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KINEMATIC TIME MIGRATION AND DEMIGRATION OF REFLECTIONS IN PRESTACK SEISMIC DATA

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Abstract. In kinematic time migration one maps the time, slope, and curvature characteristics of seismic reflection events, so-called reflection-time parameters, from the recording domain of the seismic data to the time-migration domain. The inverse process is kinematic time demigration. We generalize kinematic time migration and demigration in several respects: The reflection-time parameters may belong to arbitrary source-receiver offsets; local heterogeneity of the time-migration velocity model is accounted for; the mapping operations do not depend specifically on the type of diffraction-time function and the parametrization of the velocity model. Migration and demigration spreading matrices are obtained as byproducts of the mapping operations. These matrices yield a paraxial expression for the connection between midpoint and image-point gather locations of mapped reflection events. We assume diffractions and reflections without conversion, and that sources and receivers are located along the same measurement surface. Our framework enables the identification of a full set of first- and second-order reflection-time parameters from time-migrated seismic data followed by a kinematic demigration to the recording domain. The idea of this route is to “undo” eventual errors introduced by time migration and result in reliable estimation of recording-domain invariants, i.e., parameters insensitive to the time-migration velocity model.

1. Introduction. Time migration has been widely applied by the seismic processing industry for decades and still holds the position as the most frequently used imaging technique. Considering research and development, however, the situation is different: there, most of the resources are devoted to depth-migration methods. Although time migration has clear limitations with respect to lateral velocity variations (e.g. [34]) it also has, in particular, two great advantages over depth migration: i) time migration is normally a much faster process; ii) the problem of estimating a velocity model for time migration is, in general, well posed.

Time migration transforms seismic data from the domain of its recording coordinates to another domain in time, the time-migration domain. Both data domains are five-dimensional, assuming maximal acquisition geometry. The process has an inverse counterpart, time demigration, which transforms data from the migration domain to the recording domain. For the theoretical background of general data mapping techniques, see, e.g., [21, 44, 6, 5, 36].

The final goal of the seismic processing sequence is to obtain a well focused and accurately located image in depth. However, because of the difficulties involved in estimating a reliable depth-migration velocity model and a depth image of sufficient quality, it is often preferred to perform interpretation of geological structures on time-migrated images. In this way, the ill-posed part of the imaging process can be postponed until more information is available. This probably explains why time migration is still attractive, in spite of its known limitations. Bancroft [3] reviews time-migration approaches and divides them into two categories: i) pseudo prestack-time migration, which refers to a computationally cheap route via dip-moveout (DMO) processing, and ii) full prestack time migration. Only full prestack time migration is considered in this paper.

Full prestack time migration is conventionally conducted using an explicit two-way diffraction-time function specified in the time-migration domain. Thereby, the diffraction-time function is directly related to a migration-velocity model in this domain. The *elementary diffraction time*, i.e., the two-way time from source to receiver via a single diffraction point in the subsurface, is a function of two basic entities, source-receiver offset and migration aperture. These entities are defined specifically below. Because of its shape, the elementary diffraction-time function for 2D configurations has historically been referred to as a Cheops pyramid [33, 7]. It is based on the

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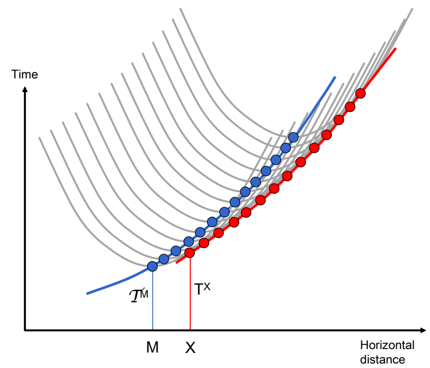

Fig. 1. *Considering a given constant offset, the collection of elementary diffraction-time functions (grey) for a given reflector in depth has the reflection-time function (red) in the recording domain as its envelope. Each output location of a classic time-migration process corresponds to minimum diffraction time between a uniquely defined diffraction point in depth and the planar horizontal datum surface. The reflection-time function (blue) in the time-migration domain represents a continuum of such elementary diffraction-time minima. Kinematic time migration is to map a local reflection event with (recording) time T^X posted in the midpoint X to a new (migrated) time $\mathcal{T}^M$ in the image-datum point M. The inverse process is called kinematic time demigration.*

classic double-square-root function, which is exact for a homogeneous isotropic medium; the so-called single-square-root function is only accurate for small offsets and apertures. It is often necessary to use traveltime expansions more sophisticated than the ones derived from the single-square-root and double-square-root representations, especially if the underlying medium is anisotropic. In [2, 43, 13] one finds examples of such expansions. There are recent approaches to time migration [14, 17] where diffraction times are computed by ray tracing. Traveltime expansions of one-way waves can be used in the context of diffractions as well as reflections. In our approach we do not restrict ourselves to short offsets.

In the recording domain, coherent local reflection events in the seismic data constitute a (hyper)surface, often referred to as the *common reflection surface (CRS)*. From such surfaces one can estimate time, slope, and curvature characteristics of the local events. (Associated reflection-time expansions were considered in [46].) These *reflection-time parameters* (or CRS parameters) can be used for multi-midpoint stacking [32] or CRS stacking [19, 26]. In CRS processing it is common to assume that the reflection-time parameters belong to zero offset; we shall however abandon this restriction in the current paper. Reflection-time parameters in the recording domain are *invariants*, i.e., they are independent with respect to the migration-velocity model in time or depth. This property makes these parameters attractive for the purpose of estimating or updating such models. Recent research is utilizing the CRS concept also in the context of time migration [31, 39]. In the time-migration domain, the CRS is generally more well behaved and more easily identified than in the recording domain, because of less noise, structures looking more like geology, and collapse of diffractions (fully or partially).

Prestack time migration and demigration have kinematic equivalents referred to as *kinematic time migration* and *kinematic time demigration*, applicable when coherent reflection events are present in the seismic data. The philosophy behind these techniques is is depicted, for a certain constant source-receiver offset, in Figure 1. A reflector in depth can be considered as a continuum of diffraction points – each point gives rise to an elementary diffraction-time function. The diffraction-time response of the entire reflector has the reflection-time function in the recording domain as its envelope. In the time-migration domain, the reflection-time function represents a continuum of all the minima of the elementary diffraction-time functions. Kinematic time migration is to map a local reflection event with (recording) time T^X posted in the midpoint X to a new (migrated) time $\mathcal{T}^M$ in the datum point M. The inverse process is kinematic time demigration. Both processes require knowledge of local slopes in the seismic data [10, 35]. In the following, a point of the type M, which specifies a common-image gather of the migrated seismic data, is referred to as an *image-datum point*.

The counterpart of kinematic time migration for mapping into depth, *kinematic depth migration* (or "map migration"), uses the same input reflection-time parameters to yield local reflector depth, dips, and curvatures [37, 28, 20, 16, 47, 24, 22, 10, 41]. It is also possible to do a corresponding time-to-depth mapping directly from the time-migration domain using image rays [18, 20]. This approach is, however, known to have more limited applicability than kinematic depth migration, as a result of the limitations inherent to conventional time migration. Some recent results in image-ray mapping are described by [45].

Whitcombe [48] introduces kinematic time demigration for the zero-offset situation and demonstrates mapping of time and slope parameters under the assumption of locally constant migration velocities. [38] describe zero-offset kinematic migration/demigration of time and slope parameters under a ray-theoretical perspective, so that the resulting mapping equations are expressed in terms of surface-to-surface ray propagator matrices of normal and image rays. [34], p. 435, points out that [48]'s equations were published relatively recently and are thus overlooked by many users. Furthermore, he states: "*Note, moreover, that these published equations must be updated to take account of time and space variability of the migration-velocity field in 3-D*". In the methodology part below, one key objective is to present such updated equations.

In this paper we extend previous approaches to kinematic time migration and demigration of reflection-time parameters so that the parameter estimation and the mapping operations can be performed for any source-receiver offset. In addition to mapping the reflection time and its slopes, we provide the option of mapping the full set of reflection-time second-derivatives. To improve accuracy the local heterogeneity of the time-migration velocity model is accounted for. The derived mapping formulas are independent of the type of diffraction-time function used and of the parametrization of the velocity model.

In typical applications of the presented methodology one would start by time-migrating the seismic data using a preliminary time-migration velocity model. The purpose is to utilize the fact that identification of seismic reflection events is generally more easily done in the migration domain than in the recording domain, even if the time-migration velocity model is not optimal. For each selected event we estimate local reflection-time parameters, which are subsequently kinematically demigrated to the recording domain. In this demigration operation one should use the same time-migration velocity model as in the originally migration of the seismic data. The idea is that this will undo eventual errors introduced by the migration and result in reliable estimation of reflection-time invariants (e.g. [1]). Knowing such invariants the ground is prepared for time-migration and depth-migration tomography. For an overview regarding velocity-model building in time or depth, see [34] and [27]. There are several recent approaches to estimating the time-migration velocity model [11, 30, 12, 29, 8, 9]. In this paper, the time-migration velocity model is assumed to be known. We discuss, however, how it can be obtained at the very end of the methodology part. Also, our paper is limited to considering kinematic (or geometric) aspects of time migration. We have work in progress which addresses also the dynamic aspects, see Appendix A.

In the following we first describe the involved coordinates of the recording and time-migration domains of the seismic data, the principles of prestack time migration, the properties of the diffraction-time function, the underlying time-migration velocity model, and the reflection-time parameters in the two data domains. Thereafter, kinematic migration and demigration is presented in the same order as they appear in the natural application sequence outlined above.

2. Coordinates of recording and migration domains. A fixed Cartesian coordinate system (ξ_1, ξ_2, ξ_3) is used for describing the three-dimensional depth domain. We use the convention of collecting the first two of these coordinates in the vector $\boldsymbol{\xi}$. The horizontal plane $\xi_3 = 0$ serves as a measurement surface where all sources and receivers of the seismic experiment are located. For an illustration of the involved lateral coordinates, see Figure 2. The recording of seismic data can then be described in terms of the five-dimensional domain $(\mathbf{s}, \mathbf{r}, t)$, where $\mathbf{s}$ and $\mathbf{r}$ are two-component

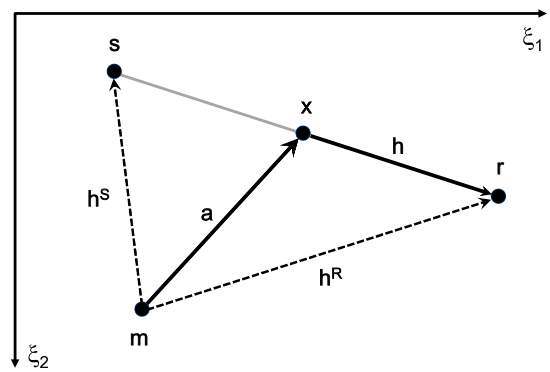

Fig. 2. *Lateral coordinates* $\mathbf{s}$*,* $\mathbf{r}$*,* $\mathbf{x}$*, and* $\mathbf{m}$ *of source point* (S)*, receiver point* (R)*, midpoint* (X)*, and image-datum point* (M)*, respectively. Also indicated are aperture* $(\mathbf{a})$*, half-offset* $(\mathbf{h})$*, source-offset* $(\mathbf{h}^S)$*, and receiver-offset* $(\mathbf{h}^R)$ *vectors.*

vectors defining the positions of any source point, S, and receiver point, R, situated along the measurement surface, and t is the recording time. The vectors $\mathbf{s}$ and $\mathbf{r}$ both belong to the Cartesian sub-coordinate system (ξ_1, ξ_2). The three coordinates of a common-source (or common-shot) gather are $(\mathbf{r}, t)$, given that the source coordinates, $\mathbf{s}$, are fixed. Conversely, for fixed receiver coordinates, $\mathbf{r}$, the coordinate space $(\mathbf{s}, t)$ constitutes a common-receiver gather.

The midpoint, X, between source and receiver shall be specified by the two-component vector $\mathbf{x}$. When introducing also half-offset coordinates, $\mathbf{h}$, such that $\mathbf{h}$ and $\mathbf{x}$ satisfy the linear transformation

$$\mathbf{h} = \frac{1}{2}(\mathbf{r} - \mathbf{s}) \ , \qquad \mathbf{x} = \frac{1}{2}(\mathbf{r} + \mathbf{s}) \ , \tag{2.1}$$

one obtains the form $(\mathbf{h}, \mathbf{x}, t)$ of the recording domain. A common-midpoint gather of the seismic data is a subset for which the midpoint vector, $\mathbf{x}$, is fixed. The internal coordinates of each common-midpoint gather are therefore $(\mathbf{h}, t)$. Likewise, for a common-offset gather the half-offset vector $\mathbf{h}$ is constant and the internal coordinates of the gather are $(\mathbf{x}, t)$. Note that one could eventually have defined the half-offset vector with opposite sign, $\mathbf{h} = (\mathbf{s} - \mathbf{r})/2$, as in [46]. If this option is preferred, one can still utilize the equations derived in this paper, but all occurrences of $\mathbf{h}$ must then be substituted by $-\mathbf{h}$.

Consider now a particular depth point, D, with lateral coordinate vector $\boldsymbol{\xi} = \mathbf{q}$ and depth $\xi_3 = z$. The common-offset depth migration domain is defined as $(\mathbf{h}, \mathbf{q}, z)$, where $\mathbf{q}$ specifies the depth migration image-datum point, Q, located vertically above point D within the depth-migration datum surface, assumed to be the plane $\xi_3 = 0$. The coordinates $(\mathbf{q}, z)$ and $(\mathbf{h}, z)$ appear in, respectively, common-offset gathers and common-image gathers of the depth-migrated seismic data.

Analogously to the above considerations, the time-migration domain is defined here as $(\mathbf{h}, \mathbf{m}, \tau)$, where τ is the migration time and the vector $\boldsymbol{\xi} = \mathbf{m}$ specifies the time-migration image-datum point, M, located in the time-migration datum surface, $\xi_3 = 0$. The image-datum points M and Q for time and depth migration usually do not coincide; this will be explained in the next section. A basic assumption behind the introduction of the time-migration domain is that the mapping between the coordinates $(\mathbf{q}, z)$ and $(\mathbf{m}, \tau)$ is one-to-one. The time variable τ is considered as a pseudo-depth parameter. As such, we have $\tau = 0$ along the datum surface. The coordinates $(\mathbf{m}, \tau)$ and $(\mathbf{h}, \tau)$ appear in, respectively, common-offset gathers and common-image gathers of the time-migrated seismic data.

The difference

$$\mathbf{a} = \mathbf{x} - \mathbf{m} \ , \tag{2.2}$$

is referred to as the aperture vector for time migration. In the migration process, the vector $\mathbf{a}$ spans the set of all input locations, $\mathbf{x}$, that contributes to the image at the output location, $\mathbf{m}$. This set of locations $\mathbf{x}$ is commonly referred to as the time-migration aperture.

For given coordinates $\mathbf{s}$, $\mathbf{r}$, and $\mathbf{m}$ we define the source-offset vector, $\mathbf{h}^S$, and the receiver-offset vector, $\mathbf{h}^R$, by

$$\mathbf{h}^S = \mathbf{s} - \mathbf{m} \ , \qquad \mathbf{h}^R = \mathbf{r} - \mathbf{m} \ . \tag{2.3}$$

Using equation 2.1 in equation 2.3 yields

$$\mathbf{h}^S = \mathbf{a} - \mathbf{h} \ , \qquad \mathbf{h}^R = \mathbf{a} + \mathbf{h} \ . \tag{2.4}$$

3. Time migration of prestack seismic data. The traveltime from source to receiver via a diffraction point in the subsurface is a fundamental entity in seismic time and depth migration. It is common to express this two-way diffraction time, T^D, as the sum of two one-way times,

$$T^D = T^S + T^R \ . \tag{3.1}$$

Here, the "source time" T^S is the traveltime of a hypothetical wave (Green's function) propagating between the source point and the diffraction point, and the "receiver time" T^R is the corresponding traveltime from the receiver point to the diffraction point. The actual arguments to be chosen for the functions T^S, T^R, and T^D will depend on the type of migration (time or depth) and the type of "common" partial images to be created. In the following our studies are limited to so-called *symmetric source- and receiver-waves*, which means that i) the underlying type of wave propagation and polarization for the one-way times T^S and T^R is the same, e.g., so that both correspond to P waves or both correspond to S waves, and ii) the sources and receivers are situated on the same measurement surface. Moreover, by considering only isolated branches of functions T^S, T^R, and T^D in a common-offset context, the diffraction time for depth migration can be specified with the arguments $T^D(\mathbf{h}, \mathbf{x}, \mathbf{q}, z)$.

