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MODELLING OF TIME-HARMONIC SEISMIC DATA WITH THE HELMHOLTZ EQUATION AND SCATTERING SERIES *

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Abstract. We study the modelling of time-harmonic seismic data with the Helmholtz equation in $\mathbb{R}^d, d \geq 2$. We follow a two-step approach. First we discuss conditions on the regularity of the underlying wavespeed model for the direct solution of the Helmholtz equation. Secondly, we extend these conditions while considering the scattering or distorted Born series supplemented with a frequency bound. This series arises upon decomposing the medium in which the (time-harmonic) waves scatter into a *heterogeneous* background (with the regularity required in the first step) and a contrast. We obtain a condition, that is, frequency bound for convergence of the series in the Morrey-Campanato norm for relative contrasts in L^∞.

Key words. Helmholtz equation, scattering series

AMS subject classifications. 86A15, 35R30

1. Introduction. We study the modelling of time-harmonic seismic data with the Helmholtz equation in $\mathbb{R}^d, d \geq 2$. We follow a two-step approach. First we discuss conditions on the regularity of the underlying wavespeed model for the (direct) solution of the Helmholtz equation. The model can contain an interface. Secondly, we extend these conditions while considering the scattering or distorted Born series supplemented with a frequency bound. Such a series arises upon decomposing the medium in which the (time-harmonic) waves scatter into a background (with the regularity required in the first step) and a contrast. We allow the contrast to be bounded and measurable on a compact set, whereas the background can be heterogeneous and of limited smoothness. The scattering problem is then formulated in terms of a Lippmann-Schwinger equation and the scattering series corresponds with the Neumann series generating its solution. The series provides a fundamental tool to analyze data by distinguishing multiple scattered waves of different orders.

The convergence of the scattering series is essentially tied to an estimate of the Born approximation. In physics, the validity of the Born approximation has been assessed through the principle of accumulation of phase, that is, the total phase lag needs to be "small". Our estimates for convergence capture this principle.

In the setting of reflection seismology, the background is tied to tomography and the (typically relatively small) contrast to inverse scattering. We note that time-harmonic seismic data can be generated by using vibrator (vibroseis) sources. The setup we consider here is motivated by scattering problems attributed to salt intrusions in sedimentary sequences on the one hand, and the recent interest in low-frequency seismic sources [11] on the other hand. We exploit recently obtained estimates [23, 8, 3, 24] to gain further insight in the scattering series, and obtain conditions for convergence, for a heterogeneous background, in Morrey-Campanato norms.

The (fixed-frequency) Helmholtz equation,

$$-(\Delta + n(x))u = f, \quad x \in \mathbb{R}^d,$$

where $n(x) = \omega^2 c(x)^{-2}$ if $c = c(x)$ denotes the wavespeed and ω the angular frequency, is directly related to the fixed-energy Schrödinger equation, $(\Delta - q(x))u = 0$ with potential q through identifying $q(x)$ with $-n(x)$; $n(x)$ is positive and bounded away from zero. (Starting from the wave

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equation, applying the Laplace transform instead of the Fourier transform, $n(x)$ becomes negative and bounded away from zero, that is, the potential becomes positive.) The Helmholtz equation is also related to the equation for Electrical Impedance Tomography (EIT)

$$L_\sigma u = \nabla \cdot \sigma \nabla u = 0;$$

here, σ is the strictly positive conductivity and needs to be sufficiently regular. To cast this equation into a Helmholtz equation, one uses the identity,

$$\sigma^{-1/2} L_\sigma(\sigma^{-1/2}) = \Delta + n, \quad n = -\frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}}.$$

Thus certain results pertaining to the EIT and the Schrödinger equation hold for the Helmholtz equation under consideration here [22, 4].

