega|$ denotes the Lebsgue measure of Ω. We assume that $\partial\Omega$ is of Lipscitz class with constants r_0 and L.

Assumption 5.5. *The conductivity γ is a piecewise constant of the form*

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x),$$

satisfying the ellipticity condition

$$K^{-1} \leq \gamma \leq K$$

for some constant K, where $\gamma_j, j = 1, \ldots N$ are unknown real numbers and D_j are known open sets in $\mathbb{R}^n$ which satisfy the following assumption.

Assumption 5.6. *$D_j, j = 1, \ldots, N$ are connected and pairwise non-overlapping open sets such that $\cup_{j=1}^N \overline{D}_j = \overline{\Omega}$ and ∂D_j are of $C^{1,\alpha}$ class with constants r_0 and L for all $j = 1, \ldots, N$. We also assume that there exists one region, say D_1 such that $\partial D_1 \cap \partial\Omega$ contains an open portion Σ_1 of $C^{1,\alpha}$ class with constants r_0 and L. For every $j \in \{2, \ldots, N\}$ there exist $j_1, \ldots, j_M \in \{1, \ldots, N\}$ such that*

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \ldots, M$,

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a open portion Σ_k of $C^{1,\alpha}$ class with constants r_0 and L.

Alessndrini and Vessella [6] show the following Lipschitz stability estimate.

THEOREM 5.7 (Lipschitz type stability). *Let Ω satisfy Assumption 5.4 and $\gamma^{(k)}, k = 1, 2$ be two real piecewise constant functions satisfy Assumption* 5.5 *and $D_j, j = 1, \ldots, N$ satisfy Assumption* 5.6. *Then, there exists a constant $C = C(n, r_0, L, A, , K, N)$ such that*

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C\|\Lambda_{\gamma^{(1)}} - \Lambda_{\gamma^{(2)}}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}. \tag{5.4}$$

5.3. Convergence. Now, we specify our preimage space as

$$X = \mathrm{span}\{\chi_{D_1}, \ldots, \chi_{D_N}\} \tag{5.5}$$

with L^p-norm, $p > 1$. With aid of this particular basis of X, we could show that F and DF are Lipschitz continuous. Moreover, assume that γ_1, γ_2 satisfy Assumption 5.5 and Ω satisfy Assumption 5.4, we have the following estimates.

$$\begin{aligned} \|F(\gamma_1) - F(\gamma_2)\|_{\mathcal{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))} &\leq C\|\gamma_1 - \gamma_2\|_{L^p(\Omega)}, \\ \|DF\|_{\mathcal{L}(X, \mathcal{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega)))} &\leq \hat{L}, \\ \|DF(\gamma_1) - DF(\gamma_2)\|_{\mathcal{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))} &\leq L\|\gamma_1 - \gamma_2\|_{L^p(\Omega)}, \end{aligned} \tag{5.6}$$

where C, $\hat{L}$ and L depend on Ω, N and ellipticity constant K. Furthermore, since X is finite dimensional, the weak topology is equivalent to the strong topology. Hence, F is a weakly sequentially closed operator.

Let Ω satisfy Assumtion 5.4, preimage space X be defined by (5.5) and F be defined by (5.1). Assume that $y = F(\gamma^\dagger)$ for some $\gamma^\dagger \in X$. Then the Assumption 4.1 and (4.3), (4.2) of Theorem 4.5 are satisfied. Hence the Landweber iteration (2.13) converges with convergence radius given by (4.6) and convergence rate given by (4.10).

6. Discussion. We discuss a modified Landweber iteration method for solving nonlinear operator equations in both Hilbert and Banach spaces. Traditionally, the gradient-type methods are often regarded as too slow for practical applications. Provided that the nonlinearity of the forward operator obeys a Hölder type stability, we could prove the convergence and give a sublinear convergence rate. With a Lipschitz type stability, the convergence rate switches to a linear one. Based on these convergence rates, we anticipate that this modified Landweber iteration is a valuable tool in solving inverse problems in both Hilbert and Banach spaces. This also motivates the study of Hölder/Lipschitz type stability in inverse problems to provide explicit reconstructions.

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