The purpose of time migration is to provide meaningful seismic images that look like geology in depth, but such that the resulting images are still constituted by traces in time. Time migration normally uses an analytic diffraction-time function. Given that a unique relationship exists between the coordinates $(\mathbf{q}, z)$ and $(\mathbf{m}, \tau)$, the diffraction-time function can be expressed in the form

$$t = T^D(\mathbf{h}, \mathbf{a}, \mathbf{m}, \tau) \ , \tag{3.2}$$

with the aperture vector $\mathbf{a}$ defined in equation 2.2.

Taking as input scalar seismic data $d(\mathbf{h}, \mathbf{x}, t)$ with frequency spectrum $D(\mathbf{h}, \mathbf{x}, \omega)$ and also a given weight function, $W(\mathbf{h}, \mathbf{x}, \mathbf{m}, \tau, \omega)$, one can outline diffraction-stack time migration for constant offset $\mathbf{h}$ and output location $(\mathbf{m}, \tau)$ by the integrations

$$\tilde{d}(\mathbf{h}, \mathbf{m}, \tau) = \int\int W^*(\mathbf{h}, \mathbf{m}+\mathbf{a}, \mathbf{m}, \tau, \omega)\, D(\mathbf{h}, \mathbf{m}+\mathbf{a}, \omega)\, \exp\left[\imath\omega T^D(\mathbf{h}, \mathbf{a}, \mathbf{m}, \tau)\right] \mathrm{d}\mathbf{a}\, \mathrm{d}\omega \ , \tag{3.3}$$

see equations A.1–A.3. Migration by conventional diffraction stack in [36, p. 195, eq. 5] uses a weight function denoted K_{DS}. Their diffraction-stack equation complies with equation 3.3 if we take

$$W^*(\mathbf{h}, \mathbf{x}, \mathbf{m}, \tau, \omega) = \frac{1}{4\pi^2}\, \imath\omega\, K_{DS}(\mathbf{h}, \mathbf{x}, \mathbf{m}, \tau) \ . \tag{3.4}$$

It is common to formulate the function T^D for a single diffraction point D such that the image-datum point M will be situated vertically above D (at the projection point Q) if the underlying depth-velocity model is homogeneous. However, when considering symmetric waves in isotropic media, [18] showed that the points M and Q generally do not coincide in the presence of lateral velocity variations. Rather, the point D will be connected to the datum surface by a generally non-straight and non-vertical image ray. This situation is depicted in Figure 3. The emergence point

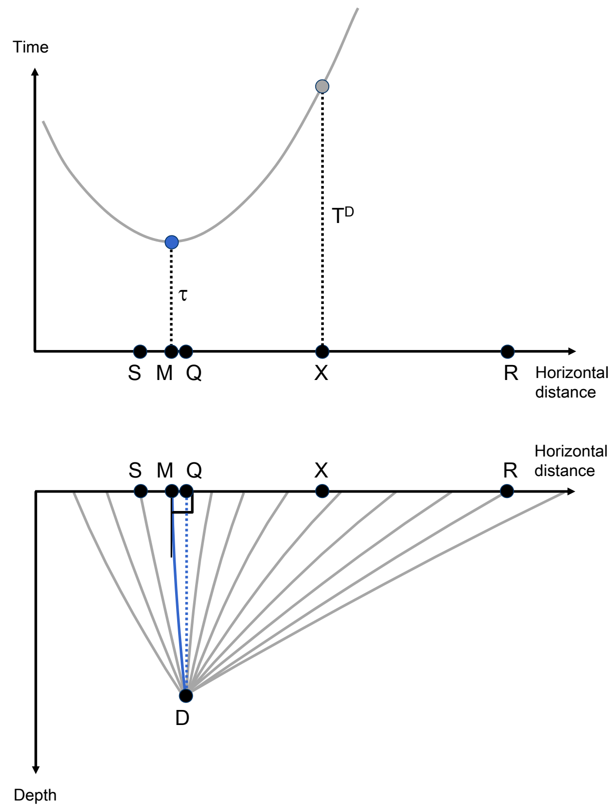

FIG. 3. *Diffraction-time concept in classic prestack time migration: (a) diffraction-time curve (grey) for a single depth point, D, with the time T^D marked (grey dot) for a midpoint (X) between selected source (S) and receiver (R) points; (b) corresponding diffraction ray paths (grey). The diffraction point D is imaged at the location M and time τ (blue dot) in the time-migration domain, for which the diffraction-time function has a minimum. The points D and M are uniquely connected by an image ray (solid blue curve). The vertical projection (dotted blue line) of point D to its true lateral position in the time-migration domain, Q, is indicated.*

M of the image ray at the datum level corresponds to minimum two-way time for the diffraction generated at the point D. In this way, we can consider the coordinate vector $\mathbf{m}$ of point M defined by stationarity of the function T^D, namely,

$$\frac{\partial T^D}{\partial \mathbf{a}}(\mathbf{h} = \mathbf{0}, \mathbf{a} = \mathbf{0}, \mathbf{m}, \tau) = \mathbf{0} \ . \tag{3.5}$$

The assumption of one-to-one correspondence $(\mathbf{q}, z) \leftrightarrow (\mathbf{m}, \tau)$ mentioned above implies that the image-ray field cannot contain caustics.

For anisotropic media there are additional concerns. In particular, if the point Q is to be situated vertically above point D in a homogeneous anisotropic medium, the slowness surface has to be laterally symmetric in any plane containing the vertical axis. As a consequence, image-ray paths will be vertical lines for constant medium parameters. When medium parameters are varying, the ray paths will still be normal to the migration-datum surface as in the isotropic situation. This inherent constraint on slowness-surface symmetry means that a classic time migration approach can be performed for transversely isotropic media with a vertical symmetry axis (VTI media) and for orthorhombic media with symmetry planes aligned with the main coordinate planes of the (ξ_1, ξ_2, ξ_3) coordinate system. The classic approach is however not adequate for media significantly violating the slowness-surface symmetry criterion, such as a tilted transversely isotropic (TTI) medium with a prominent tilt of the symmetry axis.

In the approach to kinematic time migration and demigration presented below, we do not introduce specific restrictions regarding heterogeneity or anisotropy. One basic requirement that always have to be fulfilled, however, is absence of caustics in the diffraction-time field.

4. Diffraction-time function for time migration. In this section we discuss the general properties of diffraction-time functions used in time migration.

Consistently with the above discussions it is assumed that the source- and receiver-time functions T^S and T^R in equation 3.1 are single-valued. We express them as

$$T^S(\mathbf{h}^S, \mathbf{m}, T_0^S) \ , \qquad T^R(\mathbf{h}^R, \mathbf{m}, T_0^R) \ , \tag{4.1}$$

where $\mathbf{h}^S$ and $\mathbf{h}^R$ are source- and receiver-offset vectors (equations 2.3–2.4), and T_0^S and T_0^R are source and receiver times corresponding to the situations $\mathbf{h}^S = \mathbf{0}$ and $\mathbf{h}^R = \mathbf{0}$. Moreover, the migration time τ shall be specifically defined as

$$\tau \equiv T_0^S + T_0^R \ . \tag{4.2}$$

Symmetric wave propagation and the fact that the measurement and migration-datum surfaces are identical collectively imply that $T_0^S = T_0^R = \tau/2$. As a consequence, the diffraction-time function can be written in the form

$$T^D(\mathbf{h}, \mathbf{a}, \mathbf{m}, \tau) = T^S\left(\mathbf{a} - \mathbf{h}, \mathbf{m}, \frac{\tau}{2}\right) + T^R\left(\mathbf{a} + \mathbf{h}, \mathbf{m}, \frac{\tau}{2}\right) \ . \tag{4.3}$$

The standard approach in time migration is to assume a certain analytic relationship for T^S and T^R expressed in terms of coefficients evaluated at $\mathbf{a} = \mathbf{h} = \mathbf{0}$. Such coefficients have a simple relationship to the parameters of the time-migration velocity model, which has to be known.

4.1. Diffraction-time coefficients. For the kinematic migration and demigration processes described below we need the diffraction time T^D and its partial derivatives evaluated for specific values of half offset, $\mathbf{h}$, aperture, $\mathbf{a}$, image-gather location, $\mathbf{m}$, and migration time, τ. From all such derivatives we form the diffraction-time coefficients

$$\begin{gathered}
u = \frac{\partial T^D}{\partial \tau} \ , \\
\mathbf{q}^h = \frac{\partial T^D}{\partial \mathbf{h}} \ , \qquad \mathbf{q}^a = \frac{\partial T^D}{\partial \mathbf{a}} \ , \qquad \mathbf{q}^m = \frac{\partial T^D}{\partial \mathbf{m}} \ , \\
u^\tau = \frac{\partial^2 T^D}{\partial \tau^2} \ , \\
\mathbf{u}^h = \frac{\partial^2 T^D}{\partial \mathbf{h} \partial \tau} \ , \qquad \mathbf{u}^a = \frac{\partial^2 T^D}{\partial \mathbf{a} \partial \tau} \ , \qquad \mathbf{u}^m = \frac{\partial^2 T^D}{\partial \mathbf{m} \partial \tau} \ , \\
\mathbf{U}^{hh} = \frac{\partial^2 T^D}{\partial \mathbf{h} \partial \mathbf{h}^T} \ , \qquad \mathbf{U}^{aa} = \frac{\partial^2 T^D}{\partial \mathbf{a} \partial \mathbf{a}^T} \ , \qquad \mathbf{U}^{mm} = \frac{\partial^2 T^D}{\partial \mathbf{m} \partial \mathbf{m}^T} \ . \\
\mathbf{U}^{ha} = \frac{\partial^2 T^D}{\partial \mathbf{h} \partial \mathbf{a}^T} \ , \qquad \mathbf{U}^{hm} = \frac{\partial^2 T^D}{\partial \mathbf{h} \partial \mathbf{m}^T} \ , \qquad \mathbf{U}^{am} = \frac{\partial^2 T^D}{\partial \mathbf{a} \partial \mathbf{m}^T} \ , \\
\mathbf{U}^{ah} = \mathbf{U}^{ha\,T} \ , \qquad \mathbf{U}^{mh} = \mathbf{U}^{hm\,T} \ , \qquad \mathbf{U}^{ma} = \mathbf{U}^{am\,T} \ .
\end{gathered} \tag{4.4}$$

In order to better visualize the properties of these coefficients, we introduce a seven-component column vector, $\boldsymbol{\alpha} = (h_1, h_2, a_1, a_2, m_1, m_2, \tau)^T$, containing the arguments of the diffraction-time function, so that the complete set of first and second derivatives of function T^D is given by a seven-component gradient vector and a symmetric 7×7 matrix,

$$\frac{\partial T^D}{\partial \boldsymbol{\alpha}} = \begin{pmatrix} \mathbf{q}^h \\ \mathbf{q}^a \\ \mathbf{q}^m \\ u \end{pmatrix} \ , \qquad \frac{\partial^2 T^D}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}^T} = \begin{pmatrix} \mathbf{U}^{hh} & \mathbf{U}^{ha} & \mathbf{U}^{hm} & \mathbf{u}^h \\ \cdot & \mathbf{U}^{aa} & \mathbf{U}^{am} & \mathbf{u}^a \\ \cdot & \cdot & \mathbf{U}^{mm} & \mathbf{u}^m \\ \cdot & \cdot & \cdot & u^\tau \end{pmatrix} \ . \tag{4.5}$$

For clarity, elements of the lower triangular part of matrix $\partial^2 T^D/\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T$ are only indicated by dots.

4.2. General properties of the diffraction-time function. The diffraction-time function has some general properties that are independent of the actual function representation. Considering symmetric diffractions, one immediate observation is that the function T^D has to be symmetric in the variable $\mathbf{h}$, i.e., we always have

$$T^D(\mathbf{h},\mathbf{a},\mathbf{m},\tau) = T^D(-\mathbf{h},\mathbf{a},\mathbf{m},\tau)\ . \tag{4.6}$$

The reason is reciprocity: the diffraction time is the same if source and receiver positions are interchanged. As a consequence, the partial derivatives with respect to half-offset coordinates vanishes at zero offset,

$$\frac{\partial T^D}{\partial \mathbf{h}}(\mathbf{h}=\mathbf{0},\mathbf{a},\mathbf{m},\tau) = \mathbf{0}\ . \tag{4.7}$$

Equation 4.7 has the implication that also mixed partial second derivatives involving half offset $\mathbf{h}$ vanish for $\mathbf{h}=\mathbf{0}$. The set of diffraction-time coefficients in equation 4.5 at zero offset ($\mathbf{h}=\mathbf{0}$) and generally nonzero aperture ($\mathbf{a}\neq\mathbf{0}$) can therefore be written

$$\frac{\partial T^D}{\partial\boldsymbol{\alpha}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{q}^a \\ \mathbf{q}^m \\ u \end{pmatrix}, \qquad \frac{\partial^2 T^D}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T} = \begin{pmatrix} \mathbf{U}^{hh} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cdot & \mathbf{U}^{aa} & \mathbf{U}^{am} & \mathbf{u}^a \\ \cdot & \cdot & \mathbf{U}^{mm} & \mathbf{u}^m \\ \cdot & \cdot & \cdot & u^\tau \end{pmatrix}. \tag{4.8}$$

It is inherent to conventional time migration that the diffraction-time function T^D has one and only one apex in the coordinates $\mathbf{h}$ and $\mathbf{a}$. This apex is located at $(\mathbf{h}=\mathbf{0},\mathbf{a}=\mathbf{0},\mathbf{m},\tau)$, where the gradients with respect to $\mathbf{h}$ and $\mathbf{a}$ are zero (equations 3.5 and 4.7). The situation $\mathbf{h}=\mathbf{a}=\mathbf{0}$ yields a considerably simplified set of diffraction-time coefficients. In particular, one can show that the relation

$$\frac{\partial^2 T^D}{\partial\mathbf{a}\partial\mathbf{a}^T}(\mathbf{h}=\mathbf{0},\mathbf{a}=\mathbf{0},\mathbf{m},\tau) = \frac{\partial^2 T^D}{\partial\mathbf{h}\partial\mathbf{h}^T}(\mathbf{h}=\mathbf{0},\mathbf{a}=\mathbf{0},\mathbf{m},\tau) \tag{4.9}$$

holds for any choice of diffraction-time function with the form of equation 4.3. Based on equation 4.9 we define a symmetric 2×2 matrix

$$\mathbf{S}^M(\mathbf{m},\tau) \equiv \frac{1}{4}\tau\mathbf{U}^{aa}(\mathbf{h}=\mathbf{0},\mathbf{a}=\mathbf{0},\mathbf{m},\tau) = \frac{1}{4}\tau\mathbf{U}^{hh}(\mathbf{h}=\mathbf{0},\mathbf{a}=\mathbf{0},\mathbf{m},\tau)\ , \tag{4.10}$$

which will be useful in derivations below. We observe that matrix $\mathbf{S}^M$ is a function of the three (volume) variables (m_1, m_2, τ) and has unit of squared slowness.