The problem of scattering of time-harmonic waves by obstacles in the context of the Born series, which we adapt here to seismic applications, has been studied for many years. However, here, the background is commonly assumed to be homogeneous and cannot be used to approximate the original medium. We mention the work of Colton and Kress [6], Natterer [14], and Sylvester [20]. The conditions for convergence imply estimates for the Born approximation and its use in inverse problems, see Ramm [18, 19]. Inverse scattering approaches based on the 'inversion' of the Born series have been developed by Weglein *et al.* [25], Cheney and Rose [5], Tsihrintzis and Devaney [21], and, for diffuse waves, by Moskow and Schotland [13]. These approaches originate from the work of Jost and Kohn [10], Moses [12], and Prosser [17] concerning the quantum-mechanical inverse backscattering problem.

We consider the following models. We decompose $n(x)$ into a background component and a contrast according to $c(x)^{-2} = c_0(x)^{-2}(1+\alpha(x))$; α is supported in a ball of radius R while $n \in C^2$ outside a ball of radius R_s; see Fig. 1. The background component can contain an interface, Γ say, across which $n_0(x) = \omega^2 c_0(x)^{-2}$ can jump. The boundary of the domain of interest is Σ.

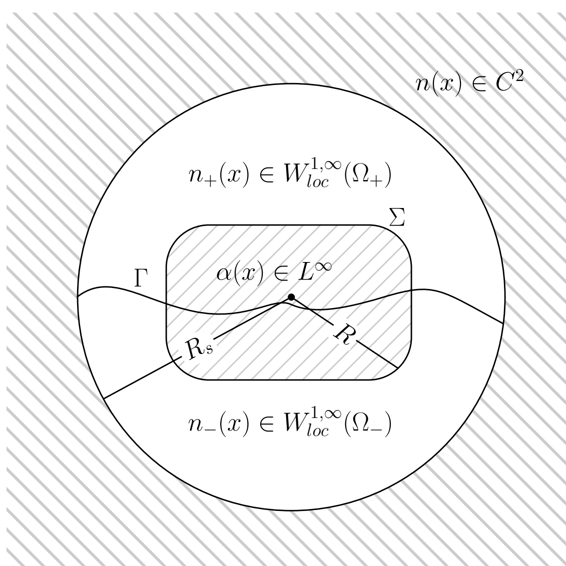


FIG. 1. *Illustration of the model assumptions and decomposition of the function $n(x)$ into its components.*

The paper is organized as follows. In the next section, we summarize results pertaining to the existence and uniqueness of solutions to the Helmholtz equation making use of the limiting absorption principle, and address how the solution models seismic data. In Section 3 we introduce the scattering series and present our main results: (i) A condition (frequency bound) for convergence of the series in the Morrey-Campanato norm for relative contrasts in L^∞ for dimension 3 and higher, and (ii) for relative contrasts in C^2, a recursive procedure coupling spectral content in the relative contrast to the mentioned frequency bound. We also obtain an estimate for the distorted Born approximation providing a measure of applicability of local linearization of the scattering problem. In Section 4 we briefly discuss the convergence of the scattering series, now in a modified Morrey-Campanato norm, for the two-dimensional case. We present a numerical verification of the condition for convergence using a model representative of salt intrusions in a geological setting in Section 5. We end with a discussion (Section 6).

2. The Helmholtz equation. In this section, we summarize results pertaining to the Helmholtz equation, which we will use in the further analysis. The Helmholtz equation is given by

$$-(\Delta + n(x))u = f, \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $n(x) = \omega^2 c(x)^{-2}$ if $c = c(x)$ denotes the wavespeed and ω the angular frequency. Without any restriction we assume that $\omega > 0$; ω is fixed. We consider $d \geq 3$.

It is a standard procedure to perturb (2.1) according to

$$(\mathrm{i}\varepsilon + \Delta + n)u_\varepsilon = -f, \quad x \in \mathbb{R}^d,\ \varepsilon > 0. \tag{2.2}$$

Because $\varepsilon \in \mathbb{R}$, then, for any $f \in L^2$, there exists a unique $u_\varepsilon \in L^2$ which solves (2.2) and

$$\|u_\varepsilon\|_{L^2} \leq \frac{1}{\varepsilon}\,\|(\mathrm{i}\varepsilon + \Delta + n)u_\varepsilon\|_{L^2}, \quad \text{or} \quad \|u_\varepsilon\|_{L^2} \leq \frac{1}{\varepsilon}\,\|f\|_{L^2}, \tag{2.3}$$

whence the problem is well posed.

