Proceedings of the Project Review, Geo-Mathematical Imaging Group (Purdue University, West Lafayette IN), Vol. 1 (2011) pp. 95-112.

RECONSTRUCTION OF THE METRIC OF A RIEMANNIAN MANIFOLD FROM LOCAL BOUNDARY DIFFRACTION TRAVEL TIMES

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Abstract. We consider a Riemannian manifold, (M, g), of dimension n with boundary ∂M. We analyze the inverse problem, originally formulated by Dix [4], of reconstructing g from boundary measurements associated with the single scattering of seismic waves on this manifold. The measurements determine the shape operator on the boundary. We develop an explicit reconstruction procedure involving the solution of certain Jacobi equations. We admit the presence of conjugate points.

Key words. geometric inverse problems, Riemannian manifold, shape operator

AMS subject classifications. 86A15, 35R30

1. Introduction. We consider a Riemannian manifold, (M, g), of dimension n with boundary ∂M. We analyze the inverse problem, originally formulated by Dix [4] and also including the more recent image-ray tomography, aiming at reconstructing g from boundary measurements associated with the single scattering (reflections) of seismic waves on this manifold. Geodesics are rays following the propagation of singularities by the parametrix corresponding with the wave operator on (M, g). (The phase velocity in this case is given by $\mathrm{v}(x, \alpha) = [\sum_{j,k=1}^{n} g^{jk}(x)\alpha_j\alpha_k]^{1/2}$, with α denoting the phase or contangent direction.) Dix developed a procedure, with a formula, for reconstructing one-dimensional wavespeed profiles in a half space with a Euclidean metric, which we generalize here to the case of multi-dimensional manifolds with general non-Euclidean metrics. The boundary ∂M is the acquisition surface. We will assume that the metric is known in a thin layer beneath the boundary, which is natural in the context of application in reflection seismology.

We introduce a complete extension, $(\widetilde{M}, \widetilde{g})$, $M \subset \widetilde{M}$, of (M, g) with $\widetilde{g}|_M = g$. We denote, for simplicity, $\widetilde{g} = g$ and assume that we are given $\widetilde{M} \setminus M$ and the metric g on $\widetilde{M} \setminus M$. We note that if the boundary of M is convex, the travel times between boundary points determine the normal derivatives, of all orders, of the metric in the boundary normal coordinates [10]; hence, in the case of a convex boundary, the smooth extension can be constructed when the travel times between the boundary points are given.

Let $\gamma_{y,\eta}(r)$ denote the geodesic with initial position $\gamma_{y,\eta}(0) = y$ and initial direction $\dot{\gamma}_{y,\eta}(0) = \eta$ on $(\widetilde{M}, g)$. (We adhere to the convention that the initial tangent vector will always be assumed to be normalized.) As usual in Riemannian geometry, for $y \in \widetilde{M}$, $\Phi(\xi) = \exp_y(\xi)$ denotes the exponential map,

$$\exp_y(\xi) = \gamma_{y,\eta}(r), \quad |\eta|_g = 1, \quad r \in \mathbb{R},\ \xi = r\eta.$$

As $\widetilde{M}$ is complete, the map $\Phi = \exp_y :\ T_y\widetilde{M} \to \widetilde{M}$ is surjective, and by Sard's theorem the set $\mathcal{C}(y) \subset \widetilde{M}$ of critical values of Φ has measure zero. Let $x_0 \in \widetilde{M} \setminus \mathcal{C}(y)$ and $\xi_0 \in T_y\widetilde{M}$ be such that $\Phi(\xi_0) = x_0$. Then ξ_0 has a neighborhood $U \subset T_y\widetilde{M}$ such that $\Phi :\ U \to W = \Phi(U)$ is a diffeomorphism; one says that $\Phi^{-1} :\ W \to U$ is a local inverse of Φ corresponding to ξ_0. In W, we define the function, $\rho(.; y, \xi_0)$, by

$$\rho(x; y, \xi_0) = |\Phi^{-1}(x)|_g,$$

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and call this function the generalized distance or travel time function from the point y associated to direction ξ_0; the wave fronts observed from a point source at y give us its level sets. The reason for this terminology is that when $\xi = r\eta \in U$ with $|\eta|_g = 1$ and $r > 0$ and $x = \Phi(\xi)$ $(= \gamma_{y,\eta}(r))$, then the travel time or distance function along the geodesic $\gamma_{y,\eta}$ is given by

$$\rho(x; y, \xi_0) = \int_0^r \sqrt{g_{jk}(\gamma_{y,\eta}(r'))\dot{\gamma}^j_{y,\eta}(r')\dot{\gamma}^k_{y,\eta}(r')}\,\mathrm{d}r'.$$

Note that here and below we use the Einstein summation convention and omit the summation symbols when no confusion is possible. Moreover,

$$d_g(x, y) = \inf\{\rho(x; y, \xi_0);\ \exp_y(\xi_0) = x\}$$

signifies the shortest travel time from y to x.

We consider the following measurements: Let $\mathcal{V} \subset \widetilde{M} \setminus M$ be an open set where we observe singly scattered waves. Typically, we have to extend the observations at an acquisition manifold of dimension $n-1$ to $\mathcal{V}$ using the smooth extension of (M, g). We assume that we are given $\widetilde{M} \setminus M$ with the metric g in it. We have points $x_s, x_r \in \mathcal{V}$ and consider $\rho(x_s; y, \xi_0) + \rho(x_r; y, \xi_0)$ assuming that $\exp_y(\xi_0) = x_0$ is in an open set containing x_s, x_r on which $\exp_y$ is invertible; the intersection of this set with $\mathcal{V}$, which does not necessarily contain x_0, is denoted by $\mathcal{V}(y, \xi_0)$. Here, y has the interpretation of diffraction point. For the further analysis, we will expand $\rho(x_s; y, \xi_0)$ in x_s and $\rho(x_r; y, \xi_0)$ in x_r within $\mathcal{V}$ about a *common* point $\widehat{x} \in \mathcal{V}(y, \xi_0)$. We introduce the level sets of the reflection times,

$$\Sigma_{\widehat{x}}(y, \xi_0; s) = \{(x_s, x_r) \in W_{\widehat{x}} \times W_{\widehat{x}};\ W_{\widehat{x}} \subset \mathcal{V}(y, \xi_0), \rho(x_s; y, \xi_0) + \rho(x_r; y, \xi_0) = s\}, \tag{1.1}$$

where $W_{\widehat{x}}$ indicates an open neighborhood of $\widehat{x}$. The ξ_0 label different branches. The point $y \in M$ can be connected to $\widehat{x}$ by a geodesic, $\gamma_{\widehat{x},\widehat{\eta}}$ say: $y = \gamma_{\widehat{x},\widehat{\eta}}(r)$ for some $r > 0$, $|\widehat{\eta}|_g = 1$; $-\dot{\gamma}_{\widehat{x},\widehat{\eta}}(r) \in \widehat{U}$, $\exp_y(\widehat{U}) = W_{\widehat{x}}$. The level sets in (1.1) define locally smooth hypersurfaces, and generate two components of data, namely, $2\rho(\widehat{x}; y, \xi_0)$ and the Hessian of $2\rho(x; y, \xi_0)$ at $x = \widehat{x}$. We suppress the factor 2 in the further analysis. The set of available points y is dense in M.

Our approach to reconstruction is as follows. We let $\widehat{x} \in \mathcal{V}$ and fix the direction $\widehat{\eta}$. We consider diffraction points along the geodesic $\gamma_{\widehat{x},\widehat{\eta}}$, $y_t = \gamma_{\widehat{x},\widehat{\eta}}(t)$. We obtain t from the measurements without knowledge of the location of y_t. We can re-interpret the measurements and obtain, for different values of the parameter t, the travel times,

$$\rho_t(x) = \{\rho(x; y_t, \xi_0);\quad y_t = \gamma_{\widehat{x},\widehat{\eta}}(t),\, x \in \mathcal{V}(y_t, \xi_0)\}; \tag{1.2}$$

$\rho_t(x)$ is a multi-valued function. We decompose the function into branches, which are smooth except for a discrete set of values of t; we will exclude these values in the further description of our procedure.

We denote $\gamma(t) = \gamma_{\widehat{x},\widehat{\eta}}(t)$. As we know $(\widetilde{M} \setminus M, g)$, we can choose a parallel, orthonormal frame of vectors F_j, $j = 1, 2, \ldots, n$, $F_n = \dot{\gamma}_{\widehat{x},\widehat{\eta}}$, on $\gamma_{\widehat{x},\widehat{\eta}}(\mathbb{R}_-) \subset \widetilde{M} \setminus M$. Then

$$H_{jk}(r, t) = \operatorname{Hess} \rho_t|_{\gamma(r)}(F_j(r), F_k(r)), \quad t > 0,\ r \le 0,$$

suppressing the dependencies on $(\widehat{x}, \widehat{\eta})$ in the notation. Then $H_{nj}(r, t) = 0$ for $j = 1, 2, \ldots, n$. For the reconstruction of g presented in this paper, we use $\partial_t^m H_{jk}(r, t)$, $m = 0, 1, 2, 3$ as the data. These are estimated, in practice, from the singular support of the observed reflections; the derivatives will be acquired by interpolation in t. In reflection seismology, the $\gamma_{\widehat{x},\widehat{\eta}}$ play the role of so-called image rays. (Image rays are obtained if $\widehat{x} \in \partial M$ and $\widehat{\eta}$ is normal to ∂M.)