We shall also utilize that the diffraction time is equal to the migration time τ at any apex location (equation 4.2), which leaves all first- and second-order partial derivatives of T^D involving $\mathbf{m}$ equal to zero. For the same reason all mixed-term second derivatives involving $\mathbf{h}$ or $\mathbf{a}$ are also zero. In summary, we therefore find that the set of diffraction-time coefficients at $\mathbf{h}=\mathbf{a}=\mathbf{0}$ has the structure

$$\frac{\partial T^D}{\partial\boldsymbol{\alpha}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ u \end{pmatrix}, \qquad \frac{\partial^2 T^D}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^T} = \begin{pmatrix} \frac{4}{\tau}\mathbf{S}^M & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cdot & \frac{4}{\tau}\mathbf{S}^M & \mathbf{0} & \mathbf{0} \\ \cdot & \cdot & \mathbf{0} & \mathbf{0} \\ \cdot & \cdot & \cdot & u^\tau \end{pmatrix}. \tag{4.11}$$

4.3. Example 1: double-square-root function. One common realization of diffraction time is the double-square-root function based on exact traveltime equations for P- or S-wave propagation in a homogeneous isotropic medium. The diffraction time is obtained as $T^D = T^S + T^R$ (see equation 4.3) with one-way times T^S and T^R specified by

$$T^S = \sqrt{\frac{\tau^2}{4} + (\mathbf{a}-\mathbf{h})^T \, \mathbf{S}^M(\mathbf{m},\tau)\,(\mathbf{a}-\mathbf{h})}\ ,\ T^R = \sqrt{\frac{\tau^2}{4} + (\mathbf{a}+\mathbf{h})^T \, \mathbf{S}^M(\mathbf{m},\tau)\,(\mathbf{a}+\mathbf{h})}\ . \tag{4.12}$$

Matrix $\mathbf{S}^M$ has been defined in equation 4.10. The double-square-root function T^D is symmetric not only in the variable $\mathbf{h}$ but also in $\mathbf{a}$, so that

$$T^D(\mathbf{h},\mathbf{a},\mathbf{m},\tau) = T^D(\mathbf{h},-\mathbf{a},\mathbf{m},\tau)\ . \tag{4.13}$$

4.4. Example 2: single-square-root function. Another example of diffraction-time functions is the single-square-root approximation. Following [20] we can express it as

$$T^D(\mathbf{h},\mathbf{a},\mathbf{m},\tau) = \sqrt{\tau^2 + 4\mathbf{a}^T\mathbf{S}^M(\mathbf{m},\tau)\,\mathbf{a} + 4\mathbf{h}^T\mathbf{S}^M(\mathbf{m},\tau)\,\mathbf{h}}\ . \tag{4.14}$$

As opposed to the double-square-toot function (which uses equation 4.12), the single-square-root function in equation 4.14 is not exact for nonzero apertures and offsets in homogeneous isotropic media. At zero aperture or at zero offset, however, the single and double square-root formulas are identical. One can see immediately that the symmetry properties in equations 4.6 and 4.13 holds for the single-square-root approximation.

5. Time-migration velocity model. The complete set of parameters required for time migration is referred to as the time-migration velocity model or to just as the time-migration velocity, in the case of a mono-parametric representation. Following common practise, the model is defined in the time-migration coordinates $(\mathbf{m},\tau)$, so that the model parameters do not depend on offset. To permit formulations of kinematic migration and demigration that do not rely on a particular model parametrization, we use the general form $\mathcal{V}_i(\mathbf{m},\tau)$, $i = 1, 2, \ldots, N$, where N is the number of parameters. One possibility is to use the symmetric 2×2 matrix $\mathbf{S}^M$, defined in equation 4.10, as a basis for defining the parameters $(\mathcal{V}_i)$. When using conventional double-square-root or single-square-root diffraction-time functions it is sufficient to consider three model parameters, e.g., $\mathcal{V}_1 = S^M_{11}$; $\mathcal{V}_2 = S^M_{22}$; $\mathcal{V}_3 = S^M_{12} = S^M_{21}$.

Matrix $\mathbf{S}^M$ has historically been related to a location- and direction-dependent time-migration velocity $V^M(\theta^a,\mathbf{m},\tau)$ corresponding to small apertures and offsets, such that [20, 25]

$$\left[V^M(\theta^a,\mathbf{m},\tau)\right]^{-2} = \mathbf{e}(\theta^a)^T\,\mathbf{S}^M(\mathbf{m},\tau)\,\mathbf{e}(\theta^a)\ . \tag{5.1}$$

Here, the directional dependency of V^M is specified in terms of the two-component unit vector $\mathbf{e}(\theta^a) = (\cos\theta^a, \sin\theta^a)^T$, which corresponds to a normalization of the aperture vector $\mathbf{a}$. Observe that it would be formally equivalent to specify V^M using instead a similar unit vector $\mathbf{e}(\theta^h)$ obtained by a normalization of the half-offset vector $\mathbf{h}$. It is not equally practical, though, as prestack time migration is commonly done separately for constant-offset sub-cubes of the seismic data set.

For a fixed location $(\mathbf{m},\tau)$ equation 5.1 yields the *time-migration ellipse* with coefficients specified by matrix $\mathbf{S}^M$. For this reason we refer to $\mathbf{S}^M$ as the *time-migration-ellipse matrix.* The restriction of function V^M to small apertures and offsets comes from its relation to second-derivatives of diffraction time evaluated at $\mathbf{h} = \mathbf{a} = \mathbf{0}$, see equation 4.10.

6. Reflection time. In the recording- and migration domains we now introduce two single-valued reflection-time functions corresponding to symmetrically reflected waves,

$$t = T(\mathbf{h},\mathbf{x})\ , \qquad \tau = \mathcal{T}(\mathbf{h},\mathbf{m})\ . \tag{6.1}$$

The first of these is historically known as a common-reflection surface (CRS) belonging to a prestack seismic data set; the second function yields the corresponding common-reflection surface in the time-migration domain. The common-reflection surfaces $T(\mathbf{h},\mathbf{x})$ and $\mathcal{T}(\mathbf{h},\mathbf{x})$ can be parameterized locally in terms of reflection-time parameters, which is the subject of the following subsections.

6.1. Reflection-time parameters in the recording domain. The vector couple $(\mathbf{h}, \mathbf{x})$ specifies traces within the recording domain. In equations below we equivalently refer to such traces using the four-component column vector $\bar{\mathbf{x}} = (h_1, h_2, x_1, x_2)^T$. For a reflection event on a given trace, $(\mathbf{h}_0, \mathbf{x}_0)$, say, we associate a number of reflection-time parameters, namely: reflection time, $T^X = T(\mathbf{h}_0, \mathbf{x}_0)$, slope (first-derivative) vectors $\mathbf{p}^h = \partial T/\partial \mathbf{h}$, $\mathbf{p}^x = \partial T/\partial \mathbf{x}$, and second-derivative matrices $\mathbf{M}^{hh} = \partial^2 T/\partial \mathbf{h}\partial \mathbf{h}^T$, $\mathbf{M}^{hx} = \partial^2 T/\partial \mathbf{h}\partial \mathbf{x}^T$, $\mathbf{M}^{xx} = \partial^2 T/\partial \mathbf{x}\partial \mathbf{x}^T$. For a better overview the latter first- and second-derivative parameters can be collected in a four component vector, $\bar{\mathbf{p}}$, and a 4×4 matrix, $\bar{\mathbf{M}}$, as follows [46, 15],

$$\bar{\mathbf{p}} = \frac{\partial T}{\partial \bar{\mathbf{x}}} = \begin{pmatrix} \mathbf{p}^h \\ \mathbf{p}^x \end{pmatrix} , \qquad \bar{\mathbf{M}} = \frac{\partial^2 T}{\partial \bar{\mathbf{x}} \partial \bar{\mathbf{x}}^T} = \begin{pmatrix} \mathbf{M}^{hh} & \mathbf{M}^{hx} \\ \mathbf{M}^{hx\,T} & \mathbf{M}^{xx} \end{pmatrix} . \tag{6.2}$$

The time T^X and the parameters in equation 6.2 are evaluated at a specific trace $(\mathbf{h}_0, \mathbf{x}_0)$. In the following, however, we consider all these parameters to be functions of $\mathbf{h}$ and $\mathbf{x}$.

Reciprocity of symmetric reflections implies that

$$T(\mathbf{h}, \mathbf{x}) = T(-\mathbf{h}, \mathbf{x}) . \tag{6.3}$$

As a consequence, the following partial derivatives involving half offset vanish at zero offset,

$$\frac{\partial T}{\partial \mathbf{h}}(\mathbf{h} = \mathbf{0}, \mathbf{x}) = \mathbf{0} , \qquad \frac{\partial^2 T}{\partial \mathbf{h} \partial \mathbf{x}^T}(\mathbf{h} = \mathbf{0}, \mathbf{x}) = \mathbf{0} . \tag{6.4}$$

These properties determine the classic CRS parameters [26, e.g.,] and yield vector $\bar{\mathbf{p}}$ and matrix $\bar{\mathbf{M}}$ as

$$\bar{\mathbf{p}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{p}^x \end{pmatrix} , \qquad \bar{\mathbf{M}} = \begin{pmatrix} \mathbf{M}^{hh} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{xx} \end{pmatrix} , \tag{6.5}$$

where all parameters correspond to $(\mathbf{h} = \mathbf{0}, \mathbf{x})$.

The reflection-time parameters constituting the second-derivative matrix $\mathbf{M}^{hh}$ evaluated for $\mathbf{h} = \mathbf{0}$ have a particular interpretation in terms of normal-moveout (NMO) velocity. To aid this interpretation we introduce a 2×2 matrix

$$\mathbf{S}^{NMO}(\mathbf{x}) \equiv \frac{1}{4} T(\mathbf{h} = \mathbf{0}, \mathbf{x})\, \mathbf{M}^{hh}(\mathbf{h} = \mathbf{0}, \mathbf{x}) . \tag{6.6}$$

The matrix $\mathbf{S}^{NMO}$ can be related to a direction-dependent and surface-specific normal-moveout velocity $v^{NMO}(\theta^h, \mathbf{x})$ associated with small offsets, such that [20, 23]

$$\left[v^{NMO}(\theta^h, \mathbf{x}) \right]^{-2} = \mathbf{e}(\theta^h)^T\, \mathbf{S}^{NMO}(\mathbf{x})\; \mathbf{e}(\theta^h) . \tag{6.7}$$

Here, $\mathbf{e}(\theta^h)$ is the unit vector $(\cos\theta^h, \sin\theta^h)^T$ corresponding to normalization of the vector $\mathbf{h}$. In the case of a fixed midpoint location, $\mathbf{x}$, equation 6.7 constitutes the *NMO ellipse*. The association of parameter v^{NMO} with small offsets comes from the connection to second-derivatives of reflection time evaluated at zero offset (equation 6.6).

The elements of matrix $\mathbf{S}^{NMO}$ determine the shape of the NMO ellipse and is therefore referred to here as the *NMO-ellipse matrix*. Matrix $\mathbf{S}^{NMO}$ is a function of the two (surface) variables (x_1, x_2) and has unit of squared slowness. It is essential to recognize that matrix $\mathbf{S}^{NMO}$ is a surface function, in contrast to the time-migration-ellipse matrix, $\mathbf{S}^M$, which is a volume function. By imposing strong limitations on the shape of (depth) reflectors, however, it is possible to consider also the NMO-ellipse matrix (and corresponding angle-dependent NMO velocity) as a volume function, i.e., as a single-valued function of the variables (x_1, x_2, t).

6.2. Reflection-time parameters in the time-migration domain. In the time-migration domain, each trace is uniquely specified by the vector couple $(\mathbf{h}, \mathbf{m})$ or by the equivalent four-component vector $\bar{\mathbf{m}} = (h_1, h_2, m_1, m_2)^T$. For a migrated reflection event on a certain trace, $(\mathbf{h}_0, \mathbf{m}_0)$, we consider the following reflection-time parameters: migrated reflection time, $\mathcal{T}^M = \mathcal{T}(\mathbf{h}_0, \mathbf{m}_0)$, slope vectors $\boldsymbol{\psi}^h = \partial\mathcal{T}/\partial\mathbf{h}$, $\boldsymbol{\psi}^m = \partial\mathcal{T}/\partial\mathbf{m}$, and second-derivative matrices $\boldsymbol{\mathcal{M}}^{hh} = \partial^2\mathcal{T}/\partial\mathbf{h}\partial\mathbf{h}^T$, $\boldsymbol{\mathcal{M}}^{hm} = \partial^2\mathcal{T}/\partial\mathbf{h}\partial\mathbf{m}^T$, $\boldsymbol{\mathcal{M}}^{mm} = \partial^2\mathcal{T}/\partial\mathbf{m}\partial\mathbf{m}^T$. As in the recording domain, it is convenient to assemble the first- and second-derivative parameters in a four-component vector and a 4×4 matrix,

$$\bar{\boldsymbol{\psi}} = \frac{\partial\mathcal{T}}{\partial\bar{\mathbf{m}}} = \begin{pmatrix} \boldsymbol{\psi}^h \\ \boldsymbol{\psi}^m \end{pmatrix} , \qquad \bar{\boldsymbol{\mathcal{M}}} = \frac{\partial^2\mathcal{T}}{\partial\bar{\mathbf{m}}\partial\bar{\mathbf{m}}^T} = \begin{pmatrix} \boldsymbol{\mathcal{M}}^{hh} & \boldsymbol{\mathcal{M}}^{hm} \\ \boldsymbol{\mathcal{M}}^{hm\,T} & \boldsymbol{\mathcal{M}}^{mm} \end{pmatrix} . \tag{6.8}$$

In the following, the parameters in equation 6.8 are considered functions of $\mathbf{h}$ and $\mathbf{m}$.

As will be proved in the derivation of equations 7.15—7.16 below, the entities $\boldsymbol{\psi}^h$ and $\boldsymbol{\mathcal{M}}^{hm}$ are zero when $\mathbf{h} = \mathbf{0}$. The natural definition of (zero-offset) CRS parameters in the time-migration domain is therefore

$$\bar{\boldsymbol{\psi}} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\psi}^m \end{pmatrix} , \qquad \bar{\boldsymbol{\mathcal{M}}} = \begin{pmatrix} \boldsymbol{\mathcal{M}}^{hh} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{M}}^{mm} \end{pmatrix} , \tag{6.9}$$

where all parameters are evaluated for $(\mathbf{h} = \mathbf{0}, \mathbf{m})$. An ideal prestack time-migration result would imply that the entities $\boldsymbol{\psi}^h$, $\boldsymbol{\mathcal{M}}^{hh}$, and $\boldsymbol{\mathcal{M}}^{hm}$ are zero for all $(\mathbf{h}, \mathbf{m})$. When this is not the case one can use observations of such parameters to update the time-migration velocity model, applying the so-called extension principle.

7. Kinematic time demigration. We consider the situation that a prestack seismic data set has been migrated using a known time-migration velocity model, $(\mathcal{V}_i)$. This model is not necessarily the optimal one, but it is assumed sufficiently accurate to yield well-defined coherent reflection events within common-offset subsets of the data. We further assume that a picking process has been applied, and that a set of reflection-time parameters, as specified in equation 6.8, has been estimated. These known reflection-time parameters shall now be mapped from the migration domain to the recording domain by kinematic time demigration, to yield the output parameter set in equation 6.2. The process relies on the computation of first- and second order partial derivatives of the known diffraction-time function, T^D, see equation 4.4. We shall need access to first- and second-order partial derivatives of each model parameter $\mathcal{V}_i$ with respect to the coordinates $\mathbf{m}$ and τ within the time-migration domain.