The right-hand side of (2.3) blows up if $\varepsilon \to 0^+$, which necessitates the introduction of norms different from the L^2 norm to analyze the problem of solving the Helmholtz equation. In case $n(x) = \text{const}$, at $\varepsilon = 0$, the following estimate holds (Agmon and Hörmander [1])

$$\|u\|_{B^*} \leq \frac{C(d)}{(1+n)^{1/2}}\,\|(\Delta + n)u\|_B, \quad \text{or} \quad \|u\|_{B^*} \leq \frac{C(d)}{(1+n)^{1/2}}\,\|f\|_B; \tag{2.4}$$

here,

$$\|f\|_B = \|f\|_{L^2(B_0)} + \sum_{j=0}^{\infty} 2^{j/2}\|f\|_{L^2(A_j)}, \qquad \|u\|_{B^*} = \max(\|u\|_{L^2(B_0)}, \sup_{j\in\mathbb{N}_0} 2^{-j/2}\|u\|_{L^2(A_j)}), \tag{2.5}$$

with $A_j = \{2^j \leq |x| \leq 2^{j+1}\}$, $j = 0, 1, 2, \ldots$ and B_0 denoting the unit ball. The duality between these two norms is expressed by

$$\int_{\mathbb{R}^d} |f(x)u(x)|\mathrm{d}x \leq \|f\|_B\|u\|_{B^*}. \tag{2.6}$$

2.1. Model assumptions. We consider models of variable wavespeed or n (wavenumber squared), containing an interface; for the existence of solutions to the Helmholtz equation we require that n is at least $W^{1,\infty}_{loc}$ (or $C^{0,1}_{loc}$) away from the interface. For uniqueness, incorporating the Sommerfeld radiation condition, we require that n is at least C^2 outside a ball of some finite radius.

We introduce two unbounded domains Ω_+ and Ω_- such that $\overline{\Omega}_+ \cup \Omega_- = \Omega_+ \cup \overline{\Omega}_- = \mathbb{R}^d$. The boundary or interface, $\Gamma = \partial\Omega_+ = \partial\Omega_-$, is a smooth (at least Lipschitz) hypersurface. We write

$$n(x) = \begin{cases} n_+(x), & x \in \Omega_+, \\ n_-(x), & x \in \Omega_-, \end{cases}$$

and denote, for $x \in \Gamma$, the jump by

$$[n](x) = n_+(x) - n_-(x).$$

We write ∇n instead of $\nabla n_+ 1_{\Omega_+} + \nabla n_- 1_{\Omega_-}$, the gradient of n outside the interface. We denote the unit normal vector at $x \in \Gamma$ directed from Ω_- to Ω_+ by $\nu(x)$.

With the notation introduced above, we state the assumptions [23, 8]

Assumption 1. *There is a constant $\gamma > 0$ such that the d^{th} component of $\nu(x)$ satisfies*

$$\nu_d(x) \geq \gamma \quad \textit{for all } x \in \Gamma.$$

Assumption 2. *We have $n_+(x) \in W^{1,\infty}_{loc}(\Omega_+)$, $n_-(x) \in W^{1,\infty}_{loc}(\Omega_-)$ and $n(x) \geq n_2 > 0$.*

Assumption 3. *The jump function $[n](x)$ has the same sign, σ, for all $x \in \Gamma$; $\sigma = -$ if $[n]$ is non-negative and $\sigma = +$ if $[n]$ is non-positive.*

Assumption 4. *We have*

$$2 \sum_{j \in \mathbb{Z}} \textit{ess sup}_{A_j} \frac{(x \cdot \nabla n(x))_-}{n(x)} = \beta_1 < \infty; \tag{2.7}$$

$$\frac{1}{\gamma} \sum_{j \in \mathbb{Z}} \textit{ess sup}_{A_j} 2^{j+1} \frac{(\partial_d n(x))_\sigma}{n(x)} = \beta_2 < \infty; \tag{2.8}$$

here, $(a)_-$ denotes the negative part of $a \in \mathbb{R}$ and $(a)_+$ denotes the positive part of a; γ is the constant appearing in Assumption 1. Moreover, $\beta_1 + \beta_2 < 1$.