We summarize the procedure originally developed by Dix [4]. To this end, we consider the Euclidean case, and a half space. We assume that we are given the travel times, and if we expand the diffraction travel time infinitesimally about $x = \widehat{x}$, we obtain

$$\rho_t(\widehat{x}) + \tfrac{1}{2} \underbrace{g_{lk}(\widehat{x}) S^k_j(t;\widehat{x})\, \mathrm{d}x^l \mathrm{d}x^j}_{(\mathrm{Hess}\,\rho_t)(\widehat{x})}, \quad \widehat{x} \in \partial M,\ t \geq 0$$

(the linear term vanishes because $\widehat{\eta}$ has been chosen normal to the boundary). Let the Euclidean coordinate normal to $\partial M = \mathbb{R}^{n-1}$ be denoted by z, and the coordinates parallel to ∂M be denoted by x'. We take $n = 2$. The expansion can then be written in the form

$$t + \tfrac{1}{2} t^{-1} S(t)\, \mathrm{d}{x'}^2.$$

The metric is given by $g_{jk}(z) = c^2(z)\delta_{jk}$, and is translationally invariant in the directions parallel to the boundary. Because of the translational invariance, only one boundary point needs to be considered, say $\widehat{x} = 0$. We introduce the function, $t(z) = \int_0^z c^{-1}(z')\, \mathrm{d}z'$, with inverse, $z(t)$; we let $\widetilde{c}(t)$ be defined through $\widetilde{c}(t(z)) = c(z)$. Then $\rho_t(0) = \int_0^t c^{-1}(z(t')) \frac{\mathrm{d}z}{\mathrm{d}t}(t')\, \mathrm{d}t' = \int_0^t c^{-1}(z(t'))\widetilde{c}(t')\, \mathrm{d}t' = t$. Then

$$S(t) = t c_{\mathrm{m}}^{-2}(t), \quad c_{\mathrm{m}}^2(t) = \int_0^t \widetilde{c}^2(t')\, \mathrm{d}t',$$

where m stands for root-mean-square. The reconstruction reads

$$\widetilde{c}(t) = \sqrt{\frac{\partial}{\partial t}(t c_{\mathrm{m}}^2(t))}, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \tfrac{1}{2}\widetilde{c}(t),$$

from which $(z(t), c(z(t)))$ follows. Since Dix, various adaptions have been considered to admit more general wavespeed functions in a half space. We mention the work of Shah [14], Hubral & Krey [8], Dubose, Jr. [5], Mann [11], and Tygel & Iversen [9].

Background and notation. We mention some general references to Riemannian geometry [6, 12, 13]. We recall the following notations in local coordinates, $(x^1, x^2, \ldots, x^n)$. The metric tensor is given by $g_{jk}(x)\mathrm{d}x^j\mathrm{d}x^k$ and the inverse of the matrix $[g_{jk}]$ is denoted by $[g^{jk}]$. The Riemannian curvature tensor, R_{ijkl}, is

$$R_{ijkl} = \frac{\partial}{\partial x^k}\Gamma^i_{jl} - \frac{\partial}{\partial x^l}\Gamma^i_{jk} + \Gamma^p_{jl}\Gamma^i_{pk} - \Gamma^p_{jk}\Gamma^i_{pl}, \quad R^p_{jkl} = g^{pi}R_{ijkl},$$

where Γ^i_{jk} are the Christoffel symbols,

$$\Gamma^i_{jk} = \frac{1}{2} g^{pi} \left(\frac{\partial g_{jp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p} \right).$$

The Ricci curvature tensor R_{ij} is given by $R_{ij} = R^k_{ijk} = g^{pq}R_{pijq}$, and R is the scalar curvature, $R = g^{ij}R_{ij}$. Finally, $\nabla_k = \nabla_{\partial_k}$ is the covariant derivative to direction $\partial_k = \frac{\partial}{\partial x^k}$, which is defined for a (1,1)-tensor A^j_l by

$$\nabla_k A^j_l = \frac{\partial}{\partial x^k} A^j_l - \Gamma^p_{kl} A^j_p + \Gamma^j_{kp} A^p_l,$$

and for a (1,0)-tensor B^l and a (0,1)-tensor B_l by

$$\nabla_k B^l = \frac{\partial}{\partial x^k} B^l + \Gamma^l_{kp} B^p, \quad \nabla_k B_l = \frac{\partial}{\partial x^k} B_l - \Gamma^p_{lk} B_p.$$

2. Jacobi and Riccati equations and Riemannian normal coordinates. We note that the relation of the Jacobi and the Riccati equations is a generally well known fact, see e.g. [3].

Let $y_1 \in \widetilde{M}$ and e_j be an orthonormal basis of $T_{y_1}\widetilde{M}$ with respect to the metric g. We use coordinates associated to this basis. Let $\xi_1 \in T_{y_1}\widetilde{M}$ be such that the differential $d\exp_{y_1}$ at ξ_1 is invertible and $x_1 = \exp_{y_1}(\xi_1)$. Then ξ_1 has a neighborhood U and x_1 has a neighborhood W in which the inverse of $\exp_{y_1} : U \to W$ gives the Riemannian normal coordinates $X_1 : W \to \mathbb{R}^n$,

$$X_1(x) = (x^1, x^2, \ldots, x^n),$$

where

$$x = \exp_{y_1}(x^j e_j) = \gamma_{y_1, \xi/|\xi|_g}(|\xi|_g), \quad \xi = x^j e_j \in T_{y_1}\widetilde{M}. \tag{2.1}$$

We say that X_1 are the Riemannian normal coordinates defined near x_1, corresponding to origin y_1, the direction ξ_1, and the basis $(e_1, e_2, \ldots, e_n)$. Then $E_j(x) = d\exp_{y_1}|_{X_1(x)} e_j$ are the coordinate vector fields in the X_1 coordinates. Usually, E_j is denoted by $\frac{\partial}{\partial x^j}$.

Let $g_{jk} = g(E_j, E_k)$ be the metric tensor in X_1 coordinates and let f be a smooth function defined near x_1. The Hessian, $(\mathrm{Hess}\, f)(V) = \nabla_V \mathrm{grad}\, f$, of f defines a bilinear form on $T_{y_1}\widetilde{M}$ given by

$$(\mathrm{Hess}\, f)(V, V') = g((\mathrm{Hess}\, f)V, V').$$

In X_1 coordinates, that is, upon substituting $V = E_j$ and $V' = E_k$, we find

$$\mathrm{Hess}\, f = \left(\frac{\partial^2}{\partial x^j \partial x^k} f - \Gamma^p_{jk} \frac{\partial}{\partial x^p} f \right) \mathrm{d}x^j \mathrm{d}x^k. \tag{2.2}$$

The (0,2)-tensor $\mathrm{Hess}\, f$ corresponds to the (1,1)-tensor $S = S^k_j(x)\, \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k}$, given by

$$\mathrm{Hess}\, f = g_{lk} S^k_j \mathrm{d}x^l \mathrm{d}x^j. \tag{2.3}$$

Then

$$(\mathrm{Hess}\, f)(V, V') = (SV, V')_g. \tag{2.4}$$

Let $\widehat{x} \in \widetilde{M}$ and $\widehat{\eta} \in T_{\widehat{x}}\widetilde{M}$, $|\widehat{\eta}|_g = 1$. Let $\gamma(t) = \gamma_{\widehat{x},\widehat{\eta}}(t)$ and set $y_1 = \gamma(t_1)$, $\xi_1 = -\dot{\gamma}(t_1)$ (then $|\xi_1|_g = 1$). We put a point source at y_1 and consider different quantities related to observations at points $\gamma(r)$, $r \in [0, t_1]$. We let $r_2 \in [0, t_1]$ and $y_2 = \gamma(r_2) = \exp_{y_1}((t_1 - r_2)\xi_1)$. We assume that r_2 is such that the differential $d\exp_{y_1}$ is invertible at $(t_1 - r_2)\xi_1$. As discussed above, there will be a neighborhood $W \subset \widetilde{M}$ of y_2 such that there are Riemannian normal coordinates, $X_1 : W \to \mathbb{R}^n$, $X_1(x) = (x^1, x^2, \ldots, x^n)$, defined near y_2, corresponding to the origin y_1, the direction ξ_1, and a basis $(e_1, e_2, \ldots, e_n)$ for which, here, $e_n = \xi_1$.