In a demigration of parameters corresponding to migrated reflection time $\mathcal{T}(\mathbf{h}, \mathbf{m})$ the output trace $(\mathbf{h}, \mathbf{x})$ is assumed to have a single-valued relationship to the input trace $(\mathbf{h}, \mathbf{m})$ such that

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m}) . \tag{7.1}$$

Because of the function relationship in equation 7.1, the aperture vector $\mathbf{a}$ in equation 2.2 will also be a function of $\mathbf{m}$ and $\mathbf{h}$, given by

$$\mathbf{a} = \hat{\mathbf{a}}(\mathbf{h}, \mathbf{m}) = \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m}) - \mathbf{m} . \tag{7.2}$$

By applying a stationary-phase argument (Appendix B) one can state two basic conditions for demigration of reflection-time parameters: Firstly, the time of the diffraction-time function must equal the reflection time corresponding to the output trace $(\mathbf{h}, \mathbf{x}) = [\mathbf{h}, \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m})]$, which means that

$$T\left(\mathbf{h}, \hat{\mathbf{x}}\right) = T^D\left[\mathbf{h}, \hat{\mathbf{x}} - \mathbf{m}, \mathbf{m}, \mathcal{T}\left(\mathbf{h}, \mathbf{m}\right)\right] . \tag{7.3}$$

Secondly, for each constant half-offset vector, $\mathbf{h}$, the diffraction-time function is required to be tangent to the reflection time branch at the output trace location on the measurement surface, $\hat{\mathbf{x}}$, i.e.,

$$\frac{\partial T}{\partial\mathbf{x}}\left(\mathbf{h}, \hat{\mathbf{x}}\right) = \frac{\partial T^D}{\partial\mathbf{a}}\left[\mathbf{h}, \hat{\mathbf{x}} - \mathbf{m}, \mathbf{m}, \mathcal{T}\left(\mathbf{h}, \mathbf{m}\right)\right] . \tag{7.4}$$

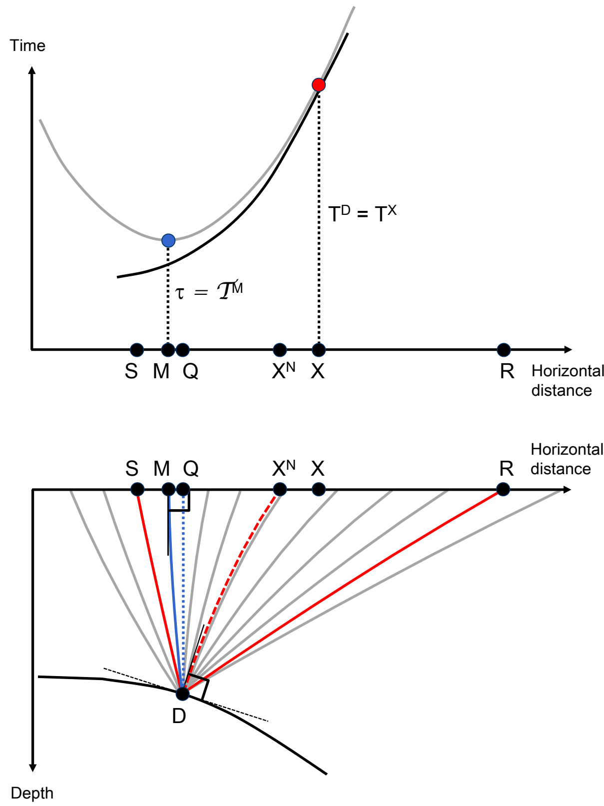

FIG. 4. *Special situation of Figure 3 where a specular reflection occurs at the point D: (a) most of the contribution to the time-migrated image at point M and time τ (blue dot) then comes from a region around point X and time T^X (red dot) on the reflection-time function (black curve) in the recording domain; (b) the reflector (black curve) through the point D and the reflection ray SDR (solid red curve) are indicated. Also shown is the normal ray $X^N DX^N$ (dashed red curve).*

The conditions in equations 7.3–7.4 are completely general and can thus be used for any type of diffraction-time function. For a visualization of the connection between the diffraction-time and reflection-time functions, see Figure 4. The flow of the kinematic time demigration process is outlined in Figure 5.

7.1. Demigrated position, reflection time, and reflection slopes. In this and the following subsections we establish a general framework for kinematic time demigration. Details of the derivations are given in Appendix C.

Using condition 7.4 in equation C.3 yields a *consistency equation*

$$\frac{\partial T^D}{\partial \mathbf{a}}\left[\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right] - \frac{\partial T^D}{\partial \mathbf{m}}\left[\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right] = \frac{\partial T^D}{\partial \tau}\left[\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right] \frac{\partial \mathcal{T}}{\partial \mathbf{m}}(\mathbf{h}, \mathbf{m}) \,, \tag{7.5}$$

which can be solved to provide the aperture vector, $\hat{\mathbf{a}} = \hat{\mathbf{x}} - \mathbf{m}$. Once we know this vector, the computation of the output time $T^X = T^D\left[\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right]$ (equation 7.3) and the output slope vector $\mathbf{p}^x = \partial T^D / \partial \mathbf{a}\left[\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right]$ (equation 7.4) is usually straightforward. The consistency equation 7.5 reflects the canonical relation A.5.

Naturally, the complexity of the algorithm required to compute $\hat{\mathbf{a}}$ will depend on the form of the diffraction-time function, T^D. Some forms of T^D yield analytical solutions for $\hat{\mathbf{a}}$; others not.

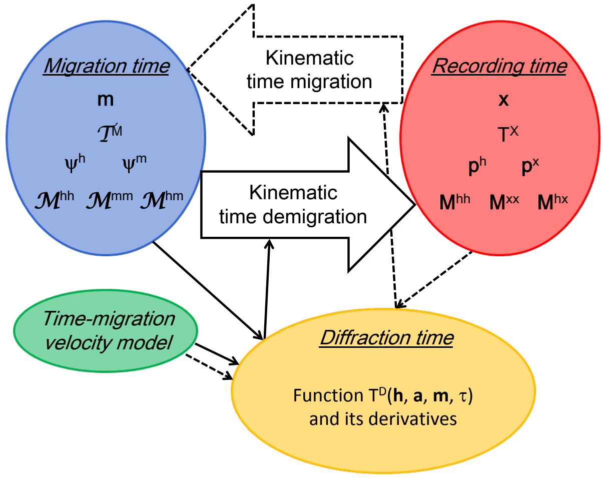

FIG. 5. *Kinematic time migration (dashed) and demigration (solid) processes for constant offset with indicated input and output reflection-time parameters. The processes estimate the aperture vector and a number of diffraction-time coefficients using the given input parameters and the known time-migration velocity model.*

In particular, as is discussed further below, $\hat{\mathbf{a}}$ can be obtained analytically under the assumption of single-square-root diffraction time and a homogeneous time-migration velocity model. For given coordinates $(\mathbf{h}, \mathbf{m})$ and time $\tau = \mathcal{T}(\mathbf{h}, \mathbf{m})$, equation 7.5 is considered in the form $f_i(\mathbf{a}) = 0$, $i = 1, 2$, with the goal of computing a root, $\mathbf{a} = \hat{\mathbf{a}}$, numerically. In this respect, one possible approach is the Newton-Raphson method, starting by assuming some trial solution for $\hat{\mathbf{a}}$ which complies with equation 7.5 only approximately, and then iterate until consistency is achieved within some predefined numerical limit. Given a physically meaningful solution $\hat{\mathbf{a}}$, one can proceed to obtain demigrated time, slopes, and second derivatives.

We have already seen that the slope vector $\mathbf{p}^x$ can be computed from equation 7.4 when vector $\hat{\mathbf{a}}$ is known. There is an alternative way to compute $\mathbf{p}^x$, provided by equation C.3, which can be restated as

$$\mathbf{p}^x = \mathbf{q}^m + u\,\boldsymbol{\psi}^m \; . \tag{7.6}$$

Equation 7.6 relates the slope $\boldsymbol{\psi}^m$ in the migration domain to the slope $\mathbf{p}^x$ in the recording domain. Computation of the diffraction-time coefficients $\mathbf{q}^m$ and u requires the knowledge of vector $\hat{\mathbf{a}}$. Similarly, equation C.5 shows that the slopes $\boldsymbol{\psi}^h$ and $\mathbf{p}^h$ are related by

$$\mathbf{p}^h = \mathbf{q}^h + u\,\boldsymbol{\psi}^h \; . \tag{7.7}$$

7.2. Demigrated reflection-time second derivatives. Assuming that the output point $(\mathbf{h}, \hat{\mathbf{x}}, T^X)$ from kinematic time demigration is known, one can map the second derivatives $\bar{\boldsymbol{\mathcal{M}}} \to \bar{\mathbf{M}}$ by the following set of equations,

$$\begin{aligned}
\mathbf{M}^{hh} &= u\,\boldsymbol{\mathcal{M}}^{hh} + \mathbf{L}^{hh} \\
&\quad - \left(u\,\boldsymbol{\mathcal{M}}^{hm} + \mathbf{L}^{hm} - \mathbf{K}^{ha}\right)\left(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}\right)^{-1}\left(u\,\boldsymbol{\mathcal{M}}^{hm} + \mathbf{L}^{hm} - \mathbf{K}^{ha}\right)^T \; , \\
\mathbf{M}^{hx} &= \mathbf{K}^{ha} - \left(u\,\boldsymbol{\mathcal{M}}^{hm} + \mathbf{L}^{hm} - \mathbf{K}^{ha}\right)\left(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}\right)^{-1}\left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^T \; , \\
\mathbf{M}^{xx} &= \mathbf{U}^{aa} - \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)\left(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}\right)^{-1}\left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^T \; .
\end{aligned} \tag{7.8}$$

The mapping relations 7.8 include a number of 2×2 matrices, defined in terms of diffraction-time coefficients (equation 4.4) and reflection-time slope $\bar{\psi}$ in the migration domain (equation 6.8), as follows,

$$\mathbf{K}^{ha} = \mathbf{U}^{ha} + \boldsymbol{\psi}^{h}\mathbf{u}^{aT} \ , \qquad \mathbf{K}^{hm} = \mathbf{U}^{hm} + \boldsymbol{\psi}^{h}\mathbf{u}^{mT} \ , \qquad \mathbf{K}^{am} = \mathbf{U}^{am} + \mathbf{u}^{a}\boldsymbol{\psi}^{mT} \ ,$$

$$\mathbf{K}^{ah} = \mathbf{K}^{ha^T} \ , \qquad \mathbf{K}^{mh} = \mathbf{K}^{hm^T} \ , \qquad \mathbf{K}^{ma} = \mathbf{K}^{am^T} \ ,$$

$$\mathbf{L}^{mm} = \mathbf{U}^{mm} + \boldsymbol{\psi}^{m}\mathbf{u}^{mT} + \mathbf{u}^{m}\boldsymbol{\psi}^{mT} + u^{\tau}\,\boldsymbol{\psi}^{m}\boldsymbol{\psi}^{mT} \ ,$$

$$\mathbf{L}^{hh} = \mathbf{U}^{hh} + \boldsymbol{\psi}^{h}\mathbf{u}^{h^T} + \mathbf{u}^{h}\boldsymbol{\psi}^{h^T} + u^{\tau}\,\boldsymbol{\psi}^{h}\boldsymbol{\psi}^{h^T} \ ,$$

$$\mathbf{L}^{hm} = \mathbf{U}^{hm} + \boldsymbol{\psi}^{h}\mathbf{u}^{mT} + \mathbf{u}^{h}\boldsymbol{\psi}^{mT} + u^{\tau}\,\boldsymbol{\psi}^{h}\boldsymbol{\psi}^{mT} \ ,$$

$$\mathbf{Y} = \mathbf{U}^{aa} + \mathbf{L}^{mm} - (\mathbf{K}^{am} + \mathbf{K}^{ma}) \ . \tag{7.9}$$

Among these, the matrices $\mathbf{L}^{mm}$, $\mathbf{L}^{hh}$, and $\mathbf{Y}$ are symmetric.

To first order, the change in the output position $\hat{\mathbf{x}}$ can be described as

$$\Delta\hat{\mathbf{x}} = \mathbf{X}^{h}\Delta\mathbf{h} + \mathbf{X}^{m}\Delta\mathbf{m} \ , \tag{7.10}$$

with 2×2 matrices $\mathbf{X}^h$ and $\mathbf{X}^h$ given by

$$\mathbf{X}^{h} = \frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{h}^{T}} = \left(\frac{\partial\hat{x}_i}{\partial h_j}\right) \ , \qquad \mathbf{X}^{m} = \frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{m}^{T}} = \left(\frac{\partial\hat{x}_i}{\partial m_j}\right) \ . \tag{7.11}$$

We refer to these as *demigration-spreading matrices*. Matrix $\mathbf{X}^h$ describes first-order changes of the *output location* in the recording domain, here: common midpoint, as a result of changing the *migration gather*, here: half offset, while the *input location* in the migration domain, the common image-datum point, is unchanged. Matrix $\mathbf{X}^m$ describes first-order changes of the same output location as a result of changing the image-datum point, while the migration gather is kept constant. In this way, $\mathbf{X}^h$ and $\mathbf{X}^m$ have similarities with paraxial matrices known from ray theory.

The mapping operation $\bar{\boldsymbol{\mathcal{M}}} \to \bar{\mathbf{M}}$ yields the demigration-spreading matrices in equation 7.11 as byproducts (see Appendix C),

$$\mathbf{X}^{h} = -\left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^{-T}\left(u\,\boldsymbol{\mathcal{M}}^{hm} + \mathbf{L}^{hm} - \mathbf{K}^{ha}\right)^{T} \ ,$$

$$\mathbf{X}^{m} = -\left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^{-T}\left(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}\right) \ . \tag{7.12}$$

From the last sub-equation 7.12 one can observe that the matrix $u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}$ is not necessarily invertible. In a caustic situation, matrix $\mathbf{X}^m$ will have determinant equal zero, and the inverse $(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y})^{-1}$ can not be computed. Application of mapping equation 7.8 is therefore limited to cases where matrix $\mathbf{X}^m$ is non-singular.

7.3. Kinematic demigration from focused migrated images. If the time migration of the seismic data resulted in perfect focusing (no residual moveout after the migration), all derivatives of the migrated reflection time $\mathcal{T}$ with respect to half offset are zero. As a consequence, the coefficient matrices in equation 7.9 are subjected to simplifications

$$\mathbf{K}^{ha} = \mathbf{U}^{ha} \ , \qquad \mathbf{K}^{hm} = \mathbf{U}^{hm} \ ,$$

$$\mathbf{L}^{hh} = \mathbf{U}^{hh} \;, \qquad \mathbf{L}^{hm} = \mathbf{U}^{hm} + \mathbf{u}^h \boldsymbol{\psi}^{mT} \;, \tag{7.13}$$

while equation 7.7, first two sub-equations 7.8, and first sub-equation 7.12 reappear in idealized versions as

$$\mathbf{p}^h = \mathbf{q}^h \;,$$

$$\mathbf{M}^{hh} = \mathbf{U}^{hh} - \left(\mathbf{U}^{hm} - \mathbf{U}^{ha} + \mathbf{u}^h \boldsymbol{\psi}^{mT}\right) \left(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}\right)^{-1} \left(\mathbf{U}^{hm} - \mathbf{U}^{ha} + \mathbf{u}^h \boldsymbol{\psi}^{mT}\right)^T \;,$$

$$\mathbf{M}^{hx} = \mathbf{K}^{ha} - \left(\mathbf{U}^{hm} - \mathbf{U}^{ha} + \mathbf{u}^h \boldsymbol{\psi}^{mT}\right) \left(u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y}\right)^{-1} \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^T \;,$$

$$\mathbf{X}^h = -\left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^{-T} \left(\mathbf{U}^{hm} - \mathbf{U}^{ha} + \mathbf{u}^h \boldsymbol{\psi}^{mT}\right)^T \;. \tag{7.14}$$

7.4. Demigration at zero offset. In the zero-offset situation the diffraction-time coefficients satisfy equation 4.8, while the reflection-time function T exhibits the properties in equation 6.5. This yields $\mathbf{u}^h = \mathbf{0}$; $\mathbf{q}^h = \mathbf{0}$; $\mathbf{p}^h = \mathbf{0}$, and also $\mathbf{U}^{ha} = \mathbf{U}^{hm} = \mathbf{0}$; $\mathbf{M}^{hx} = \mathbf{0}$. Using equation 7.7 it follows that

$$\boldsymbol{\psi}^h = \mathbf{0} \;. \tag{7.15}$$

In other words, at zero offset the slope with respect to offset of the reflection-time function in the migration domain is zero, regardless of the type of diffraction-time function and the parametrization of the time-migration velocity model. A further consequence is that $\mathbf{K}^{ha} = \mathbf{K}^{hm} = \mathbf{L}^{hm} = \mathbf{0}$, and from equations 7.8 and 7.12 we therefore obtain

$$\boldsymbol{\mathcal{M}}^{hm} = \mathbf{0} \;, \qquad \mathbf{M}^{hh} = u\,\boldsymbol{\mathcal{M}}^{hh} + \mathbf{U}^{hh} \;, \qquad \mathbf{X}^h = \mathbf{0} \;. \tag{7.16}$$

For a perfectly focused migration result at zero offset, we have $\boldsymbol{\mathcal{M}}^{hh} = \mathbf{0}$ and the expression for matrix $\mathbf{M}^{hh}$ in equation 7.16 simplifies to

$$\mathbf{M}^{hh} = \mathbf{U}^{hh} \;. \tag{7.17}$$

8. Kinematic time migration. Our approach to kinematic time migration is structured in a similar way as the kinematic demigration approach described above. A schematic overview is depicted in Figure 5. The input reflection-time parameters to kinematic migration are exactly those that were output from kinematic demigration, namely, the time function T and its first- and second-derivative parameters in equation 6.2. These parameters are assumed known for a certain trace $(\mathbf{h}, \mathbf{x})$ in the recording domain. However, since the time-migration velocity model is specified in the migration domain, not in the recording domain, the coefficients of the diffraction-time function T^D are inherently expressed in terms of the *output point* $(\mathbf{h}, \mathbf{m}, \tau)$ of the kinematic migration and not the input point, $(\mathbf{h}, \mathbf{x}, t)$. In order to solve this fundamental problem, it is necessary to compute the output point before proceeding to kinematic mapping of first- and second-order reflection-time parameters. Apart from in trivial situations (e.g., assuming homogeneous time-migration velocity and single-square-root diffraction time) the computation of the output point will have to be done numerically.