Remark 2.1. *Assumption 3 and 4 can be understood as conditions on the rays of geometrical optics. Assumption 3 and (2.8) ensure that the energy flows from one side of the interface, Γ, to the other. Equation (2.8) becomes a weaker assumption if the interface is close to a hyperplane (γ approaches 1) [8].*

Equation (2.7) is the virial condition, a reinforced version of the so-called non-trapping condition, which, together with Assumption 2, ensures that the rays of geometric optics leave any compact set at a nonzero speed. To be precise, let $(X(t), \Xi(t))$ denote the Hamilton flow on $T^*\mathbb{R}^d$ associated with the symbol, $\xi^2 - n(x)$, of the Helmholtz operator (cf. (2.1)), that is,

$$\begin{aligned} \dot{X}(t) &= \Xi(t) &, \quad X(0) &= x, \\ \dot{\Xi}(t) &= \tfrac{1}{2} \nabla_x n(X(t)) &, \quad \Xi(0) &= \xi. \end{aligned} \tag{2.9}$$

We have that $|\Xi(t)|^2 = n(X(t))$ determining the ray velocity. Assumptions 2 and 4 then imply that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(X(t) \cdot \Xi(t)) &= \Xi(t)^2 + \tfrac{1}{2} X(t) \cdot \nabla_x n(X(t)) \\ &= n(X(t)) + \tfrac{1}{2} X(t) \cdot \nabla_x n(X(t)) \\ &\geq (1 - \tfrac{\beta_1}{4})\, n(X(t)) \\ &> \tfrac{3}{4} n_2. \end{aligned} \tag{2.10}$$

Hence, for $|t|$ sufficiently large, we have that $|X(t)\cdot\Xi(t)| \geq \frac{3n_2}{4}|t|$. It follows that all the trajectories $X(t)$ satisfy

$$X(t)^2 \geq \frac{3n_2}{4}t^2$$

for $|t|$ sufficient large; this is stronger than the non-trapping condition,

$$|X(t)| \to \infty \text{ as } t \to \pm\infty$$

[3].

2.2. Morrey-Campanato estimates. We consider the Morrey-Campanato norm,

$$\|\!|u|\!\|^2 = \sup_{y\in\mathbb{R}^d, R>0} \frac{1}{R}\int_{|x-y|<R} |u(x)|^2 \mathrm{d}x, \tag{2.11}$$

and its dual norm

$$N(f) = \sum_{j\in\mathbb{Z}} 2^{(j+1)/2}\|f\|_{L^2(A_j)}. \tag{2.12}$$

The duality is expressed by

$$\begin{aligned} \left|\int f(x)u(x)\mathrm{d}x\right| &\leq \int |f(x)u(x)|\,\mathrm{d}x = \sum_{j\in\mathbb{Z}}\int_{A_j}|f(x)u(x)|\mathrm{d}x \\ &\leq \sum_{j\in\mathbb{Z}}\left(\int_{A_j} 2^{j+1}|f(x)|^2\mathrm{d}x\right)^{1/2}\left(\int_{A_j} 2^{-(j+1)}|u(x)|^2\mathrm{d}x\right)^{1/2} \leq N(f)\,\|\!|u|\!\|. \end{aligned} \tag{2.13}$$

We have

THEOREM 2.2 (Morrey-Campanato estimate [23, 8]). *Let $d \geq 3$. If Assumptions 1-4 hold true then there exists a constant $\tilde{C} = \tilde{C}(d,\gamma,\beta_1,\beta_2)$ such that the solution of (2.2) satisfies,*

$$\|\!|\nabla u_\varepsilon|\!\|^2 + \|\!|n^{1/2}u_\varepsilon|\!\|^2 \leq \tilde{C}(d,\gamma,\beta_1,\beta_2)(\varepsilon + \|n\|_{L^\infty})N(n^{-1/2}f)^2, \tag{2.14}$$

for all $\varepsilon > 0$.