We can then extend the parameter r, defined on γ, to a function $r(x)$ defined on $W \ni y_2$ by setting

$$r(x) = t_1 - |X_1(x)|, \quad x \in W, \tag{2.5}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. We note that $X_1(y_2) = (0, \ldots, 0, t_1 - r_2)$ so that $r(y_2) = r_2$. Also,

$$r(x) = t_1 - \rho(x; y_1, (t_1 - r_2)\xi_1). \tag{2.6}$$

Moreover, for $x \in \gamma \cap W$, $r(x)$ is given in the X_1 coordinates by $r(x) = t_1 - x^n$, and we see that that the n:th coordinate vector corresponding to the X_1 coordinates satisfies

$$E_n(x) = \dot{\gamma}_{y_1, e_n}(t_1 - r(x)) = -\partial_r \quad \text{for } x \in \gamma \cap W. \tag{2.7}$$

We consider the function $\rho(.) = \rho(.; y_1, (t_1 - r_2)\xi_1) = t_1 - r(.) : W \to \mathbb{R}$. We have $\partial_r = \nabla r(x) = -\nabla \rho(x) = -\partial_\rho$, and

$$(\mathrm{Hess}\,\rho)(V) = -\nabla_V \partial_r = \nabla_V \frac{\partial}{\partial x^n}. \tag{2.8}$$

As in [12], we refer to the (1,1)-tensor S corresponding to $\mathrm{Hess}\,\rho$ as the shape operator of the level sets of ρ. As the function ρ is a generalized distance function, that is, $|\nabla \rho|_g = 1$, it follows from the radial curvature (Riccati) equation [12, sect. 4.2, Thm 2] that

$$\nabla_{\partial_\rho} S + S^2 = -R_{\partial_\rho}, \tag{2.9}$$

or, equivalently,

$$-\nabla_{\partial_r} S + S^2 = -R_{\partial_r}, \tag{2.10}$$

where $R_{\partial_r} V = R(V, \partial_r)\partial_r$ is an operator associated with the curvature R of $(\widetilde{M}, g)$, which in local coordinates is given by

$$R\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)\frac{\partial}{\partial x_k} = R^p_{ijk}\frac{\partial}{\partial x_p}. \tag{2.11}$$

The Jacobi field $J(r) \in T_{\gamma(r)}\widetilde{M}$ along the geodesic γ is a vector field satisfying

$$\nabla^2_{\partial_r} J(r) + R(J(r), \partial_r)\partial_r = 0, \quad r \in [0, t_1]. \tag{2.12}$$

As by (2.7) $\partial_r = -\frac{\partial}{\partial x^n}$ on γ, it follows from [7, p. 36]) that if $J(r)|_{r=t_1} = 0$, then $\nabla_{-\partial_r} J(r) = S(r)\,J(r)$, that is,

$$-\nabla_{\partial_r} J(r) = S(r)\,J(r), \quad r \in [0, t_1]. \tag{2.13}$$

Our strategy for reconstruction is to show that from the data we can reconstruct operator $S(r)$ on γ using Riccati equation (2.10). The curvature tensor on γ is then obtained using Jacobi equation (2.12) and (2.13). We begin with defining the data.

3. Reconstruction of the metric along one geodesic.

3.1. Wave fronts. As before, we assume that we are given $(\widetilde{M} \setminus M, g)$. Here, we revisit the data which we can assume to be also given. To specify these in the general setting, we let $x_0 \in \widetilde{M}\setminus M$, $y \in M^{\mathrm{int}}$, and $\xi_0 \in T_y M$ define a direction such that $\exp_y(\xi_0) = x_0 \notin \mathcal{C}(y)$; we denote by $\mathcal{V}(y, \xi_0) \subset \widetilde{M}\setminus M$ (relaxing for now the requirement that $\mathcal{V}(y, \xi_0) \subset \mathcal{V}$ in the introduction) a neighborhood of x_0 on which the generalized distance function, $\rho(.; y, \xi_0)$, is defined as a smooth function.

The data, in principle, provide the smooth subsets, $\Sigma(y, \xi_0; s)$, of the level sets of the functions $\rho(.; y, \xi_0) : \mathcal{V}(y, \xi_0) \to \mathbb{R}$, given by

$$\Sigma(y, \xi_0; s) = \{x \in \mathcal{V}(y, \xi_0);\ \exp_y(\xi_0) = x_0 \notin \mathcal{C}(y),\ \rho(x; y, \xi_0) = s\}, \quad s > 0;$$

that is,

$$\mathrm{Data}_1 = \{(s, \Sigma(y, \xi_0; s));\ y \in M^{\mathrm{int}},\ \xi_0 \in T_y M,\ s > 0\}. \tag{3.1}$$

We can re-organize the data as follows: There is a $\widehat{x} \in \Sigma(y, \xi_0; t)$ and $\widehat{\eta} \in T_{\widehat{x}}\widetilde{M}$, $|\eta|_g = 1$ directed such that

$$(t, \Sigma(y_t, \xi_t; t)), \quad \text{with } y_t = \gamma_{\widehat{x},\widehat{\eta}}(t),\ \xi_t = -\dot{\gamma}_{\widehat{x},\widehat{\eta}}(t)$$

(then $|\xi_t|_g = 1$) reproduce the data. We note that $\Sigma(y_t, \xi_t; t)$ is contained in a subset of $\widetilde{M}\setminus M$, changing shape with t. As discussed in the introduction, data processing provides, in practice, estimates of $\rho(\widehat{x}; y_t, \xi_t)$ and its Hessian at $\widehat{x} \in \mathcal{V}(y, \xi_0)$ determining an approximation to $\Sigma(y_t, \xi_t; t)$ in the neighborhood of $\widehat{x}$. We denote $\gamma(t) = \gamma_{\widehat{x},\widehat{\eta}}(t)$.

3.2. Modelling the data. We aim to reconstruct g along γ in suitable coordinates. As we are given $(\widetilde{M} \setminus M, g)$, we can choose a parallel, not necessarily orthonormal, frame of vectors, F_j, $j = 1, 2, \dots, n$, $F_n = \dot\gamma$ on $\gamma \cap (\widetilde{M} \setminus M)$. By parallel translation along γ, that is, $\nabla_{\dot\gamma(r)} F_j(r) = 0$, we can define this frame on the whole geodesic $\gamma(\mathbb{R})$. This frame defines the coframe, f^k, $j = 1, 2, \dots, n$, on the geodesic γ, in a unique way such that [1]

$$\langle f^k, F_j \rangle = \delta^k_j. \tag{3.2}$$

We consider the Hessian of $\rho_t(.) = \rho(.; y_t, \xi')$ with $\xi' \in T_{y_t}M$ being the vector $\xi' = (t - r)\xi_t$, evaluated at $x = \gamma(r)$:

$$H_{jk}(r, t) = \mathrm{Hess}\, \rho_t|_{\gamma(r)}(F_j(r), F_k(r)), \quad r \in \mathbb{R} \text{ is such that } \gamma(r) \notin \mathcal{C}(y_t); \tag{3.3}$$

as $x \notin \mathcal{C}(y_t)$, the function ρ_t is smooth near x. We note that $\exp_{y_t}(\xi') = \gamma(r)$. Let $S_t(r)$ be the (1,1)-tensor at $\gamma(r) \notin \mathcal{C}(y_t)$ corresponding to $\mathrm{Hess}\, \rho_t|_{\gamma(r)}$,

$$H_{jk}(r, t) = (S_t(r) F_j(r), F_k(r))_g, \tag{3.4}$$

cf. (2.4). That is, $S_t(r)$ is the shape operator associated to level sets of ρ_t.

We show that we can find the operators $S_t(r)$ for small r. To begin with, $S_t(r)$ is known for $-r_0 < r \le 0$, while $r_0 > 0$ is sufficiently small such that $S_t(r)$ does not blow up for $0 < r < r_0$. We consider the second fundamental form on the ball $|r| < r_0$. We write $S_t(r)$ in terms of parallel frames, that is,

$$S_t(r) = \mathbf{s}^k_j(r, t)\, f^j(r) \otimes F_k(r), \quad \mathbf{s}^k_j(r, t) = \langle S_t(r) F_j(r), f^k(r) \rangle. \tag{3.5}$$

and discuss how to obtain the Hessians from the mentioned level sets. As we are given $(\widetilde{M} \setminus M, g)$, we can compute for $r \in \mathbb{R}$, $|r| < r_0$, the second fundamental form, $\Pi_{\Sigma(r)}$, of the surface $\Sigma(r) := \Sigma(y_t, \xi_t; t - r)$ in $\widetilde{M} \setminus M$ at the point $\gamma(r)$. That is,

$$\Pi_{\Sigma(r)}|_{\gamma(r)} = \sum_{j,k=1}^{n-1} H_{jk}(r, t)\, f^j(r) f^k(r). \tag{3.6}$$

As $\Sigma(r)$ are the level sets of ρ_t and $|\nabla \rho_t|_g = 1$, we have $H_{nn}(r, t) = 0$ and $H_{nj}(r, t) = 0$ for $j = 1, 2, \dots, n - 1$. Thus

$$\Pi_{\Sigma(r)}|_{\gamma(r)} = \sum_{j,k=1}^{n} H_{jk}(r, t)\, f^j(r) f^k(r). \tag{3.7}$$

We have

$$H_{jk}(r, t) = \Pi_{\Sigma(r)}|_{\gamma(r)}(F_j, F_k) = g(\nabla_{F_j} N, F_k)|_{\gamma(r)}, \tag{3.8}$$

where $N = \nabla \rho(.; y_t, \xi_t)$ be the normal vector field of surfaces $\Sigma(r)$ with respect to the metric g. (We recall that the surface $\Sigma(r)$ is contained in a neighborhood where the generalized distance function is smooth.)

An explicit construction can be obtained upon choosing local coordinates, $Z(x) = (z^1, z^2, \dots, z^n)$ say, on $\widetilde{M}$, and represent $N(z) = N^a(z) \frac{\partial}{\partial z^a}$ and $F_j = F^b_j \frac{\partial}{\partial z^b}$. Then

$$\begin{aligned} g(\nabla_{F_j} N, F_k)|_{\gamma(r)} &= g\Big(F^l_j \frac{\partial N^p}{\partial z^l} \frac{\partial}{\partial z^p} - \Gamma^p_{ab} N^a F^b_j \frac{\partial}{\partial z^p}, F^q_k \frac{\partial}{\partial z^q}\Big)\Big|_{z = Z(\gamma(r))} \\ &= g_{pq}\Big(F^l_j \frac{\partial N^p}{\partial z^l} - \Gamma^p_{ab} N^a F^b_j\Big) F^q_k\Big|_{z = Z(\gamma(r))}, \quad |r| < r_0, \end{aligned} \tag{3.9}$$

[1] Here, $\langle ., . \rangle$ denotes the usual pairing of $T_y\widetilde{M}$ and $T^*_y\widetilde{M}$.