In a kinematic migration corresponding to a reflection-time branch $T(\mathbf{h}, \mathbf{x})$ in the recording domain the output trace $(\mathbf{h}, \mathbf{m})$ is assumed to have a single-valued relationship to the input trace $(\mathbf{h}, \mathbf{x})$ such that

$$\mathbf{m} = \hat{\mathbf{m}}(\mathbf{h}, \mathbf{x}) \;. \tag{8.1}$$

The aperture function $\hat{\mathbf{a}}$ introduced in equation 7.2 is redefined accordingly to read

$$\mathbf{a} = \hat{\mathbf{a}}(\mathbf{h}, \mathbf{x}) = \mathbf{x} - \hat{\mathbf{m}}(\mathbf{h}, \mathbf{x}) \ . \tag{8.2}$$

The migration output location, $\hat{\mathbf{m}}$, will generally have to be found numerically, while the mapping of all first and second-derivative parameters can be performed by means of analytical formulas. For kinematic migration under the assumption of perfect focusing the results are essentially the same as for kinematic demigration, see equations 7.13, 7.14, and 7.17.

8.1. Migrated position, reflection time, and reflection slopes. The basic conditions in equations 7.3–7.4 formulated for kinematic demigration have to be satisfied also for kinematic migration. Hence, the output time $\hat{\tau}$ and the corresponding aperture vector $\hat{\mathbf{a}}$ have to comply with the relations

$$T(\mathbf{h}, \mathbf{x}) = T^D(\mathbf{h}, \hat{\mathbf{a}}, \mathbf{x} - \hat{\mathbf{a}}, \hat{\tau}) \ ,$$

$$\frac{\partial T}{\partial \mathbf{x}}(\mathbf{h}, \mathbf{x}) = \frac{\partial T^D}{\partial \mathbf{a}}(\mathbf{h}, \hat{\mathbf{a}}, \mathbf{x} - \hat{\mathbf{a}}, \hat{\tau}) \ . \tag{8.3}$$

The latter system of three component equations can be worked on iteratively until one finds a solution for $\hat{\tau}$ and $\hat{\mathbf{a}}$ (and therefore also $\hat{\mathbf{m}} = \mathbf{x} - \hat{\mathbf{a}}$). Thereby, we are ready to compute all the diffraction-time coefficients in equation 4.4. The sought reflection time in the migration domain is retrieved as the solution $\mathcal{T}^M = \hat{\tau}$.

From equations 7.6 and 7.7 it is clear that slopes $\bar{\boldsymbol{\psi}}$ in the migration domain can be computed from slopes $\bar{\mathbf{p}}$ in the recording domain using

$$\boldsymbol{\psi}^h = \frac{1}{u}(\mathbf{p}^h - \mathbf{q}^h) \ , \qquad \boldsymbol{\psi}^m = \frac{1}{u}(\mathbf{p}^x - \mathbf{q}^m) \ . \tag{8.4}$$

Knowing the slopes $\bar{\boldsymbol{\psi}}$, and also the diffraction-time coefficients in equation 4.4, we have sufficient information to compute all the mapping-coefficient 2×2 matrices in equation 7.9.

8.2. Migrated reflection-time second derivatives. The next step is to map reflection-time second derivatives $\bar{\mathbf{M}} \to \bar{\boldsymbol{\mathcal{M}}}$ from the recording domain to the migration domain. We achieve this by reversing the transformations in equation 7.8 and taking advantage of intermediate results in Appendix C. This yields

$$\begin{aligned} \boldsymbol{\mathcal{M}}^{hh} = \frac{1}{u} \Big[& \mathbf{M}^{hh} - \mathbf{L}^{hh} \\ & - \left(\mathbf{M}^{hx} - \mathbf{K}^{ha}\right) \left(\mathbf{M}^{xx} - \mathbf{U}^{aa}\right)^{-1} \left(\mathbf{M}^{hx} - \mathbf{K}^{ha}\right)^T \Big] \ , \end{aligned}$$

$$\boldsymbol{\mathcal{M}}^{hm} = \frac{1}{u} \left[\mathbf{L}^{hm} - \mathbf{K}^{ha} + \left(\mathbf{M}^{hx} - \mathbf{K}^{ha}\right) \left(\mathbf{M}^{xx} - \mathbf{U}^{aa}\right)^{-1} \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right) \right] \ ,$$

$$\boldsymbol{\mathcal{M}}^{mm} = -\frac{1}{u} \left[\mathbf{Y} + \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^T \left(\mathbf{M}^{xx} - \mathbf{U}^{aa}\right)^{-1} \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right) \right] \ . \tag{8.5}$$

The change of output position $\hat{\mathbf{m}}$ is given to first order by

$$\Delta \hat{\mathbf{m}} = \boldsymbol{\mathcal{X}}^h \Delta \mathbf{h} + \boldsymbol{\mathcal{X}}^x \Delta \mathbf{x} \tag{8.6}$$

where the two 2×2 *migration-spreading matrices* $\boldsymbol{\mathcal{X}}^h$ and $\boldsymbol{\mathcal{X}}^x$ have the definitions

$$\boldsymbol{\mathcal{X}}^h = \frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{h}^T} = \left(\frac{\partial \hat{m}_i}{\partial h_j} \right) \ , \qquad \boldsymbol{\mathcal{X}}^x = \frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{x}^T} = \left(\frac{\partial \hat{m}_i}{\partial x_j} \right) \ . \tag{8.7}$$

The matrices $\boldsymbol{\mathcal{X}}^h$ and $\boldsymbol{\mathcal{X}}^x$ describe first-order changes of the common image-datum point as a result of changing, respectively, the half-offset vector and the common-midpoint location, while the other entity (midpoint and half offset) is kept constant.

In view of equations 7.10 and 8.6 the migration-spreading matrices $\boldsymbol{\mathcal{X}}^h$ and $\boldsymbol{\mathcal{X}}^x$ can be expressed in terms of the corresponding demigration-spreading matrices as

$$\boldsymbol{\mathcal{X}}^h = -\left(\mathbf{X}^m\right)^{-1} \mathbf{X}^h \ , \qquad \boldsymbol{\mathcal{X}}^x = \left(\mathbf{X}^m\right)^{-1} \ . \tag{8.8}$$

Using equations 7.12, 8.8, as well as Appendix C, we formulate the migration-spreading matrices in terms of reflection-time parameters of the recording domain as

$$\boldsymbol{\mathcal{X}}^h = \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^{-1} \left(\mathbf{M}^{hx} - \mathbf{K}^{ha}\right)^T \ ,$$

$$\boldsymbol{\mathcal{X}}^x = \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^{-1} \left(\mathbf{M}^{xx} - \mathbf{U}^{aa}\right) \ . \tag{8.9}$$

9. Kinematic time migration/demigration using single/double-square-root diffraction times. Up to now, we have considered completely general schemes for kinematic time migration and demigration, which are independent of the choice of diffraction-time function and the parametrization of the time-migration velocity model. In the following, we turn to considering kinematic migration/demigration with specific conventionally used functions, namely, the single and double square-root approximations. For the double-square-root approximation at arbitrary offsets (equation 4.12), it is not practical to write explicit expressions for the mapping operations, as these will become very extensive. Instead, we refer to the general mapping framework described above, where one should insert diffraction-time coefficients as specified in Appendix D.

9.1. Migration/demigration with single-square-root diffraction time. We consider kinematic migration an demigration using the single-square-root approximation to diffraction time (equation 4.14). All required coefficients for this function are given in Appendix E.

9.1.1. Position and reflection time. Inserting the expressions for the partial derivatives given by equations E.2, E.3, and E.4 into the consistency equation 7.5 we obtain the relation

$$4S^M_{kj}\hat{a}_j - \mathcal{T}^M\psi^m_k - 2\left(\frac{\partial S^M_{ij}}{\partial m_k} + \psi^m_k\frac{\partial S^M_{ij}}{\partial \tau}\right)\left(\hat{a}_i\hat{a}_j + h_ih_j\right) = 0 \ . \tag{9.1}$$

Equation 9.1 can be used for demigration of the lateral input position $\mathbf{m}$ in the migration domain, by finding the aperture vector $\hat{\mathbf{a}}$ that corresponds to the time $\mathcal{T}^M$ and slope $\boldsymbol{\psi}^m$ of a reflection event identified in this domain. We observe that the equation is second order with respect to the components of vector $\hat{\mathbf{a}}$, which implies two potential roots. One of these is to be classified as "non-physical". If the variations of matrix $\mathbf{S}^M$ with $(\mathbf{m},\tau)$ are neglected one obtains the solution

$$\hat{\mathbf{a}} = \frac{1}{4}\mathcal{T}^M\mathbf{S}^{M^{-1}}\boldsymbol{\psi}^m \ . \tag{9.2}$$

The latter result, pertaining to a homogeneous time-migration velocity model, corresponds to Whitcombe's [48] equation 3 in the 2D situation and to Söllner and Andersen's [38] equation 8 in the 3D situation. When vector $\hat{\mathbf{a}}$ is known we use equation 4.14 to obtain the demigrated reflection time, T^X.

For kinematic migration it is generally required to solve the equation system 8.3, thus implying a simultaneous estimation of aperture vector $\hat{\mathbf{a}}$ and migrated reflection time, $\mathcal{T}^M$. However, if the time-migration velocity model is homogeneous one can first compute $\hat{\mathbf{a}}$ by combining the last sub-equation 8.3 with equation E.2 and subsequently the migrated time $\mathcal{T}^M$ using equation 4.14.

9.1.2. Reflection-time slopes and second derivatives; spreading matrices. Having obtained the aperture vector $\hat{\mathbf{a}}$ and the output (migrated/demigrated) reflection time, it is fairly straightforward to obtain all corresponding output slopes and second derivatives, by combining the above general formulations of kinematic migration/demigration with the explicit coefficients for the single-square-root diffraction-time function given in Appendix E.

For better insight and clarity, it may be instructive to neglect variations of matrix $\mathbf{S}^M$ with $(\mathbf{m}, \tau)$. As a result, we get simple relations i) between slope vectors in the recording and migration domains,

$$T^X \mathbf{p}^h = \mathcal{T}^M \boldsymbol{\psi}^h + 4\mathbf{S}^M \mathbf{h} \,, \qquad T^X \mathbf{p}^x = \mathcal{T}^M \boldsymbol{\psi}^m \,, \tag{9.3}$$

ii) between second-derivative matrices in the two domains,

$$\begin{aligned} T^X \mathbf{M}^{hh} + \mathbf{p}^h \mathbf{p}^{h\,T} = {} & \mathcal{T}^M \boldsymbol{\mathcal{M}}^{hh} + \boldsymbol{\psi}^h \boldsymbol{\psi}^{h\,T} + 4\mathbf{S}^M \\ & + \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{hm} + \boldsymbol{\psi}^h \boldsymbol{\psi}^{m\,T}\right) \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{mm} + \boldsymbol{\psi}^m \boldsymbol{\psi}^{m\,T} + 4\mathbf{S}^M\right)^{-1} \\ & \times \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{hm} + \boldsymbol{\psi}^h \boldsymbol{\psi}^{m\,T}\right)^T , \end{aligned}$$

$$T^X \mathbf{M}^{hx} + \mathbf{p}^h \mathbf{p}^{x\,T} = \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{hm} + \boldsymbol{\psi}^h \boldsymbol{\psi}^{m\,T}\right) \left[\mathbf{I} + \frac{1}{4}\mathbf{S}^{M\,-1} \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{mm} + \boldsymbol{\psi}^m \boldsymbol{\psi}^{m\,T}\right)\right]^{-1} ,$$

$$\left[\mathbf{I} - \frac{1}{4}\mathbf{S}^{M\,-1} \left(T^X \mathbf{M}^{xx} + \mathbf{p}^x \mathbf{p}^{x\,T}\right)\right] \left[\mathbf{I} + \frac{1}{4}\mathbf{S}^{M\,-1} \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{mm} + \boldsymbol{\psi}^m \boldsymbol{\psi}^{m\,T}\right)\right] = \mathbf{I} \,, \tag{9.4}$$

and iii) for the demigration-spreading matrices,

$$\mathbf{X}^h = \frac{1}{4}\mathbf{S}^{M\,-1} \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{hm} + \boldsymbol{\psi}^h \boldsymbol{\psi}^{m\,T}\right) \,, \; \mathbf{X}^m = \mathbf{I} + \frac{1}{4}\mathbf{S}^{M\,-1} \left(\mathcal{T}^M \boldsymbol{\mathcal{M}}^{mm} + \boldsymbol{\psi}^m \boldsymbol{\psi}^{m\,T}\right) \,. \tag{9.5}$$

To obtain corresponding migration-spreading matrices, see equations 8.8–8.9. For idealized migration focusing, we observe from equation 9.5 that matrix $\mathbf{X}^h$ is zero for all offsets, while matrix $\mathbf{X}^m$ is invariant with offset. We emphasize that these properties of the demigration spreading matrices are not general— they belong specifically to the single-square-root function. The latter is therefore unable to take into account reflection-point smearing in the migration process, while the double-square-root function handles such smearing correctly when the medium is homogeneous and isotropic.

In the situation of demigration the slope vector $\mathbf{p}^x$ may be computed by the last equation 9.3. Alternatively, it can be obtained by combining equations 7.4 and E.2, which yields

$$\mathbf{p}^x = \frac{4}{T^X}\mathbf{S}^M \hat{\mathbf{a}} \,. \tag{9.6}$$

Equation 9.6 is, in contrast to the last equation 9.3, exact in the context of the single-square-root diffraction-time approximation.