When ε is small, estimate (2.14) implies

$$\|\!|\nabla u_\varepsilon|\!\|^2 + \|\!|n^{1/2}u_\varepsilon|\!\|^2 \leq C(d,\gamma,\beta_1,\beta_2)N(f)^2. \tag{2.15}$$

With the aid of the uniform energy estimate (2.15), applying the limiting absorption principle, that is, letting $\varepsilon \to 0^+$, one obtains the existence of solutions to (2.1). The limiting function, u, satisfies the following Morrey-Campanato energy estimate:

COROLLARY 2.3 ([8]). *Let $d \geq 3$. If Assumptions 1-4 hold true then the solution of (2.1) satisfies the Morrey-Campanato estimate,*

$$\|\!|\nabla u|\!\|^2 + \|\!|n^{1/2}u|\!\|^2 \leq C(d,\gamma,\beta_1,\beta_2)N(f)^2. \tag{2.16}$$

The norm $N(\cdot)$ in the right-hand side of (2.16) is not translation invariant. To remedy this, we consider a different norm for the energy estimate of the Helmholtz equation. One says that a measurable function b is a block if it is supported in a ball B_R with radius R in such a way that

$$\left(\int_{B_R}|b(x)|^2\mathrm{d}x\right)^{1/2} \leq R^{-1/2}.$$

In fact, a block is an atom without cancelation. The block space $\mathcal{B}$ consists of measurable functions f which can be written as

$$f = \sum_{k=1}^{\infty} \lambda_k b_k \quad \text{a.e.}$$

with $\sum |\lambda_k| < \infty$ and the b_k denoting blocks;

$$\|f\|_{\mathcal{B}} = \inf \left\{ \sum |\lambda_k| \mid f = \sum \lambda_k b_k \right\}, \tag{2.17}$$

where the infimum is taken over all possible decompositions of f into blocks. With the aid of the translation operator, we obtain the following corollary,

COROLLARY 2.4. *Let $d \geq 3$. If Assumptions 1-4 hold true then the solution of (2.1) satisfies the Morrey-Campanato estimate,*

$$|\!|\!|\nabla u|\!|\!|^2 + |\!|\!|n^{1/2}u|\!|\!|^2 \leq C(d, \gamma, \beta_1, \beta_2) \|f\|^2_{\mathcal{B}}. \tag{2.18}$$

Proof. Assume that $f = \sum \lambda_k b_k$ is a decomposition of f into blocks and let $\operatorname{supp} b_k \subset B_{R_k}(x_k)$. It is straightforward to check that $N(\tau_{x_k} b_k) = \sum_{j\in\mathbb{Z}} 2^{\frac{j+1}{2}} \|\tau_{x_k} b_k\|_{L^2(A_j)} \leq \frac{\sqrt{2}}{\sqrt{2}-1}$, where $\tau_h : u(x) \mapsto u(x-h)$ is the translation operator. Let u_k be the solution to the equation

$$-(\Delta + \tau_{x_k} n(x)) u_k(x) = \tau_{x_k} b_k.$$

Then $u = \sum \lambda_k \tau_{-x_k} u_k$ solves the equation

$$-(\Delta + n(x)) u(x) = f.$$

Using Corollary 2.3, we find that

$$\begin{aligned} |\!|\!|\nabla u|\!|\!| + |\!|\!|n^{1/2}u|\!|\!| &= |\!|\!|\sum \lambda_k \nabla(\tau_{-x_k} u_k)|\!|\!| + |\!|\!|\sum \lambda_k \tau_{-x_k}((\tau_{x_k} n^{1/2}) u_k)|\!|\!| \\ &\leq \sum |\lambda_k| \left(|\!|\!|\nabla u_k|\!|\!| + |\!|\!|(\tau_{x_k} n^{1/2}) u_k|\!|\!| \right) \leq C \sum |\lambda_k| N(\tau_{x_k} b_k) \leq C' \sum |\lambda_k|; \end{aligned} \tag{2.19}$$