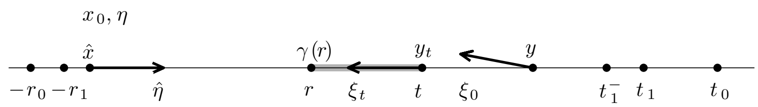

FIG. 1. *Notation, following the geodesic γ.*

where $\Gamma^p_{jk} = \Gamma^p_{jk}(z)$ are the Christoffel symbols of $(\widetilde{M} \setminus M, g)$ in the Z-coordinates.

In summary, with $\widehat{x} \in \Sigma(r)$ and $\widehat{\eta}$ being the unit normal vector of $\Sigma(r)$ at $\widehat{x}$, $y_t = \gamma_{\widehat{x},\widehat{\eta}}(t)$. Using $(\widetilde{M} \setminus M, g)$ we can evaluate the second fundamental forms, $\mathrm{II}_{\Sigma(r)}|_{\gamma(r)}$, at the points $\gamma(r) = \gamma_{\widehat{x},\widehat{\eta}}(r)$ for $|r|$ small enough. This gives us the functions $H_{jk}(r,t)$, and further the functions $\mathbf{s}^k_j(r,t)$, for $|r|$ being small enough while t is such that $\gamma(t) \in M$. We set

$$\text{Data}_2 = \{(t, (\mathbf{s}^k_j(r,t))^n_{j,k=1});\ y_t \in M^{\text{int}},\ |r| \text{ small},\ t > 0\}. \tag{3.10}$$

The notation used throughout the paper is illustrated in Figure 1.

3.3. Dirichlet-to-Neumann map. We apply the analysis presented in the previous section to points y_t on the geodesic γ, for $t > 0$.

The shape operator, S_t, of level sets of ρ_t relative to a point y_t satisfies (cf. (2.10), identifying $(t-r)\xi_1$ with $(t-r)\xi_t$)

$$-\nabla_{\partial_r} S_t + S_t^2 = -R_{\partial_r}. \tag{3.11}$$

We use the parallel frame F_j introduced in the previous subsection, and denote by $\widehat{g}_{jk}$ the inner products

$$\widehat{g}_{jk} = (F_j, F_k)_g = g(F_j, F_k).$$

Note that these inner products are constant along γ. Moreover we use a Jacobi field, $r \mapsto J(r,t)$ on γ which vanishes at y_t, that is $J(r,t)|_{r=t} = 0$. The field $J(r,t)$ is written in the frame F_j as

$$J(r,t) = \mathbf{j}^k(r,t)\, F_k(r), \quad \mathbf{j}^k(r,t) = \langle J(r,t), f^k(r)\rangle. \tag{3.12}$$

Hence,

$$-\partial_r \mathbf{j}^k(r,t) = \mathbf{s}^k_p(r,t)\mathbf{j}^p(r,t) \tag{3.13}$$

(cf. (2.13)).

We also introduce linearly independent Jacobi fields, $J_{(m)}(r,t)$, $m = 1, 2, \ldots, n$, on γ vanishing at y_t,

$$J_{(m)}(r,t)|_{r=t} = 0.$$

They satisfy (cf. (2.13))

$$-\nabla_{\partial_r} J_{(m)}(.,t) = S_t J_{(m)}(.,t), \tag{3.14}$$

and solve the equations (cf. (2.12))

$$\nabla^2_{\partial_r} J_{(m)}(.,t) + R(J_{(m)}(.,t), \partial_r)\partial_r = 0. \tag{3.15}$$

We choose

$$\nabla_{\partial_r} J_{(m)}(r,t)|_{r=t} = F_m(t). \tag{3.16}$$

We introduce matrices $\mathbf{j}^k_{(m)}(r)$ such that

$$J_{(m)}(r,t) = \mathbf{j}^k_{(m)}(r,t)\, F_k(r), \quad \mathbf{j}^k_{(m)}(r,t) = \langle J_{(m)}(r,t), f^k(r)\rangle, \tag{3.17}$$

satisfying

$$-\partial_r \mathbf{j}^k_{(m)}(r,t) = \mathbf{s}^k_p(r,t)\mathbf{j}^p_{(m)}(r,t), \tag{3.18}$$

and write

$$\mathbf{r}^k_j(r) = \langle R(f^k(r), \dot\gamma(r))\, \dot\gamma(r), F_j(r)\rangle. \tag{3.19}$$

The Riccati equation becomes

$$-\partial_r \mathbf{s}^k_j(r,t) + \mathbf{s}^k_p(r,t)\mathbf{s}^p_j(r,t) = -\mathbf{r}^k_j(r), \tag{3.20}$$

and the Jacobi equations attain the form

$$\partial_r^2 \mathbf{j}^k_{(m)}(r,t) + \mathbf{r}^k_p(r)\mathbf{j}^p_{(m)}(r,t) = 0, \quad r \in [0,t], \tag{3.21}$$

supplemented with the initial data

$$\mathbf{j}^k_{(m)}(r,t)|_{r=t} = 0, \tag{3.22}$$

and

$$\partial_r \mathbf{j}^k_{(m)}(r,t)|_{r=t} = \delta^k_m, \tag{3.23}$$

for example, generating a linearly independent set corresponding with a (data generating) point source at y_t. We also use other initial conditions in the further analysis.

Lemma 3.1. *Let $\mathbf{S}(r,t) = (\mathbf{s}^j_k(r,t))^n_{j,k=1}$ be the matrices defined in (3.5), $t_0 > 0$ and i_0 be the injectivity radius of $(\widetilde{M}, g)$ at $\gamma(t_0)$. Let $t, r \in [t_0 - i_0/2, t_0]$ and $\mathbf{K}(r,t) = \mathbf{S}(r,t)^{-1}$. Then*

$$\mathbf{K}(r,t) = (t-r)\, I + \frac{(t-r)^3}{3}\,\mathbf{R}(t) + \mathcal{O}((t-r)^4), \quad \mathbf{R}(t) = (\mathbf{r}^j_k(t))^n_{j,k=1}, \tag{3.24}$$

where $\mathcal{O}((t-r)^4)$ is estimated in a norm on the space of matrices $\mathbb{R}^{n\times n}$.

Proof. We use Jacobi fields $\mathbf{j}^k_{(m)}(s; t_-, t_+)F_k(s)$ on $\gamma([t_-, t_+])$, $m = 1, 2, \ldots, n$ satisfying

$$\partial_s^2 \mathbf{j}^k_{(m)}(s; t_-, t_+) + \mathbf{r}^k_p(s)\mathbf{j}^p_{(m)}(s; t_-, t_+) = 0, \quad s \in [t_-, t_+], \tag{3.25}$$

supplemented with the initial data

$$\mathbf{j}^k_{(m)}(s; t_-, t_+)|_{s=t_-} = \delta^k_m, \quad \mathbf{j}^k_{(m)}(s; t_-, t_+)|_{s=t_+} = 0. \tag{3.26}$$

We will consider these when $t_+ = t$ and $t_- = r$, and $t - r = \varepsilon > 0$ is small. To this end, we introduce notations

$$v^k_{(m)}(z,t) = \mathbf{j}^k_{(m)}(t-z; t-\varepsilon, t), \tag{3.27}$$

$$\rho^j_k(z,t) = \mathbf{r}^j_k(t-z), \tag{3.28}$$

where $z \in [0,\varepsilon]$. Equations (3.25)-(3.26) attain the form

$$\begin{aligned} &\partial_z^2 v^k_{(m)}(z,t) + \rho^k_p(z,t) v^p_{(m)}(z,t) = 0, \quad z \in [0,\varepsilon], \\ &\qquad v^k_{(m)}(z,t)|_{z=0} = 0, \quad v^k_{(m)}(z,t)|_{z=\varepsilon} = \delta^k_m. \end{aligned} \tag{3.29}$$

We then let

$$w^k_{(m);\varepsilon}(y,t) = v^k_{(m)}(\varepsilon y, t), \tag{3.30}$$

$$\sigma^j_{k;\varepsilon}(y,t) = \rho^j_k(\varepsilon y, t), \tag{3.31}$$

where $y \in [0,1]$. We drop the subscripts and superscripts for simplicity of notation and view w and σ as matrices. We freeze t. Then

$$\begin{aligned} \partial_y^2 w(y,t) + \varepsilon^2 \sigma(y,t) w(y,t) = 0, \quad y \in [0,1], \\ w(y,t)|_{y=0} = 0, \quad w(y,t)|_{y=1} = I. \end{aligned} \tag{3.32}$$

The supremum of the norm of the Riemannian curvature is bounded by the Poincaré constant, over $r, t \in [t_0 - i_0/2, t_0]$. It follows that

$$\|(\partial_y^2 + \varepsilon^2 \sigma(y,t))^{-1}\|_{L^2([0,1]) \to H^1_0([0,1])} \le C_6. \tag{3.33}$$