9.2. Migration/demigration at zero offset. We recapitulate from above (see equations 7.15–7.17) that the zero-offset situation always yields $\boldsymbol{\psi}^h = \mathbf{p}^h = \mathbf{0}$, $\boldsymbol{\mathcal{M}}^{hm} = \mathbf{M}^{hx} = \mathbf{0}$, and $\mathbf{X}^h = \boldsymbol{\mathcal{X}}^h = \mathbf{0}$. The double-square-root and single-square-root diffraction-time functions are identical at zero offset, with only one exception: The coefficient matrix $\mathbf{U}^{hh}$ is different for the two approximations, as shown in Appendix F. This difference has important consequences with respect to applicability, which is discussed in more detail below.

Migration/demigration of common midpoint/image-gather locations can be done with the procedure described in the previous subsection, after substituting $\mathbf{h} = \mathbf{0}$ into equation 9.1. For migration/demigration of reflection slopes and second-derivatives one can use the general framework with diffraction coefficients from Appendix F. If the time-migration velocity model is assumed homogeneous, slope mapping $\boldsymbol{\psi}^m \leftrightarrow \mathbf{p}^x$ and second-derivative mapping $\boldsymbol{\mathcal{M}}^{mm} \leftrightarrow \mathbf{M}^{xx}$ can be conducted using relevant sub-relations in equations 9.3 and 9.4. The demigration-spreading matrix $\mathbf{X}^m$ is given in equation 9.5.

9.3. Relating migration and normal-moveout ellipses. When considering the double-square-root approximation (equations 3.1 and 4.12 in combination), it is of interest to relate time-migration velocity at the common-image gather location to normal-moveout velocity at the common-midpoint location. For this we need diffraction-time coefficients at zero offset, which are specified in Appendix F.

Consider now equation F.1 for the coefficient matrix $\mathbf{U}^{hh}$, and also equation 7.17, which corresponds to a perfectly focused migration result at zero offset. Taking into account definitions of the migration- and NMO-ellipse matrices $\mathbf{S}^M$ and $\mathbf{S}^{NMO}$ in equations 4.10 and 6.6, we find the relation

$$\mathbf{S}^M = \mathbf{S}^{NMO} + \frac{1}{4}\mathbf{p}^x\mathbf{p}^{xT} \; . \tag{9.7}$$

Equation 9.7 can be used to estimate matrix $\mathbf{S}^M$ from matrix $\mathbf{S}^{NMO}$ or vice versa. If we instead are using the single-square-root approximation, equation 9.4 has the implication that

$$\mathbf{S}^M = \mathbf{S}^{NMO} \; . \tag{9.8}$$

In other words, the single-square-root function predicts the NMO-ellipse matrix $\mathbf{S}^{NMO}$ at the common-midpoint location to be equal to the time-migration-ellipse matrix $\mathbf{S}^M$ at the common image-gather location. Moreover, for any given direction, θ, the surface-specific NMO velocity is predicted equal to the corresponding time-migration velocity. As these results are obviously not exact even for an homogeneous isotropic medium, they demonstrate that the single-square-root approximation should be used with care. One cannot use the single-square-root approximation for estimating NMO velocity from time-migration velocity or vice versa. One can, however, use it for kinematic migration/demigration between the recording and migration domains as long as offsets and/or apertures are small.

10. Estimation of the time-migration velocity model. We outline two routes by which the parameters of the time-migration velocity model can be estimated, using the methodology of kinematic time migration and demigration.

The first route is driven by known reflection-time parameters in the time-migration domain. One starts by time-migrating the seismic data set using a preliminary time-migration velocity model. In the migrated data set, reflection-time parameters are identified for a number of key reflections. Using the same preliminary time-migration velocity model as before, one kinematically demigrate the retrieved reflection-time parameters so that they become represented in the recording domain. One can now formulate an inversion scheme by which the time-migration velocity model is systematically updated by minimizing the slope parameter $\boldsymbol{\psi}^h$ for a range of offsets. If desirable, minimization of the second-derivative parameters $\boldsymbol{\mathcal{M}}^{hh}$ and $\boldsymbol{\mathcal{M}}^{hm}$ may be included in the procedure, as well.

The second route is driven by known reflection-time parameters in the recording domain. Assume, for example, that one takes as diffraction-time function the double-square-root function (equations 4.3 and 4.12 in combination). Let us further assume that for a given location $\mathbf{x}$ and zero offset, $\mathbf{h} = \mathbf{0}$, the reflection time T^X, the slope vector $\mathbf{p}^x$, and the NMO-ellipse matrix, $\mathbf{S}^{NMO}$, are known. Estimation of the latter from seismic data requires that observations are available for at least three different directions of the half-offset vector. A corresponding time-migration-ellipse matrix, $\hat{\mathbf{S}}^M$, can then be estimated using equation 9.7, but we do not yet know its location $(\hat{\mathbf{m}}, \hat{\tau})$. For that, we can first use equation 9.6, which is exact for the single- and double-square-root functions at zero offset, to obtain the aperture vector $\hat{\mathbf{a}}$ and then the image-gather location $\hat{\mathbf{m}} = \mathbf{x} - \hat{\mathbf{a}}$. Thereafter, we use equation 4.14 with $\mathbf{h} = \mathbf{0}$ to compute the migration time, $\hat{\tau}$. Having applied this direct estimation procedure to yield a number of samples $(\hat{\mathbf{m}}, \hat{\tau}, \hat{\mathbf{S}}^M)$, the function $\mathbf{S}^M(\mathbf{m}, \tau)$ is finally established by regularization.

11. Conclusions. We presented a generalization of kinematic time migration and demigration for arbitrary source-receiver offsets and a class of heterogeneous, anisotropic velocity models. We developed a description beyond the propagation of singularities, namely, by including reflection curvatures. The results tie in with earlier work on estimating geological dips and curvatures from time-migrated (zero-offset) reflections [45]. Also, the migration/demigration procedure aids in geometry driven compressive data regularization. Finally we obtained a procedure to estimate time-migration velocity ellipses in a heterogeneous, anisotropic model setting, which signify the "data" for nonlinear velocity inversion. Applications of parameters insensitive to the time-migration velocity model include multi-midpoint stacking, kinematic depth migration, time-migration tomography, and depth-migration tomography.

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Appendix A. Operators for time migration and demigration.

The extended time demigration operator, F, associated with equation 3.3 can be written in the form $d(\mathbf{h},\mathbf{x},t) = (F\tilde{d})(\mathbf{h},\mathbf{x},t)$, where

$$d(\mathbf{h},\mathbf{x},t) = \iiint W(\mathbf{h},\mathbf{x},\mathbf{m},\tau,\omega)\exp[\imath\phi(\mathbf{h},\mathbf{x},t,\mathbf{m},\tau,\omega)]\,\tilde{d}(\mathbf{h},\mathbf{m},\tau)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\tau\,\mathrm{d}\omega, \tag{A.1}$$

with phase function

$$\phi(\mathbf{h},\mathbf{x},t,\mathbf{m},\tau,\omega) = -\omega\,[T^D(\mathbf{h},\mathbf{x}-\mathbf{m},\mathbf{m},\tau) - t] \tag{A.2}$$

and amplitude $W(\mathbf{h}, \mathbf{x}, \mathbf{m}, \tau, \omega)$. In principle, a can be derived from the Born approximation and asymptotic ray theory, while $\tilde{d}$ can be related to reflectivity using image-ray coordinates. In the absence of caustics, operator F is asymptotically invertible.

The extended time migration operator, F^*, is obtained by taking the adjoint of the extended time demigration operator and is given by

(A.3)
$$\begin{aligned}\tilde{d}(\mathbf{h}, \mathbf{m}, \tau) &= \iiint W(\mathbf{h}, \mathbf{x}, \mathbf{m}, \tau, \omega)^* \exp[-\imath\phi(\mathbf{h}, \mathbf{x}, t, \mathbf{m}, \tau, \omega)]\, d(\mathbf{h}, \mathbf{x}, t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t\, \mathrm{d}\omega \\ &= \iiint W(\mathbf{h}, \mathbf{m}+\mathbf{a}, \mathbf{m}, \tau, \omega)^* \exp[-\imath\phi(\mathbf{h}, \mathbf{m}+\mathbf{a}, t, \mathbf{m}, \tau, \omega)]\, d(\mathbf{h}, \mathbf{m}+\mathbf{a}, t)\, \mathrm{d}\mathbf{a}\, \mathrm{d}t\, \mathrm{d}\omega,\end{aligned}$$

introducing (relative) aperture coordinates. The amplitude corrections are obtained upon considering the composition F^*F and constructing its asymptotic inverse.

The propagation of singularities by F follows from evaluating

(A.4)
$$\left(\mathbf{h}, \mathbf{x}, t, \frac{\partial\phi}{\partial\mathbf{h}}, \frac{\partial\phi}{\partial\mathbf{x}}, \frac{\partial\phi}{\partial t}; \mathbf{h}, \mathbf{m}, \tau, -\frac{\partial\phi}{\partial\mathbf{h}}, -\frac{\partial\phi}{\partial\mathbf{m}}, -\frac{\partial\phi}{\partial\tau}\right) \quad \text{at } \frac{\partial\phi}{\partial\omega} = 0,$$

and is described by the so-called canonical relation,

(A.5)
$$\begin{aligned}\Lambda = \Bigg\{ \Bigg[&\mathbf{h}, \mathbf{x}, T^D(\mathbf{h}, \mathbf{x}-\mathbf{m}, \mathbf{m}, \tau), -\omega\frac{\partial T^D}{\partial\mathbf{h}}(\mathbf{h}, \mathbf{x}-\mathbf{m}, \mathbf{m}, \tau), -\omega\frac{\partial T^D}{\partial\mathbf{a}}(\mathbf{h}, \mathbf{x}-\mathbf{m}, \mathbf{m}, \tau), \omega; \\ &\mathbf{h}, \mathbf{m}, \tau, \omega\frac{\partial T^D}{\partial\mathbf{h}}(\mathbf{h}, \mathbf{x}-\mathbf{m}, \mathbf{m}, \tau), \omega\frac{\partial T^D}{\partial\mathbf{m}}(\mathbf{h}, \mathbf{x}-\mathbf{m}, \mathbf{m}, \tau), \omega\frac{\partial T^D}{\partial\tau}(\mathbf{h}, \mathbf{x}-\mathbf{m}, \mathbf{m}, \tau)\Bigg]\Bigg\}.\end{aligned}$$

Here, $\partial T^D/\partial\tau$ signifies the "coordinate stretch". Λ is the graph of an invertible transformation, Σ say; Σ does not affect $\mathbf{h}$.

The propagation of singularities reads that $(\mathbf{h}, \mathbf{x}, t, \mathbf{p}^h, \mathbf{p}^x, \omega)$ belongs to the wavefront set of the prestack data if there exists a $(\mathbf{h}, \mathbf{m}, \tau, \boldsymbol{\psi}^h, \boldsymbol{\psi}^m, \upsilon)$ such that $(\mathbf{h}, \mathbf{x}, t, \mathbf{p}^h, \mathbf{p}^x, \omega; \mathbf{h}, \mathbf{m}, \tau, \boldsymbol{\psi}^h, \boldsymbol{\psi}^m, \upsilon)$ belongs to Λ and $(\mathbf{h}, \mathbf{m}, \tau, \boldsymbol{\psi}^h, \boldsymbol{\psi}^m, \upsilon)$ is contained in the wavefront of the extended time image. The scalar υ can be interpreted as a frequency variable associated with the time-migration domain.

If the singular support of the extended time image can be described (locally) by a level set function $\mathcal{T}$, that is, $\tau = \mathcal{T}(\mathbf{h}, \mathbf{m})$, one can define a map from $(\mathbf{h}, \mathbf{m})$ to $\mathbf{x}$, namely through a projection of the elements of the wavefront set of the data obtained from $[\mathbf{h}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m}), \partial\mathcal{T}(\mathbf{h}, \mathbf{m})/\partial\mathbf{h}, \partial\mathcal{T}(\mathbf{h}, \mathbf{m})/\partial\mathbf{m}, -1]$; in the main text this map is denoted by $\hat{\mathbf{x}}$. The singular support of the prestack data can be correspondingly described (locally) by a level set function T, that is, $t = T(\mathbf{h}, \mathbf{x})$, so that

(A.6)
$$T\left[\mathbf{h}, \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m})\right] = T^D\left[\mathbf{h}, \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m}) - \mathbf{m}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right]$$

according to the canonical relation; moreover, we read off

(A.7)
$$-\frac{\partial T}{\partial\mathbf{x}}\left[\mathbf{h}, \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m})\right] = -\frac{\partial T^D}{\partial\mathbf{a}}\left[\mathbf{h}, \hat{\mathbf{x}}(\mathbf{h}, \mathbf{m}) - \mathbf{m}, \mathbf{m}, \mathcal{T}(\mathbf{h}, \mathbf{m})\right].$$

The propagation of singularities by time migration is straightforwardly described by Λ^*. If the singular support of the prestack data can be described (locally) by a level set function T, that is, $t = T(\mathbf{h}, \mathbf{x})$, one can define a map from $(\mathbf{h}, \mathbf{x})$ to $\mathbf{m}$, namely through a projection of the elements of the wavefront set of the extended image obtained from $[\mathbf{h}, \mathbf{x}, T(\mathbf{h}, \mathbf{x}), -\partial T(\mathbf{h}, \mathbf{x})/\partial\mathbf{h}, -\partial T(\mathbf{h}, \mathbf{x})/\partial\mathbf{x}, 1]$;

in the main text this map is denoted by $\hat{\mathbf{m}}$. The singular support of the extended image can be correspondingly described (locally) by a level set function $\mathcal{T}$, that is, $\tau = \mathcal{T}(\mathbf{h}, \mathbf{m})$, so that

$$T(\mathbf{h}, \mathbf{x}) = T^D\left[\mathbf{h}, \mathbf{x} - \hat{\mathbf{m}}(\mathbf{h}, \mathbf{x}), \hat{\mathbf{m}}(\mathbf{h}, \mathbf{x}), \mathcal{T}(\mathbf{h}, \hat{\mathbf{m}}(\mathbf{h}, \mathbf{x}))\right] \tag{A.8}$$

according to the canonical relation A.5.

Appendix B. Basic conditions for kinematic migration and demigration.
We provide further insight into equations A.6–A.7. Consider a reflected wave

$$d(\mathbf{h}, \mathbf{x}, t) = A(\mathbf{h}, \mathbf{x})\, s\left[(t - T(\mathbf{h}, \mathbf{x})\right] \tag{B.1}$$

with amplitude $A(\mathbf{h}, \mathbf{x})$, traveltime $T(\mathbf{h}, \mathbf{x})$, and wavelet $s(t)$. We let $S(\omega)$ denote the Fourier transform of $s(t)$. When the time-migration equation 3.3 is applied to this data for fixed coordinates $(\mathbf{h}, \mathbf{m}, \tau)$, we obtain

$$\begin{aligned} \tilde{d}(\mathbf{h}, \mathbf{m}, \tau) = \iint & W^*(\mathbf{h}, \mathbf{m}+\mathbf{a}, \mathbf{m}, \tau, \omega)\, A(\mathbf{h}, \mathbf{m}+\mathbf{a}) \\ \times & \;\; S(\omega)\, \exp\left\{\imath\omega\left[T^D(\mathbf{h}, \mathbf{a}, \mathbf{m}, \tau) - T(\mathbf{h}, \mathbf{m}+\mathbf{a})\right]\right\} \mathrm{d}\mathbf{a}\, \mathrm{d}\omega \ , \end{aligned} \tag{B.2}$$

since $\mathbf{x} = \mathbf{m} + \mathbf{a}$. Furthermore, since $\mathbf{m}$ is constant, we have $\partial/\partial\mathbf{x} = \partial/\partial\mathbf{a}$. Applying the method of stationary phase [42, 4, 40, e.g.] it is clear that the stationary point $\hat{\mathbf{a}}(\mathbf{h}, \mathbf{m})$ of the aperture integral satisfies the equations

$$T\left(\mathbf{h}, \mathbf{m}+\hat{\mathbf{a}}\right) = T^D\left(\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \tau\right) \ , \tag{B.3}$$

$$\frac{\partial T}{\partial \mathbf{x}}\left(\mathbf{h}, \mathbf{m}+\hat{\mathbf{a}}\right) = \frac{\partial T^D}{\partial \mathbf{a}}\left(\mathbf{h}, \hat{\mathbf{a}}, \mathbf{m}, \tau\right) \ . \tag{B.4}$$

Equations B.3–B.4 coincide with equations A.6–A.7 and are equations 7.3–7.4 in the main text.