This implies that there is a $C_6 > 0$ for all t and ε such that $r, t \in [t_0 - i_0/2, t_0]$,

$$\|w(.,t)\|_{H^1_0([0,1])} \le C_7. \tag{3.34}$$

We expand $\sigma(y,t)$,

$$\sigma(y,t) = \sigma(0,t) + \mathcal{E}_\sigma(y,t), \tag{3.35}$$

with

$$\|\mathcal{E}_\sigma(.,t)\|_{L^\infty([0,1])} \le C_1 \varepsilon, \tag{3.36}$$

where C_1 is the supremum of $\|\nabla R\|_g$ on the geodesic $\gamma([t_0 - i_0/2, t_0])$ times the maximum (over $l = 1, \ldots, n$) of $|f^l|_g$ times the maximum (over $k = 1, \ldots, n$) of $|F_k|_g$. (This reflects essentially that the metric is C^3.) We expand w accordingly,

$$w(y,t) = w^0(y,t) + \mathcal{E}_w(y,t), \tag{3.37}$$

where

$$\begin{aligned} \partial_y^2 w^0(y,t) + \varepsilon^2 \sigma(0,t) w^0(y,t) = 0, \quad y \in [0,1], \\ w^0(y,t)|_{y=0} = 0, \quad w^0(y,t)|_{y=1} = I. \end{aligned} \tag{3.38}$$

We observe that there is a constant $C_4 > 0$, such that for all $r, t \in [t_0 - i_0/2, t_0]$,

$$\|(\partial_y^2 + \varepsilon^2 \sigma(0,t))^{-1}\|_{L^2([0,1]) \to H^1_0([0,1])} \le C_4. \tag{3.39}$$

This implies that there is a $C_5 > 0$ for all t and ε such that $r, t \in [t_0 - i_0/2, t_0]$,

$$\|w^0(.,t)\|_{H^1_0([0,1])} \le C_5.$$

Now

$$\begin{aligned} \partial_y^2 \mathcal{E}_w(y,t) + \varepsilon^2 \sigma(0,t) \mathcal{E}_w(y,t) = -\varepsilon^2 \mathcal{E}_\sigma(y,t) w(y,t), \quad y \in [0,1], \\ \mathcal{E}_w(y,t)|_{y=0} = 0, \quad \mathcal{E}_w(y,t)|_{y=1} = 0. \end{aligned}$$

Using (3.36), (3.34) and (3.39), we find that there is a C_2 such that

$$\|\mathcal{E}_w(.,t)\|_{H^1_0([0,1])} \le C_2 \varepsilon^3$$

for ε sufficiently small.

Denoting $\lambda = \lambda(t) = \sqrt{\sigma(0,t)} \in \mathbb{C}^{n\times n}$ (we can use any branch of the matrix square root) we get

$$w^0(y,t) = [\sin(\varepsilon\lambda(t))]^{-1}\sin(\varepsilon\lambda(t)y).$$

Expanding this solution in ε yields

$$w^0(y,t) = y\ \Big(\ I + \frac{1}{6}\varepsilon^2\lambda(t)^2(1-y^2) + \mathcal{O}(\varepsilon^4 y^4)\ \Big)$$

whence

$$w(y,t) = y\ \Big(\ I + \frac{1}{6}\varepsilon^2\lambda(t)^2(1-y^2)\ \Big) + \mathcal{E}_{w;1}(y,t), \tag{3.40}$$

where $\|\mathcal{E}_{w;1}(.,t)\|_{H^1([0,1])} \le C_3\varepsilon^3$, and

$$\partial_y w(y) = I + \frac{1}{6}\varepsilon^2\lambda(t)^2 - \frac{1}{2}\varepsilon^2\lambda(t)^2y^2 + \partial_y\mathcal{E}_{w;1}(y,t), \tag{3.41}$$

where $\|\partial_y\mathcal{E}_{w;1}(.,t)\|_{L^2([0,1])} \le C_4\varepsilon^3$.

We recall (3.27) and differentiate,

$$\partial_z v^k_{(m)}(z,t)|_{z=\varepsilon} = -\partial_s \mathbf{j}^k_{(m)}(s;t-\varepsilon,t)|_{s=t-\varepsilon}.$$

Because $v^k_{(m)}(z,t)|_{z=0} = \mathbf{j}^k_{(m)}(t;t-\varepsilon,t) = 0$,

$$-\partial_s\mathbf{j}^k_{(m)}(s;t-\varepsilon,t) = \mathbf{s}^k_p(s,t)\mathbf{j}^p_{(m)}(s;t-\varepsilon,t)$$

(cf. (3.18)). Using this identity at $s = t-\varepsilon$, we find that

$$\partial_z v(z,t)|_{z=\varepsilon} = \mathbf{S}(t-\varepsilon,t)v(z,t)|_{z=\varepsilon}. \tag{3.42}$$

However, $v(z,t)|_{z=\varepsilon} = I$, so that, with $\widetilde{\mathbf{S}}(t-\varepsilon,t) := (\widehat{g}_{lk}\mathbf{s}^k_m(t-\varepsilon,t))^n_{l,m=1}$,

$$\begin{aligned}\widetilde{\mathbf{S}}(t-\varepsilon,t) &= \mathbf{S}(t-\varepsilon,t)v(\varepsilon,t)\cdot v(\varepsilon,t) = \partial_z v(z,t)|_{z=\varepsilon}\cdot v(\varepsilon,t)\\ &= \int_0^\varepsilon (\partial_z v(z,t)\cdot\partial_z v(z,t) + \partial_z^2 v(z,t)\cdot v(z,t))\,dz,\end{aligned}$$

where $v\cdot v$ stands for $v^k_{(l)}v^j_{(m)}\widehat{g}_{km}$. Substituting (3.30) yields

$$\begin{aligned}\widetilde{\mathbf{S}}(t-\varepsilon,t) &= \int_0^1 (\varepsilon^{-1}\partial_y w(y,t)\cdot\varepsilon^{-1}\partial_y w(y,t) + \varepsilon^{-2}\partial_y^2 w(y,t)\cdot w(y,t))\,\varepsilon dy\\ &= \varepsilon^{-1}\int_0^1 (\partial_y w(y,t)\cdot\partial_y w(y,t) - \varepsilon^2\sigma(y,t)w(y,t)\cdot w(y,t))\,dy,\end{aligned}$$

using Jacobi equation (3.32).

Inserting expansions (3.40) and (3.35) gives

$$\begin{aligned}\widetilde{\mathbf{S}}(t-\varepsilon,t) = \varepsilon^{-1}\int_0^1 \Big(&\Big(I + \frac{1}{6}\varepsilon^2\lambda(t)^2 - \frac{1}{2}\varepsilon^2\lambda(t)^2y^2\Big)\cdot\Big(I + \frac{1}{6}\varepsilon^2\lambda(t)^2 - \frac{1}{2}\varepsilon^2\lambda(t)^2y^2\Big)\\ &- \varepsilon^2\sigma(0,t)y^2\Big(I + \frac{1}{6}\varepsilon^2\lambda(t)^2(I-y^2)\Big)\cdot\Big(I + \frac{1}{6}\varepsilon^2\lambda(t)^2(I-y^2)\Big)\Big)dy + \mathcal{O}(\varepsilon^2)\end{aligned}$$

so that

$$\mathbf{S}(t-\varepsilon,t) = \varepsilon^{-1}\int_0^1 \Big(\Big(I+\frac{1}{3}\varepsilon^2\lambda(t)^2-\varepsilon^2\lambda(t)^2y^2\Big)-\varepsilon^2\sigma(0,t)y^2\Big)dy+\mathcal{O}(\varepsilon^2)$$
$$= \varepsilon^{-1}I-\frac{\varepsilon}{3}\sigma(0,t)+\mathcal{O}(\varepsilon^2).$$

Then, as $\rho(0,t)=\sigma(0,t)$, we obtain

$$\mathbf{S}(t-\varepsilon,t)^{-1} = \varepsilon\left(\, I+\frac{\varepsilon^2}{3}\rho(0,t)+\mathcal{O}(\varepsilon^3)\,\right),$$

so that

$$\mathbf{K}(r,t) = (t-r)\,I+\frac{(t-r)^3}{3}\mathbf{R}(t)+\mathcal{O}((t-r)^4)$$

using that $\rho(0,t)=\mathbf{R}(t)$. $\square$

In fact, (3.21) with (3.22)-(3.23) determine Data$_2$, at $r=0$ or for $r\le 0$, through (3.18). Alternatively, we can view Data$_2$, that is, $\mathbf{s}^j_k(r,t)$ for all $t>0$, $r=0$, $j,k=1,\dots,n$, as the Dirichlet-to-Neumann matrices, $\Lambda_t:\ (\mathbf{h}^l)_{l=1}^n\mapsto((\Lambda_t\mathbf{h})^k)_{k=1}^n$, $t>0$, for the Jacobi equations, that is,

$$\partial_r^2\mathbf{j}^k(r,t)+\mathbf{r}^k_p(r)\mathbf{j}^p(r,t)=0,\quad r\in[0,t], \tag{3.43}$$

$$\mathbf{j}^k(r,t)|_{r=t}=0,\quad \mathbf{j}^k(r,t)|_{r=0}=\mathbf{h}^k, \tag{3.44}$$

so that $(\Lambda_t\mathbf{h})^k=-\partial_r\mathbf{j}^k(r,t)|_{r=0}$ using that these equations have a unique solution.

THEOREM 3.2. *Functions $\mathbf{s}^j_k(r,t)$, $t>0$, $-r_0<r\le 0$, determine uniquely functions $\mathbf{r}^j_k(r)$, $r>0$.*

Proof. We are given the matrices $\mathbf{S}(r,t)=(\mathbf{s}^j_k(r,t))^n_{j,k=1}$, $t>0$, $-r_0<r\le 0$. Using Lemma 3.1, it follows that the curvature matrix, $\mathbf{R}(r)=(\mathbf{r}^j_k(r))^n_{j,k=1}$, satisfies

$$\mathbf{R}(r)=\frac{1}{6}\partial_t^3\mathbf{K}(r,t)|_{t=r}; \tag{3.45}$$

similarly, $\mathbf{R}(r)=-\frac{1}{6}\partial_r^3\mathbf{K}(r,t)|_{r=t}$.

Using

$$\partial_r\mathbf{S}(r,t)=\mathbf{S}(r,t)^2+\mathbf{R}(r)$$

(cf. (3.20)) we find that

$$\begin{aligned}\partial_r\mathbf{K}(r,t) &= -(\mathbf{S}(r,t))^{-1}\partial_r\mathbf{S}(r,t)(\mathbf{S}(r,t))^{-1}\\ &= -(\mathbf{S}(r,t))^{-1}(\mathbf{S}(r,t)^2+\mathbf{R}(r))(\mathbf{S}(r,t))^{-1}\\ &= -I+(\mathbf{S}(r,t))^{-1}\mathbf{R}(r)(\mathbf{S}(r,t))^{-1}\\ &= -I+\mathbf{K}(r,t)\mathbf{R}(r)\mathbf{K}(r,t).\end{aligned}$$

We let ∂_t act on the final equation above, and obtain

$$\begin{aligned}\partial_r((\partial_t\mathbf{K})(r,t)) &= \partial_t(-I+\mathbf{K}(r,t)\mathbf{R}(r)\mathbf{K}(r,t))\\ &= (\partial_t\mathbf{K})(r,t)\mathbf{R}(r)\mathbf{K}(r,t)+\mathbf{K}(r,t)\mathbf{R}(r)(\partial_t\mathbf{K})(r,t)).\end{aligned}$$

Computing the second and third t-derivatives in a similar manner, and denoting $V = V(r,t) = (V^j(r,t))_{j=0}^3$, $V^j(r,t) = \partial_t^j \mathbf{K}(r,t)$ and $\mathbf{R} = \mathbf{R}(r)$, we obtain the equations

$$\partial_r V^0 = -I + V^0 \mathbf{R} V^0, \tag{3.46}$$

$$\partial_r V^1 = V^1 \mathbf{R} V^0 + V^0 \mathbf{R} V^1, \tag{3.47}$$

$$\partial_r V^2 = V^2 \mathbf{R} V^0 + V^0 \mathbf{R} V^2 + 2V^1 \mathbf{R} V^1, \tag{3.48}$$

$$\partial_r V^3 = V^3 \mathbf{R} V^0 + V^0 \mathbf{R} V^3 + 2V^2 \mathbf{R} V^1 + 2V^1 \mathbf{R} V^2. \tag{3.49}$$

Since $\mathbf{R}$ depends on V^3, this system is "closed". We define the operator, $\mathcal{T}$, according to

$$(\mathcal{T}V)(r) = V^3(r,r), \tag{3.50}$$

so that, with (3.45), $\mathbf{R} = \mathbf{R}(r) = \frac{1}{6}(\mathcal{T}V)(r)$. Hence,

$$\begin{aligned}
\partial_r V^0 &= -I + \frac{1}{6} V^0 (\mathcal{T}V) V^0, \\
\partial_r V^1 &= \frac{1}{6}(V^1 (\mathcal{T}V) V^0 + V^0 (\mathcal{T}V) V^1), \\
\partial_r V^2 &= \frac{1}{6}(V^2 (\mathcal{T}V) V^0 + V^0 (\mathcal{T}V) V^2 + 2V^1 (\mathcal{T}V) V^1), \\
\partial_r V^3 &= \frac{1}{6}(V^3 (\mathcal{T}V) V^0 + V^0 (\mathcal{T}V) V^3 + 2V^2 (\mathcal{T}V) V^1 + 2V^1 (\mathcal{T}V) V^2).
\end{aligned}$$

We write this as

$$\partial_r V(r,t) = F(V(r,t), (\mathcal{T}V)(r)),$$

where the map F is a polynomial of its variables. We then introduce

$$\mathcal{F} : \ W(r,t) \mapsto F(W(r,t), (\mathcal{T}W)(r))$$

so that the system of nonlinear differential equations attains the form

$$\partial_r V(r,t) = (\mathcal{F}V)(r,t). \tag{3.51}$$

Assuming that we are given $\mathbf{S}(r,t)|_{r=0}$ with $t > 0$, we know the initial data

$$V_0(t) = V(0,t) = (\partial_t^j (\mathbf{S}(0,t))^{-1})_{j=0}^3. \tag{3.52}$$

We now address whether the initial value problem (3.51)-(3.52) has a unique solution.

Let us now assume the $t_0 > 0$ is such that the geodesic $\gamma([0,t_0])$ has no focal points, that is, both matrices $\mathbf{S}(r,t)$ and $\mathbf{K}(r,t)$ are bounded on $(r,t) \in [0,t_0]^2$. (The Dirichlet-to-Neumann map eigenvalues of the Jacobi equations are bounded as well.) Let $\mathcal{K}$ be a prior bound on the Riemannian curvature, that is, $\|R\|_g \leq \mathcal{K}$ on $\gamma([0,t_0])$. Boundedness is then guaranteed if [2] $t - r \leq \pi/(4\sqrt{\mathcal{K}})$. Let $0 < t_1 < t_0$ and

$$Y_{t_1} = C([0,t_1]_r; C([0,t_1]_t; \mathbb{R}^{n\times n})^4)$$

equipped with the norm

$$\|V\|_{Y_{t_1}} := \sup_{r\in[0,t_1]} \|V(r,.)\|_{C([0,t_1];\mathbb{R}^{n\times n})^4} = \sup_{(r,t)\in[0,t_1]^2} \max_{j\in\{0,\ldots,3\}} \|V^j(r,t)\|_{\mathbb{R}^{n\times n}}.$$

[2] We use [12, Cor. 2.4] implying that $\mathbf{S}(r,t) \geq \frac{\cos(\sqrt{\mathcal{K}}(t-r))}{\sin(\sqrt{\mathcal{K}}(t-r))} \mathcal{K} I$.

It is immediate that

$$|V^3(r,r)| \le \sup_{(r,t)\in[0,t_1]^2} \|V^3(r,t)\|_{\mathbb{R}^{n\times n}} \le \|V\|_{Y_{t_1}}.$$

If $B_{t_1}(\mathcal{R}) \subset Y_{t_1}$ is the zero centered ball of radius $\mathcal{R}$ in Y_{t_1}, because $\mathcal{F}$ contains no differentiation, we find that

$$\mathcal{F}:\ \overline{B}_{t_1}(\mathcal{R}) \to Y_{t_1}$$

is (locally) Lipschitz, with Lipschitz constant $3\mathcal{R}^2$, that is, the Lipschitz constant does not depend on t_1.

We reformulate the differential equations in integral form, $HV = V$, with

$$H:\ Y_{t_1} \to Y_{t_1}, \quad (HW)(r,t) = V_0(t) + \int_0^r \mathcal{F}(W(r',t))dr', \quad r,t \in [0,t_1].$$

Clearly, $H:\ \overline{B}_{t_1}(\mathcal{R}) \to Y_{t_1}$ is (locally) Lipschitz, with Lipschitz constant $t_1\, 3\mathcal{R}^2$. For H to be a contraction, we need that

$$t_1\, 3\mathcal{R}^2 < 1.$$

To guarantee that $H(\overline{B}_{t_1}(\mathcal{R})) \subset \overline{B}_{t_1}(\mathcal{R})$, we require that

$$\|V_0\|_{C([0,t_0];\mathbb{R}^{n\times n})^4} + t_1\,(1+3\mathcal{R}^2) < \mathcal{R}.$$

We choose [3]

$$\mathcal{R} = 2\,\|V_0\|_{C([0,t_0];\mathbb{R}^{n\times n})^4}.$$

Then we have the condition,

$$t_1 = \frac{1}{2}\min\left(\frac{\pi}{4\sqrt{\mathcal{K}}}, \frac{1}{3\mathcal{R}^2}, \frac{\mathcal{R}}{2\,(1+3\mathcal{R}^2)}\right).$$

By the Banach fixed point theorem, H has a unique fixed point in $\overline{B}_{t_1}(\mathcal{R})$. Thus (3.51)-(3.52) has a unique solution $V \in Y_{t_1}$.

Now we complete the proof of the theorem as follows. Let us choose $t_1 > 0$ sufficiently small such that equation (3.51) has a solution. Solving $V(r,t)$ using equation (3.51) gives us also the curvature matrix $\mathbf{R}(r)$ for $r \in [0,t_1]$ by applying $\mathcal{T}$. For given $t > t_1$, let $r_1 = r_1(t)$, $0 < r_1 < r_0$, be such that $\mathbf{S}(r_1,t)$ does not blow up. Using $\mathbf{R}(r)$ and the given matrices $\mathbf{S}(-r_1,t)$, we find on the interval $r \in [-r_1,t_1]$ the solutions $\mathbf{j}^k(r,t)$ of the Jacobi equations with some given vectors $\mathbf{h}^k$, namely, by solving the Cauchy problems

$$\partial_r^2\mathbf{j}^k(r,t) + \mathbf{r}^k_p(r)\mathbf{j}^p(r,t) = 0, \quad r \in [-r_1,t_1], \tag{3.53}$$

$$\mathbf{j}^k(r,t)|_{r=-r_1} = \mathbf{h}^k, \quad \partial_r\mathbf{j}^k(r,t)|_{r=-r_1} = -\mathbf{s}^k_l(-r_1,t)\mathbf{h}^l. \tag{3.54}$$

Thus we obtain some solutions for the Jacobi equations (3.21)-(3.23) on the interval $r \in [-r_1,t_1]$. We choose $\mathbf{h}^k$ to be a basis in $T_{\gamma(-r_1)}\widetilde{M}$. We consider any $0 < t_1^- \le t_1$ be such that the $\mathbf{j}^k(r,t)|_{r=t_1^-}$ are linearly independent; then, $-\partial_r\mathbf{j}^k(r,t) = \mathbf{S}(r,t)\mathbf{j}^k(r,t)$ (cf. (3.18)) at $r = t_1^-$ determine $\mathbf{S}(t_1^-,t)$. This yields a new, "downward continued" dataset. We used the fact that matrices $\mathbf{S}(r,t)$ correspond to the Dirichlet-to-Neumann operator for Jacobi equations (3.21)-(3.23).