Appendix C. Transformations between derivatives of reflection time in the migration and recording domains.
The topic of this appendix is derivation of transformations between derivatives of reflection time in the migration and recording domains, both being considered in their common-offset representations.

C.1. First-order derivatives. Consider first equation 7.2. Differentiation with respect to half-offset coordinates h_k and position components m_k in the time-migration domain yields

$$\frac{\partial \hat{a}_i}{\partial h_k} = \frac{\partial \hat{x}_i}{\partial h_k} \ , \qquad \frac{\partial \hat{a}_i}{\partial m_k} = \frac{\partial \hat{x}_i}{\partial m_k} - \delta_{ik} \ . \tag{C.1}$$

Now we take derivatives of the basic condition in equation 7.3 with respect to position components m_k. The result is

$$\frac{\partial \hat{x}_i}{\partial m_k}\frac{\partial T}{\partial x_i} = \frac{\partial T^D}{\partial m_k} + \frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial T^D}{\partial \tau} + \frac{\partial \hat{a}_i}{\partial m_k}\frac{\partial T^D}{\partial a_i} \ . \tag{C.2}$$

In the last equation we recognize the components, $\partial\hat{x}_i/\partial m_k$, of the demigration-spreading matrix $\mathbf{X}^m$ (equation 7.11). Using equation 7.4 and the last sub-equation C.1 in equation C.2 yields

$$\frac{\partial T}{\partial x_i} = \frac{\partial T^D}{\partial m_i} + \frac{\partial \mathcal{T}}{\partial m_i}\frac{\partial T^D}{\partial \tau} \ . \tag{C.3}$$

This equation relates the first derivatives of reflection time $\partial\mathcal{T}/\partial m_i$ and $\partial T/\partial x_i$.

Differentiation of equation 7.3 with respect to half-offset components h_k yields

$$\frac{\partial \hat{x}_i}{\partial h_k}\frac{\partial T}{\partial x_i} + \frac{\partial T}{\partial h_k} = \frac{\partial T^D}{\partial h_k} + \frac{\partial \mathcal{T}}{\partial h_k}\frac{\partial T^D}{\partial \tau} + \frac{\partial \hat{a}_i}{\partial h_k}\frac{\partial T^D}{\partial a_i} \ . \tag{C.4}$$

Applying the first equation C.1 and the basic condition 7.4, we find that the derivatives $\partial \mathcal{T}/\partial h_i$ and $\partial T/\partial h_i$ are related by the equation

$$\frac{\partial T}{\partial h_i} = \frac{\partial T^D}{\partial h_i} + \frac{\partial \mathcal{T}}{\partial h_i}\frac{\partial T^D}{\partial \tau} \ . \tag{C.5}$$

C.2. Second-order derivatives. The basic condition in equation 7.4 is differentiated with respect to coordinates m_k to yield

$$\frac{\partial \hat{x}_j}{\partial m_k}\frac{\partial^2 T}{\partial x_j \partial x_i} = \frac{\partial^2 T^D}{\partial m_k \partial a_i} + \frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau \partial a_i} + \frac{\partial \hat{a}_j}{\partial m_k}\frac{\partial^2 T^D}{\partial a_j \partial a_i} \ . \tag{C.6}$$

We also differentiate equation C.3 with respect to m_k, which gives

$$\begin{aligned}\frac{\partial \hat{x}_j}{\partial m_k}\frac{\partial^2 T}{\partial x_j \partial x_i} = {} & \frac{\partial^2 T^D}{\partial m_k \partial m_i} + \frac{\partial^2 \mathcal{T}}{\partial m_k \partial m_i}\frac{\partial T^D}{\partial \tau} \\ & + \frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau \partial m_i} + \frac{\partial \mathcal{T}}{\partial m_i}\frac{\partial^2 T^D}{\partial \tau \partial m_k} + \frac{\partial \mathcal{T}}{\partial m_i}\frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau^2} \\ & + \frac{\partial \hat{a}_j}{\partial m_k}\left(\frac{\partial^2 T^D}{\partial a_j \partial m_i} + \frac{\partial \mathcal{T}}{\partial m_i}\frac{\partial^2 T^D}{\partial a_j \partial \tau}\right) \ .\end{aligned} \tag{C.7}$$

Using that the right-hand sides of equations C.6 and C.7 have to be equal, and also taking into account equation C.1, we obtain

$$\begin{aligned}\left(\frac{\partial^2 T^D}{\partial a_i \partial a_j} - \frac{\partial^2 T^D}{\partial m_i \partial a_j} - \frac{\partial \mathcal{T}}{\partial m_i}\frac{\partial^2 T^D}{\partial a_j \partial \tau}\right)\frac{\partial \hat{x}_j}{\partial m_k} = {} & \frac{\partial^2 \mathcal{T}}{\partial m_i \partial m_k}\frac{\partial T^D}{\partial \tau} + \frac{\partial^2 T^D}{\partial a_i \partial a_k} + \frac{\partial^2 T^D}{\partial m_i \partial m_k} \\ & + \frac{\partial^2 T^D}{\partial \tau \partial m_i}\frac{\partial \mathcal{T}}{\partial m_k} + \frac{\partial^2 T^D}{\partial \tau \partial m_k}\frac{\partial \mathcal{T}}{\partial m_i} + \frac{\partial \mathcal{T}}{\partial m_i}\frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau^2} \\ & - \left(\frac{\partial^2 T^D}{\partial a_i \partial m_k} + \frac{\partial^2 T^D}{\partial \tau \partial a_i}\frac{\partial \mathcal{T}}{\partial m_k}\right) \\ & - \left(\frac{\partial^2 T^D}{\partial a_k \partial m_i} + \frac{\partial^2 T^D}{\partial \tau \partial a_k}\frac{\partial \mathcal{T}}{\partial m_i}\right) \ .\end{aligned} \tag{C.8}$$

Using the definitions in equation 4.5 and 7.9, the equivalent matrix form of equation C.8 is

$$-\left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^T \mathbf{X}^m = u\,\boldsymbol{\mathcal{M}}^{mm} + \mathbf{Y} \ . \tag{C.9}$$

Equation C.9 can be used for computation of the demigration-spreading matrix $\mathbf{X}^m$; the final expression for it is given in equation 7.12.

A minor rearrangement of terms in equation C.6 yields

$$\left(\frac{\partial^2 T}{\partial x_i \partial x_j} - \frac{\partial^2 T^D}{\partial a_i \partial a_j}\right)\frac{\partial \hat{x}_j}{\partial m_k} = \frac{\partial^2 T^D}{\partial a_i \partial m_k} + \frac{\partial^2 T^D}{\partial \tau \partial a_i}\frac{\partial \mathcal{T}}{\partial m_k} - \frac{\partial^2 T^D}{\partial a_i \partial a_k} \ , \tag{C.10}$$

or in matrix form,

$$\left(\mathbf{M}^{xx} - \mathbf{U}^{aa}\right)\mathbf{X}^m = \left(\mathbf{K}^{am} - \mathbf{U}^{aa}\right)^T \ . \tag{C.11}$$

By combining equations C.9 and C.11 matrix $\mathbf{X}^m$ is eliminated, and we obtain the explicit formula for matrix $\mathbf{M}^{xx}$ in equation 7.8.

Our next task is to find an expression for the mixed-term second-derivative matrix belonging to the recording domain, $\mathbf{M}^{hx} = \left(\partial^2 T/\partial h_i \partial x_j\right)$. We proceed by differentiating equation C.5 with respect to m_k,

$$\begin{aligned}
\frac{\partial \hat{x}_j}{\partial m_k}\frac{\partial^2 T}{\partial x_j \partial h_i} = {} & \frac{\partial^2 T^D}{\partial m_k \partial h_i} + \frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau \partial h_i} + \frac{\partial \hat{a}_j}{\partial m_k}\frac{\partial^2 T^D}{\partial a_j \partial h_i} \\
& + \frac{\partial^2 \mathcal{T}}{\partial m_k \partial h_i}\frac{\partial T^D}{\partial \tau} \\
& + \frac{\partial \mathcal{T}}{\partial h_i}\left(\frac{\partial^2 T^D}{\partial \tau \partial m_k} + \frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau^2} + \frac{\partial \hat{a}_j}{\partial m_k}\frac{\partial^2 T^D}{\partial a_j \partial \tau}\right) \ ,
\end{aligned} \tag{C.12}$$

which can be restated as

$$\begin{aligned}
\left(\frac{\partial^2 T}{\partial h_i \partial x_j} - \frac{\partial^2 T^D}{\partial h_i \partial a_j} - \frac{\partial \mathcal{T}}{\partial h_i}\frac{\partial^2 T^D}{\partial a_j \partial \tau}\right)\frac{\partial \hat{x}_j}{\partial m_k} = {} & + \frac{\partial^2 \mathcal{T}}{\partial h_i \partial m_k}\frac{\partial T^D}{\partial \tau} + \frac{\partial^2 T^D}{\partial h_i \partial m_k} \\
& + \frac{\partial^2 T^D}{\partial \tau \partial h_i}\frac{\partial \mathcal{T}}{\partial m_k} + \frac{\partial \mathcal{T}}{\partial h_i}\frac{\partial^2 T^D}{\partial \tau \partial m_k} \\
& + \frac{\partial \mathcal{T}}{\partial h_i}\frac{\partial \mathcal{T}}{\partial m_k}\frac{\partial^2 T^D}{\partial \tau^2} \\
& - \frac{\partial^2 T^D}{\partial h_i \partial a_k} - \frac{\partial \mathcal{T}}{\partial h_i}\frac{\partial^2 T^D}{\partial a_k \partial \tau} \ .
\end{aligned} \tag{C.13}$$

Applying the coefficient definitions in equation 4.5 and 7.9, we rewrite equation C.13 in matrix form as follows,

$$\left(\mathbf{M}^{hx} - \mathbf{K}^{ha}\right)\mathbf{X}^m = u\,\boldsymbol{\mathcal{M}}^{hm} + \mathbf{L}^{hm} - \mathbf{K}^{ha} \ . \tag{C.14}$$

Combination of the last result with equation C.9 eliminates matrix $\mathbf{X}^m$ and yields the middle sub-equation 7.8 for matrix $\mathbf{M}^{hx}$ given in the main text.

Differentiation of equation 7.4 with respect to half-offset component h_k gives

$$\frac{\partial \hat{x}_j}{\partial h_k}\frac{\partial^2 T}{\partial x_j \partial x_i} + \frac{\partial^2 T}{\partial h_k \partial x_i} = \frac{\partial^2 T^D}{\partial h_k \partial a_i} + \frac{\partial \mathcal{T}}{\partial h_k}\frac{\partial^2 T^D}{\partial \tau \partial a_i} + \frac{\partial \hat{a}_j}{\partial h_k}\frac{\partial^2 T^D}{\partial a_j \partial a_i} \ , \tag{C.15}$$

which can be rewritten as

$$\left(\frac{\partial^2 T}{\partial x_i \partial x_j} - \frac{\partial^2 T^D}{\partial a_i \partial a_j}\right)\frac{\partial \hat{x}_j}{\partial h_k} = -\frac{\partial^2 T}{\partial x_i \partial h_k} + \frac{\partial^2 T^D}{\partial a_i \partial h_k} + \frac{\partial^2 T^D}{\partial \tau \partial a_i}\frac{\partial \mathcal{T}}{\partial h_k} \tag{C.16}$$

with the corresponding matrix form

$$\left(\mathbf{M}^{xx} - \mathbf{U}^{aa}\right)\mathbf{X}^h = -\left(\mathbf{M}^{hx} - \mathbf{K}^{ha}\right)^T \ . \tag{C.17}$$

We observe that the demigration-spreading matrix $\mathbf{X}^h$ (see equation 7.11) can be determined from equation C.17. Utilizing also the last two relations in equation 7.8 leads to the first sub-equation 7.12, which relates matrix $\mathbf{X}^h$ solely to parameters and coefficients belonging to the time-migration domain.

To obtain the second-derivative matrix $\mathbf{M}^{hh} = \left(\partial^2 T/\partial h_i \partial h_j\right)$, we differentiate equation C.5 with respect to h_k,

$$\begin{aligned}
\frac{\partial \hat{x}_j}{\partial h_k}\frac{\partial^2 T}{\partial x_j \partial h_i} + \frac{\partial^2 T}{\partial h_k \partial h_i} = {} & \frac{\partial^2 T^D}{\partial h_i \partial h_k} + \frac{\partial \mathcal{T}}{\partial h_k}\frac{\partial^2 T^D}{\partial \tau \partial h_i} + \frac{\partial \hat{a}_j}{\partial h_k}\frac{\partial^2 T^D}{\partial a_j \partial h_i} \\
& + \frac{\partial^2 \mathcal{T}}{\partial h_k \partial h_i}\frac{\partial T^D}{\partial \tau} \\
& + \frac{\partial \mathcal{T}}{\partial h_i}\left(\frac{\partial^2 T^D}{\partial \tau \partial h_k} + \frac{\partial \mathcal{T}}{\partial h_k}\frac{\partial^2 T^D}{\partial \tau^2} + \frac{\partial \hat{a}_j}{\partial h_k}\frac{\partial^2 T^D}{\partial a_j \partial \tau}\right) \ ,
\end{aligned} \tag{C.18}$$

which can be restated as

$$
\begin{aligned}
\frac{\partial^2 T}{\partial h_i \partial h_k} = & \frac{\partial^2 \mathcal{T}}{\partial h_i \partial h_k} \frac{\partial T^D}{\partial \tau} + \frac{\partial^2 T^D}{\partial h_i \partial h_k} \\
& + \frac{\partial^2 T^D}{\partial \tau \partial h_i} \frac{\partial \mathcal{T}}{\partial h_k} + \frac{\partial \mathcal{T}}{\partial h_i} \frac{\partial^2 T^D}{\partial \tau \partial h_k} \\
& + \frac{\partial \mathcal{T}}{\partial h_i} \frac{\partial \mathcal{T}}{\partial h_k} \frac{\partial^2 T^D}{\partial \tau^2} \\
& - \left(\frac{\partial^2 T}{\partial h_i \partial x_j} - \frac{\partial^2 T^D}{\partial h_i \partial a_j} - \frac{\partial \mathcal{T}}{\partial h_i} \frac{\partial^2 T^D}{\partial a_j \partial \tau} \right) \frac{\partial \hat{x}_j}{\partial h_k} \ ,
\end{aligned} \tag{C.19}
$$

or equivalently,

$$
\mathbf{M}^{hh} = u \, \boldsymbol{\mathcal{M}}^{hh} + \mathbf{L}^{hh} - \left(\mathbf{M}^{hx} - \mathbf{K}^{ha} \right) \mathbf{X}^h \ . \tag{C.20}
$$

Taking into account equation C.14 and already derived demigration spreading matrices in equation 7.12, we obtain the formula for matrix $\mathbf{M}^{hh}$ in equation 7.8 .

Appendix D. Coefficients of the double-square-root function.

The double-square-root function is given by the combination of equation 3.1 with equation 4.12. In this appendix we specify all its partial derivatives up to second order.