Iterating this construction, and using the fact that the propagation steps (in r) are uniformly bounded by below (that is, in the case when the Riemannian curvature of the manifold is uniformly bounded), we conclude that we can find the curvature tensor in Fermi coordinates on the whole geodesic, $\gamma(\mathbb{R})$. □

[3] Using [12, Cor. 2.4], equations (3.46)-(3.49), Lemma 3.1, and Gronwall's lemma, we can obtain the following basic "forward" estimates: $\|V^0(r,t)\|_{\mathbb{R}^{n\times n}} \lesssim \mathcal{K}^{-1}$, $\|V^1(r,t)\|_{\mathbb{R}^{n\times n}} \lesssim e^{\tau_1/3}$, $\|V^2(r,t)\|_{\mathbb{R}^{n\times n}} \lesssim 2\mathcal{K}e^{4\tau_1}$, and $\|V^3(r,t)\|_{\mathbb{R}^{n\times n}} \lesssim 8\mathcal{K}^3(1+\tau_1 e^{8\tau_1})e^{\tau_1}$ for $r,t \in [0,t_0]$ and $t - r \le \tau_1$, $\tau_1 = \tau_1(\mathcal{K}) = \pi/(4\sqrt{\mathcal{K}})$; the maximum of these results in an estimate for $\|V_0\|_{C([0,t_0];\mathbb{R}^n\times\mathbb{R}^n)^4}$ in terms of $\mathcal{K}$.

3.4. Reconstruction of the metric. In this subsection, we use the "fundamental matrix" associated with the Jacobi equations. We consider the Jacobi fields, $Y_{(m)}(r,r')$, $m=1,2,\dots,2n$, $r,r'\in\mathbb{R}$ on γ, satisfying

$$\nabla^2_{\partial_r}Y_{(m)}+R(Y_{(m)},\partial_r)\partial_r=0,$$

and numbered so that

$$\begin{aligned}&Y_{(m)}(r',r')=0,\ \dot{Y}_{(m)}(r',r')=\nabla_{\partial_r}Y_{(m)}(r',r')=F_m,\quad \text{for } m=1,2,\dots,n,\\ &Y_{(m)}(r',r')=F_{m-n},\ \dot{Y}_{(m)}(r',r')=\nabla_{\partial_r}Y_{(m)}(r',r')=0,\quad \text{for } m=n+1,2,\dots,2n.\end{aligned}$$

Let $\mathbf{y}^k_{(m)}(r,r')$ be such that

$$Y_{(m)}(r,r')=\mathbf{y}^k_{(m)}(r,r')\,F_k(r). \tag{3.55}$$

All the $\mathbf{y}^k_{(m)}(r,r')$ are found by solving the equations

$$\begin{aligned}&\partial_r^2\mathbf{y}^k_{(m)}(r,r')+\mathbf{r}^k_p(r)\mathbf{y}^p_{(m)}(r,r')=0,\\ &\mathbf{y}^k_{(m)}(r,r')|_{r=r'}=0,\quad \partial_r\mathbf{y}^k_{(m)}(r,r')|_{r=r'}=\delta^k_m,\quad \text{for } m=1,2,3,\dots,n,\\ &\mathbf{y}^k_{(n+m)}(r,r')|_{r=r'}=\delta^k_m,\quad \partial_r\mathbf{y}^k_{(n+m)}(r,r')|_{r=r'}=0,\quad \text{for } m=1,2,3,\dots,n,\end{aligned}$$

$r\in\mathbb{R}$, in the usual way; $(\mathbf{y}^k_{(m)}(r,r'),\partial_r\mathbf{y}^k_{(m)}(r,r'))^T_{m=1,\dots,2n}$ represents the fundamental matrix.

THEOREM 3.3. *Let $t>0$, $y_t=\gamma(t)$, and $L_t=\{r\in\mathbb{R};\ \gamma(r)\notin\mathcal{C}(y_t)\}$. Then functions $\mathbf{r}^k_j(r)$, $r>0$ determine uniquely the metric g on $\gamma(L_t)$, in the Riemannian normal coordinates having the origin at y_t and corresponding to such a basis that at $\gamma(0)$ the coordinate vector fields satisfy $\frac{\partial}{\partial x^j}=F_j(0)$, $j=1,2,\dots,n$.*

Proof. Given $\mathbf{r}^k_j(r)$, $r>0$, using equations (3.21) and (3.18), we can determine $\mathbf{s}^k_j(r,t)$ and a general basis of Jacobi fields, $\mathbf{j}^k_{(m)}(r,t)$, for all $r,t\in\mathbb{R}$, vanishing at y_t. We consider, here, particular Jacobi fields, $\widetilde{J}_{(m)}(r,t)=\sum_{k=1}^n\widetilde{\mathbf{j}}^k_{(m)}(r,t)\,F_k(r)$, constructed below.

We fix $r'\le 0$, for example, $r'=0$. We write $Y_{(m)}(r)=Y_{(m)}(r,r')$, and consider a general Jacobi field in the separated representation,

$$\widetilde{J}_{(m)}(r,t)=\sum_{p=1}^{2n}Y_{(p)}(r)a^p_m(t),\quad m=1,2,\dots,n,$$

or

$$\widetilde{\mathbf{j}}^k_{(m)}(r,t)=\sum_{p=1}^{2n}\mathbf{y}^k_{(p)}(r)a^p_m(t).$$

As we have constructed $\mathbf{y}^k_{(p)}(r)$ from $\mathbf{r}^k_p(r)$, we can impose the conditions,

$$\widetilde{J}_{(m)}(t,t)=0,\quad m=1,2,\dots,n, \tag{3.56}$$

$$\widetilde{J}_{(m)}(0,t)=F_m(0),\quad m=1,2,\dots,n, \tag{3.57}$$

or equivalently,

$$\widetilde{\mathbf{j}}^k_{(m)}(t,t)=\sum_{p=1}^{2n}\mathbf{y}^k_{(p)}(t)a^p_m(t)=0,\quad m=1,2,\dots,n,$$

$$\widetilde{\mathbf{j}}^k_{(m)}(0,t)=\sum_{p=1}^{2n}\mathbf{y}^k_{(p)}(0)a^p_m(t)=\delta^k_m,\quad m=1,2,\dots,n,$$

as linear equations for coefficients $a^p_m(t)$, $m = 1, 2, \ldots, n$. We use below the coefficients $a^p_m(t)$ obtained by solving these equations. We note that then

$$\widetilde{J}_{(n)}(r,t) = t^{-1}Y_{(n)}(r) = t^{-1}(r-t)\dot{\gamma}(r),$$

where the factor t^{-1} yields a normalization at $r = 0$.

Let us denote

$$T_{y_t}M \ni B_m = \nabla_{\partial_r}\widetilde{J}_{(m)}(r,t)|_{r=t}, \quad m = 1, 2, \ldots, n$$

and consider coordinates Z defined on a neighborhood of $\gamma([0,t])$, closely related to the normal coordinates at y_t, given by

$$Z(x) = (z^1, \ldots, z^n) \quad \text{if } x = \exp_{y_t}\Big(\sum_{m=1}^n z^m B_m\Big).$$

On γ, we have

$$\frac{\partial}{\partial z^m}\bigg|_{\gamma(r)} = \widetilde{J}_{(m)}(r,t), \quad m = 1, 2, \ldots, n$$

and in particular (cf. (3.57)),

$$\frac{\partial}{\partial z^m}\bigg|_{\gamma(0)} = \widetilde{J}_{(m)}(0,t) = F_m(0), \quad m = 1, 2, \ldots, n. \tag{3.58}$$

Using these Jacobi fields, we now obtain the metric:

$$\begin{aligned}
g\Big(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\Big)\bigg|_{\gamma(r)} &= g(\widetilde{J}_{(j)}(r,t), \widetilde{J}_{(k)}(r,t)) \\
&= \sum_{p=1}^n\sum_{q=1}^n g(\widetilde{\mathbf{j}}^p_{(j)}(r,t)\, F_p(r), \widetilde{\mathbf{j}}^q_{(k)}(r,t)\, F_q(r)) \\
&= \sum_{p=1}^n\sum_{q=1}^n \widetilde{\mathbf{j}}^p_{(j)}(r,t)\widetilde{\mathbf{j}}^q_{(k)}(r,t)\, g(F_p(r), F_q(r)) \\
&= \sum_{p=1}^n\sum_{q=1}^n \widetilde{\mathbf{j}}^p_{(j)}(r,t)\widetilde{\mathbf{j}}^q_{(k)}(r,t)\, \widehat{g}_{pq},
\end{aligned}$$

where $g(F_p(r), F_q(r)) = g(F_p(0), F_q(0)) = \widehat{g}_{pq}$. At the points $\gamma(r)$, $r \in L_t$ the vector fields $\frac{\partial}{\partial z^j}$, $j = 1, 2, \ldots, n$ form a basis. As we know the metric tensor in $\widetilde{M}\setminus M$ and the numbers $\widehat{g}_{pq}$, we can we find the metric tensor at points $\gamma(r)$, $r \in L_t$ in the Z coordinates, that is, in the Riemannian normal coordinates having the origin at y_t and corresponding to a basis $(B_m)_{m=1}^n$ having the property that at $\gamma(0)$ equation (3.58) holds. □

4. Generalization of reconstruction of the metric tensor using a family of geodesics – Reconstruction beyond conjugate points. We return to the set up given in Subsections 3.1-3.2, where we assumed that we are given $(\widetilde{M}\setminus M, g)$, $\mathcal{V} \subset \widetilde{M}\setminus M$, and data, Data_1.

Let $\widehat{x} \in \mathcal{V}$ and $\widehat{\eta} \in T_{\widehat{x}}\widetilde{M}$ be a unit vector. Let $\widehat{t}_2, \widehat{t}_1 > 0$, $\widehat{t}_2 > \widehat{t}_1$, define points $\widehat{y}_j = \gamma_{\widehat{x},\widehat{\eta}}(\widehat{t}_j) \in M$, $j = 1, 2$. We assume that $\widehat{t}_2 - \widehat{t}_1$ is smaller than the injectivity radius of $(\widetilde{M}, g)$ at $\widehat{y}_2$ and that $\widehat{x} \notin \mathcal{C}(\widehat{y}_2) \cup \mathcal{C}(\widehat{y}_1)$. We note that the last condition for a given point $\widehat{y}_2$ may be satisfied by a small perturbation of $\widehat{x}$: We set $\zeta = -\dot{\gamma}_{\widehat{x},\widehat{\eta}}(\widehat{t}_2)$ and consider the element $(t, \Sigma(\widehat{y}_2, \zeta; t)) \in \mathrm{Data}_1$, with

$$t = \widehat{t}_2 + r_1$$

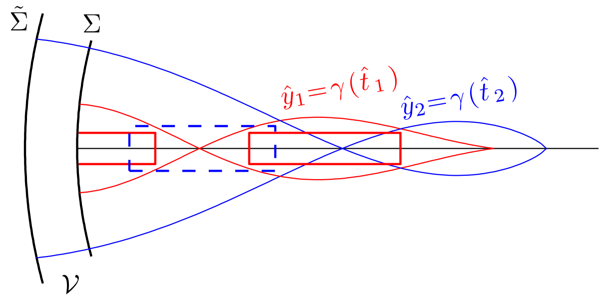

FIG. 2. *Reconstruction procedure in the case of conjugate points.*

$(r_1 > 0)$ such that

$$\Sigma(\widehat{y}_2, \zeta; t) \ni \widehat{x}_1 = \gamma_{\widehat{x},\widehat{\eta}}(-r_1) \in \mathcal{V}$$

satisfies $\widehat{x}_1 \notin \mathcal{C}(\widehat{y}_2)$ and that $\widehat{\eta}_1 = \dot{\gamma}_{\widehat{x},\widehat{\eta}}(-r_1)$ is normal to the front $\Sigma(\widehat{y}_2, \zeta; t)$ $(|\widehat{\eta}_1|_g = 1)$. We write $\Sigma = \Sigma(\widehat{y}_2, \zeta; t)$.

Our aim is to construct the metric tensor along $\gamma_{\widehat{x},\widehat{\eta}}$ near $\widehat{y}_1$ in the Riemannian normal coordinates with origin at $\widehat{y}_2$; see Figure 2. We now choose a neighborhood $W^{\Sigma}_{\widehat{x}_1} \subset \Sigma$ of $\widehat{x}_1$ with local coordinates $Y(z) = (y^1(z), \dots, y^{n-1}(z))$, $z \in W^{\Sigma}_{\widehat{x}_1}$; using these coordinates, we set $F_k = \frac{\partial}{\partial y^k}$, $k = 1, \dots, n-1$. Moreover, we let $F_n(z)$ be the unit normal vector on Σ at z. Then the $F_k(z)$, $k = 1, \dots, n$ define a smoothly varying frame on Σ. We consider the geodesics $\gamma(r; z) = \gamma_{z, F_n(z)}(r)$, $r \in \mathbb{R}$, $z \in W^{\Sigma}_{\widehat{x}_1}$, and extend the vectors $F_k(z)$, $k = 1, \dots, n$, to vector fields $F_k(r, z)$ along the geodesics $\gamma(r; z)$ using parallel translation. Then for $\widehat{r}_2 = \widehat{t}_2 + r_1$ we have $\gamma(\widehat{r}_2; z) = \widehat{y}_2$, for all $z \in W^{\Sigma}_{\widehat{x}_1}$.

We let $J_{(k)}(r, z)$, $z \in W^{\Sigma}_{\widehat{x}_1}$ be the Jacobi fields along $\gamma(r; z)$ vanishing at $r = \widehat{r}_2$ for which $J_{(k)}(0, z) = F_k(z)$. We note that $\widehat{y}_1 = \gamma_{\widehat{x},\widehat{\eta}}(\widehat{t}_1 + r_1)$ and write $\widehat{r}_1 = \widehat{t}_1 + r_1$. As $\widehat{y}_1 \notin \mathcal{C}(\widehat{y}_2)$, the Jacobi fields $J_{(k)}(\widehat{r}_1, z)$ are linearly independent at $\widehat{y}_1$, and thus there is an $\varepsilon_1 > 0$ and a neighborhood $W_1 \subset W^{\Sigma}_{\widehat{x}_1}$ of $\widehat{x}_1$ such that

$$\Psi: \ \mathcal{W} = (\widehat{r}_1 - \varepsilon_1, \widehat{r}_1 + \varepsilon_1) \times W_1 \to \mathcal{U} = \Psi(\mathcal{W}), \quad (r, z) \mapsto \gamma(r; z),$$

is a diffeomorphism; Ψ can be viewed as generating generalized boundary (Σ) normal coordinates on a neighborhood, $\mathcal{U} \subset \widetilde{M}$, of $\widehat{y}_1$.

By Theorem 3.3, we can determine the metric tensor g along any geodesic $\gamma(r; z)$ in the sense that we know the inner products $(J_{(j)}(r, z), J_{(k)}(r, z))_g$ for all $j, k = 1, 2, \dots, n$ and $(r, z) \in \mathcal{W}$. This precisely means that we can find the metric tensor in the coordinates $\Psi^{-1}: \mathcal{U} \to \mathcal{W}$. Due to this, we can find the metric tensor also in Fermi-type coordinates

$$(t, a_1, \dots, a_n) \to \exp_{\gamma(t)} \left[\sum_{j=1}^{n-1} a_j F_j(t) \right]$$

in neighborhoods of geodesics, obtained by parallel translation along the geodesics $(\nabla_{\dot{\gamma}} F_j = 0)$. Now, by varying $\widehat{t}_2$ we can find neighborhoods $\mathcal{U}$ which cover the whole geodesic γ. Indeed, the distance of the conjugate points to a given point y_t is uniformly bounded when t is in any compact interval. Thus the metric tensor can be reconstructed in the neighborhood of the whole γ despite the presence of conjugate points.

5. Discussion.

On conformal curvature tensors – Anisotropy. After we have reconstructed the metric tensor, we may ask if the obtained structure corresponds to any conformally Euclidean, that is, isotropic metric in some coordinates. To this end, we use the conformal curvature tensors. A metric g_{ij} in a domain $M \subset \mathbb{R}^n$ is conformally flat if there is a scalar function $a(x) > 0$ such that the curvature of tensor of $a(x)g_{ij}(x)$ is identically zero.

We consider a general metric tensor g_{ij} and recall some facts concerning its conformal flatness:

(1) Assume that $n = 3$. Then the conformal covariant, given in terms of curvature tensors, is given by

$$R_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(g_{ik}\nabla_j R - g_{ij}\nabla_k R).$$

Here, R_{ijk} defines a tensor that can be considered as a vector-valued 2-form $R_{ijk}dx^j \wedge dx^k$. Operating with the Hodge operator $*$ to this 2-form, we obtain the Cotton-York tensor,

$$\mathrm{C}_{ij} = g^{kp} g^{lq} \nabla_k (R_{li} - \frac{1}{4} R\, g_{li}) \epsilon_{pqj},$$

where ϵ_{pqj} is the Levi-Civita permutation symbol.

(2) Assume that $n \geq 4$. Then the Weyl tensor is given by

$$\mathrm{W}_{ijkl} = R_{ijkl} + \frac{1}{n-2}(g_{il}R_{kj} + g_{jk}R_{kj} - g_{ik}R_{lj} - g_{jl}R_{ki}) + \frac{1}{(n-1)(n-2)}(g_{ik}g_{lj} - g_{il}g_{kj})R.$$

The metric g is conformally flat if and only if in dimension $n = 3$ the Cotton-York tensor vanishes, and in dimension $n = 4$ the Weyl tensor vanishes, see [6, p. 92] or [2, 15, 1]. Thus we can detect if the medium is anisotropic.

Redundancy. Because the reconstruction procedure applies to geodesics taking off under different directions from any observation point, we can obtain many independent reconstructions of the metric tensor at any point in M^{int}; the difference of these constructions can be used in a regularization procedure when constructing the metric tensor with noise prone data.

Acknowledgments. This research was supported in part under NSF CMG grant 1025318-DMS and in part by the members, ConcoPhillips, ExxonMobil, PGS, Statoil and Total, of the Geo-Mathematical Imaging Group.

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