D.1. First-order partial derivatives.

$$
\begin{aligned}
\frac{\partial T^S}{\partial h_k} &= -\frac{1}{T^S} S^M_{kj} (a_j - h_j) \ , \\
\frac{\partial T^R}{\partial h_k} &= \frac{1}{T^R} S^M_{kj} (a_j + h_j) \ ,
\end{aligned} \tag{D.1}
$$

$$
\begin{aligned}
\frac{\partial T^S}{\partial a_k} &= \frac{1}{T^S} S^M_{kj} (a_j - h_j) \ , \\
\frac{\partial T^R}{\partial a_k} &= \frac{1}{T^R} S^M_{kj} (a_j + h_j) \ ,
\end{aligned} \tag{D.2}
$$

$$
\begin{aligned}
\frac{\partial T^S}{\partial m_k} &= \frac{1}{2T^S} \frac{\partial S^M_{ij}}{\partial m_k} (a_i - h_i)(a_j - h_j) \ , \\
\frac{\partial T^R}{\partial m_k} &= \frac{1}{2T^R} \frac{\partial S^M_{ij}}{\partial m_k} (a_i + h_i)(a_j + h_j) \ ,
\end{aligned} \tag{D.3}
$$

$$
\begin{aligned}
\frac{\partial T^S}{\partial \tau} &= \frac{1}{T^S} \left[\frac{\tau}{4} + \frac{1}{2} \frac{\partial S^M_{ij}}{\partial \tau} (a_i - h_i)(a_j - h_j) \right] \ , \\
\frac{\partial T^R}{\partial \tau} &= \frac{1}{T^R} \left[\frac{\tau}{4} + \frac{1}{2} \frac{\partial S^M_{ij}}{\partial \tau} (a_i + h_i)(a_j + h_j) \right] \ .
\end{aligned} \tag{D.4}
$$

D.2. Second-order partial derivatives.

$$
\begin{aligned}
\frac{\partial^2 T^S}{\partial h_k \partial h_l} &= \frac{1}{T^S} \left(S^M_{kl} - \frac{\partial T^S}{\partial h_k} \frac{\partial T^S}{\partial h_l} \right) \ , \\
\frac{\partial^2 T^R}{\partial h_k \partial h_l} &= \frac{1}{T^R} \left(S^M_{kl} - \frac{\partial T^R}{\partial h_k} \frac{\partial T^R}{\partial h_l} \right) \ ,
\end{aligned} \tag{D.5}
$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial a_k \partial a_l} &= \frac{1}{T^S}\left(S^M_{kl} - \frac{\partial T^S}{\partial a_k}\frac{\partial T^S}{\partial a_l}\right) \ , \\ \frac{\partial^2 T^R}{\partial a_k \partial a_l} &= \frac{1}{T^R}\left(S^M_{kl} - \frac{\partial T^R}{\partial a_k}\frac{\partial T^R}{\partial a_l}\right) \ , \end{aligned} \tag{D.6}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial m_k \partial m_l} &= \frac{1}{T^S}\left[\frac{1}{2}\frac{\partial^2 S^M_{ij}}{\partial m_k \partial m_l}(a_i - h_i)(a_j - h_j) - \frac{\partial T^S}{\partial m_k}\frac{\partial T^S}{\partial m_l}\right] \ , \\ \frac{\partial^2 T^R}{\partial m_k \partial m_l} &= \frac{1}{T^R}\left[\frac{1}{2}\frac{\partial^2 S^M_{ij}}{\partial m_k \partial m_l}(a_i + h_i)(a_j + h_j) - \frac{\partial T^R}{\partial m_k}\frac{\partial T^R}{\partial m_l}\right] \ , \end{aligned} \tag{D.7}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial h_k \partial a_l} &= -\frac{1}{T^S}\left(S^M_{kl} + \frac{\partial T^S}{\partial h_k}\frac{\partial T^S}{\partial a_l}\right) \ , \\ \frac{\partial^2 T^R}{\partial h_k \partial a_l} &= \frac{1}{T^R}\left(S^M_{kl} - \frac{\partial T^R}{\partial h_k}\frac{\partial T^R}{\partial a_l}\right) \ , \end{aligned} \tag{D.8}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial h_k \partial m_l} &= -\frac{1}{T^S}\left[\frac{\partial S^M_{kj}}{\partial m_l}(a_j - h_j) + \frac{\partial T^S}{\partial h_k}\frac{\partial T^S}{\partial m_l}\right] \ , \\ \frac{\partial^2 T^R}{\partial h_k \partial m_l} &= \frac{1}{T^R}\left[\frac{\partial S^M_{kj}}{\partial m_l}(a_j + h_j) - \frac{\partial T^R}{\partial h_k}\frac{\partial T^R}{\partial m_l}\right] \ , \end{aligned} \tag{D.9}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial a_k \partial m_l} &= \frac{1}{T^S}\left[\frac{\partial S^M_{kj}}{\partial m_l}(a_j - h_j) - \frac{\partial T^S}{\partial a_k}\frac{\partial T^S}{\partial m_l}\right] \ , \\ \frac{\partial^2 T^R}{\partial a_k \partial m_l} &= \frac{1}{T^R}\left[\frac{\partial S^M_{kj}}{\partial m_l}(a_j + h_j) - \frac{\partial T^R}{\partial a_k}\frac{\partial T^R}{\partial m_l}\right] \ , \end{aligned} \tag{D.10}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial h_k \partial \tau} &= -\frac{1}{T^S}\left[\frac{\partial S^M_{kj}}{\partial \tau}(a_j - h_j) + \frac{\partial T^S}{\partial h_k}\frac{\partial T^S}{\partial \tau}\right] \ , \\ \frac{\partial^2 T^R}{\partial h_k \partial \tau} &= \frac{1}{T^R}\left[\frac{\partial S^M_{kj}}{\partial \tau}(a_j + h_j) - \frac{\partial T^R}{\partial h_k}\frac{\partial T^R}{\partial \tau}\right] \ , \end{aligned} \tag{D.11}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial a_k \partial \tau} &= \frac{1}{T^S}\left[\frac{\partial S^M_{kj}}{\partial \tau}(a_j - h_j) - \frac{\partial T^S}{\partial a_k}\frac{\partial T^S}{\partial \tau}\right] \ , \\ \frac{\partial^2 T^R}{\partial a_k \partial \tau} &= \frac{1}{T^R}\left[\frac{\partial S^M_{kj}}{\partial \tau}(a_j + h_j) - \frac{\partial T^R}{\partial a_k}\frac{\partial T^R}{\partial \tau}\right] \ , \end{aligned} \tag{D.12}$$

$$\begin{aligned} \frac{\partial^2 T^S}{\partial m_k \partial \tau} &= \frac{1}{T^S}\left[\frac{1}{2}\frac{\partial^2 S^M_{ij}}{\partial m_k \partial \tau}(a_i - h_i)(a_j - h_j) - \frac{\partial T^S}{\partial m_k}\frac{\partial T^S}{\partial \tau}\right] \ , \\ \frac{\partial^2 T^R}{\partial m_k \partial \tau} &= \frac{1}{T^R}\left[\frac{1}{2}\frac{\partial^2 S^M_{ij}}{\partial m_k \partial \tau}(a_i + h_i)(a_j + h_j) - \frac{\partial T^R}{\partial m_k}\frac{\partial T^R}{\partial \tau}\right] \ , \end{aligned} \tag{D.13}$$

$$\frac{\partial^2 T^S}{\partial \tau^2} = \frac{1}{T^S}\left[\frac{1}{4} + \frac{1}{2}\frac{\partial^2 S_{ij}^M}{\partial \tau^2}(a_i - h_i)(a_j - h_j) - \left(\frac{\partial T^S}{\partial \tau}\right)^2\right] ,$$

$$\frac{\partial^2 T^R}{\partial \tau^2} = \frac{1}{T^R}\left[\frac{1}{4} + \frac{1}{2}\frac{\partial^2 S_{ij}^M}{\partial \tau^2}(a_i + h_i)(a_j + h_j) - \left(\frac{\partial T^R}{\partial \tau}\right)^2\right] . \tag{D.14}$$

Appendix E. Coefficients of the single-square-root function.

The single-square-root function, see equation 4.14, has first- and second order partial derivatives as specified in the following.

E.1. First-order partial derivatives.

$$q_k^h = \frac{\partial T^D}{\partial h_k} = \frac{4}{T^D} S_{kj}^M h_j , \tag{E.1}$$

$$q_k^a = \frac{\partial T^D}{\partial a_k} = \frac{4}{T^D} S_{kj}^M a_j , \tag{E.2}$$

$$q_k^m = \frac{\partial T^D}{\partial m_k} = \frac{2}{T^D}\frac{\partial S_{ij}^M}{\partial m_k}(a_i a_j + h_i h_j) , \tag{E.3}$$

$$u = \frac{\partial T^D}{\partial \tau} = \frac{1}{T^D}\left[\tau + 2\frac{\partial S_{ij}^M}{\partial \tau}(a_i a_j + h_i h_j)\right] . \tag{E.4}$$

E.2. Second-order partial derivatives.

$$U_{kl}^{hh} = \frac{\partial^2 T^D}{\partial h_k \partial h_l} = \frac{1}{T^D}\left(4 S_{kl}^M - \frac{\partial T^D}{\partial h_k}\frac{\partial T^D}{\partial h_l}\right) , \tag{E.5}$$

$$U_{kl}^{aa} = \frac{\partial^2 T^D}{\partial a_k \partial a_l} = \frac{1}{T^D}\left(4 S_{kl}^M - \frac{\partial T^D}{\partial a_k}\frac{\partial T^D}{\partial a_l}\right) , \tag{E.6}$$

$$U_{kl}^{mm} = \frac{\partial^2 T^D}{\partial m_k \partial m_l} = \frac{1}{T^D}\left[2\frac{\partial^2 S_{ij}^M}{\partial m_k \partial m_l}(a_i a_j + h_i h_j) - \frac{\partial T^D}{\partial m_k}\frac{\partial T^D}{\partial m_l}\right] , \tag{E.7}$$

$$U_{kl}^{ha} = \frac{\partial^2 T^D}{\partial h_k \partial a_l} = -\frac{1}{T^D}\frac{\partial T^D}{\partial h_k}\frac{\partial T^D}{\partial a_l} , \tag{E.8}$$

$$U_{kl}^{hm} = \frac{\partial^2 T^D}{\partial h_k \partial m_l} = \frac{1}{T^D}\left(4\frac{\partial S_{kj}^M}{\partial m_l} h_j - \frac{\partial T^D}{\partial h_k}\frac{\partial T^D}{\partial m_l}\right) , \tag{E.9}$$

$$U_{kl}^{am} = \frac{\partial^2 T^D}{\partial a_k \partial m_l} = \frac{1}{T^D}\left(4\frac{\partial S_{kj}^M}{\partial m_l} a_j - \frac{\partial T^D}{\partial a_k}\frac{\partial T^D}{\partial m_l}\right) , \tag{E.10}$$

$$u_k^h = \frac{\partial^2 T^D}{\partial h_k \partial \tau} = \frac{1}{T^D}\left(4\frac{\partial S_{kj}^M}{\partial \tau} h_j - \frac{\partial T^D}{\partial h_k}\frac{\partial T^D}{\partial \tau}\right) , \tag{E.11}$$

$$u_k^a = \frac{\partial^2 T^D}{\partial a_k \partial \tau} = \frac{1}{T^D}\left(4\frac{\partial S^M_{kj}}{\partial \tau} a_j - \frac{\partial T^D}{\partial a_k}\frac{\partial T^D}{\partial \tau}\right) , \tag{E.12}$$

$$u_k^m = \frac{\partial^2 T^D}{\partial m_k \partial \tau} = \frac{1}{T^D}\left[2\frac{\partial^2 S^M_{ij}}{\partial m_k \partial \tau}(a_i a_j + h_i h_j) - \frac{\partial T^D}{\partial m_k}\frac{\partial T^D}{\partial \tau}\right] , \tag{E.13}$$

$$u^\tau = \frac{\partial^2 T^D}{\partial \tau^2} = \frac{1}{T^D}\left[1 + 2\frac{\partial^2 S^M_{ij}}{\partial \tau^2}(a_i a_j + h_i h_j) - \left(\frac{\partial T^D}{\partial \tau}\right)^2\right] . \tag{E.14}$$

Appendix F. Coefficients of the single/double square-root functions at zero offset. At zero offset, the first and second derivatives of the single and double square-root forms of the diffraction-time function are identical, with one important exception. When using the double-square-root form we obtain

$$U_{kl}^{hh} = \frac{\partial^2 T^D}{\partial h_k \partial h_l} = \frac{1}{T^D}\left(4 S^M_{kl} - \frac{\partial T^D}{\partial a_k}\frac{\partial T^D}{\partial a_l}\right) , \tag{F.1}$$

while the corresponding relation for the single-square-root function is

$$U_{kl}^{hh} = \frac{\partial^2 T^D}{\partial h_k \partial h_l} = \frac{4}{T^D} S^M_{kl} . \tag{F.2}$$

Relations for the remaining derivatives, which are identical to the two representations in the zero-offset situation, are listed in the following.

F.1. First-order partial derivatives.

$$q_k^h = \frac{\partial T^D}{\partial h_k} = 0 , \tag{F.3}$$

$$q_k^a = \frac{\partial T^D}{\partial a_k} = \frac{4}{T^D} S^M_{kj} a_j , \tag{F.4}$$

$$q_k^m = \frac{\partial T^D}{\partial m_k} = \frac{2}{T^D}\frac{\partial S^M_{ij}}{\partial m_k} a_i a_j , \tag{F.5}$$

$$u = \frac{\partial T^D}{\partial \tau} = \frac{1}{T^D}\left(\tau + 2\frac{\partial S^M_{ij}}{\partial \tau} a_i a_j\right) . \tag{F.6}$$

F.2. Second-order partial derivatives.

$$U_{kl}^{aa} = \frac{\partial^2 T^D}{\partial a_k \partial a_l} = \frac{1}{T^D}\left(4 S^M_{kl} - \frac{\partial T^D}{\partial a_k}\frac{\partial T^D}{\partial a_l}\right) , \tag{F.7}$$

$$U_{kl}^{mm} = \frac{\partial^2 T^D}{\partial m_k \partial m_l} = \frac{1}{T^D}\left(2\frac{\partial^2 S^M_{ij}}{\partial m_k \partial m_l} a_i a_j - \frac{\partial T^D}{\partial m_k}\frac{\partial T^D}{\partial m_l}\right) , \tag{F.8}$$

$$U_{kl}^{ha} = \frac{\partial^2 T^D}{\partial h_k \partial a_l} = 0 , \tag{F.9}$$

$$U_{kl}^{hm} = \frac{\partial^2 T^D}{\partial h_k \partial m_l} = 0 \ , \tag{F.10}$$

$$U_{kl}^{am} = \frac{\partial^2 T^D}{\partial a_k \partial m_l} = \frac{1}{T^D} \left(4 \frac{\partial S_{kj}^M}{\partial m_l} a_j - \frac{\partial T^D}{\partial a_k} \frac{\partial T^D}{\partial m_l} \right) \ , \tag{F.11}$$

$$u_k^h = \frac{\partial^2 T^D}{\partial h_k \partial \tau} = 0 \ , \tag{F.12}$$

$$u_k^a = \frac{\partial^2 T^D}{\partial a_k \partial \tau} = \frac{1}{T^D} \left(4 \frac{\partial S_{kj}^M}{\partial \tau} a_j - \frac{\partial T^D}{\partial a_k} \frac{\partial T^D}{\partial \tau} \right) \ , \tag{F.13}$$

$$u_k^m = \frac{\partial^2 T^D}{\partial m_k \partial \tau} = \frac{1}{T^D} \left(2 \frac{\partial^2 S_{ij}^M}{\partial m_k \partial \tau} a_i a_j - \frac{\partial T^D}{\partial m_k} \frac{\partial T^D}{\partial \tau} \right) \ , \tag{F.14}$$

$$u^\tau = \frac{\partial^2 T^D}{\partial \tau^2} = \frac{1}{T^D} \left[1 + 2 \frac{\partial^2 S_{ij}^M}{\partial \tau^2} a_i a_j - \left(\frac{\partial T^D}{\partial \tau} \right)^2 \right] \ . \tag{F.15}$$