taking the infimum, we obtain (2.18). ◻

Estimate (2.18) will be used in Subsection 3.1, while estimate (2.4) will be used in Section 4, to establish convergence of the scattering (or distorted Born) series. To put the different estimates in perspective, we note the following continuous inclusions (cf. (2.5), (2.11)-(2.12) and (2.17)):

$$\begin{aligned} L^{2,\delta}(\mathbb{R}^d) &\subset L^2(\mathbb{R}^d, \|\cdot\|_B) \subset L^2(\mathbb{R}^d, \|\cdot\|_{\mathcal{B}}) \subset L^2(\mathbb{R}^d) \\ &\subset L^2(\mathbb{R}^d, |\!|\!|\cdot|\!|\!|) \subset L^2(\mathbb{R}^d, \|\cdot\|_{B^*}) \subset L^{2,-\delta}(\mathbb{R}^d) \end{aligned} \tag{2.20}$$

for each $\delta > \frac{1}{2}$.

It is possible to introduce the limiting absorption principle using a complex angular frequency: We replace $n(x)$ by $(\omega + \mathrm{i}\tau)^2 c(x)^{-2}$ with $0 < \tau < \tau_0 \ll \omega$. Let $c(x) \geq c_1 > 0$. We set $n_\tau(x) = (\omega^2 - \tau^2) c(x)^{-2}$. We replace ε by τ upon identifying ε with $2\omega\tau c(x)^{-2} \leq 2\omega\tau c_1^{-2}$. The estimate (2.14) changes to

$$|\!|\!|\nabla u_\tau|\!|\!|^2 + |\!|\!|n_\tau^{1/2} u_\tau|\!|\!|^2 \leq \tilde{C}'(d, \gamma, \beta_1, \beta_2)\, \frac{\omega^2 - \tau^2 + 2\omega\tau}{c_1^2}\, N(n_\tau^{-1/2} f)^2.$$

When τ is small, this implies the estimate

$$|\!|\!|\nabla u_\tau|\!|\!|^2 + |\!|\!|n_\tau^{1/2} u_\tau|\!|\!|^2 \leq C'(d, \gamma, \beta_1, \beta_2)\, N(f)^2,$$

which is uniform with respect to τ, whence the limiting absorption principle can be applied.

2.3. Sommerfeld radiation condition. Assumptions 1, 2, 3 and 4 guarantee that the energy estimate in Theorem 2.2 holds, which implies the existence of a solution to the Helmholtz equation (2.1) by applying limiting absorption principle. If the following assumption is satisfied, the solution can satisfy the Sommerfeld radiation condition and become unique.

ASSUMPTION 5 ([24]). *The function $n(x)$ admits the decomposition*

$$n(x) = \lambda + p(x), \quad \lambda > 0, \tag{2.21}$$

with $p(x) \in C^2(\mathbb{R}^d \backslash \{|x| \le R_s\})$ for some $R_s > 0$, a real-valued, bounded function which satisfies the estimates

$$|\partial_x^\alpha p(x)| \le C|x|^{-|\alpha|}, \quad \text{for all } |x| \ge R_s,\ |\alpha| \le 2. \tag{2.22}$$

For the existence of a solution, we can allow $n(x)$ to jump across the entire hypersurface Γ. However, to satisfy the Sommerfeld condition, one uses the existence of the solution $\phi(x)$ of the eikonal equation $|\nabla\phi|^2 = n$ for $|x| > R_s$ (R_s sufficiently large); see Barles [2] if n is C^2. We hence need to assume that the jump $[n]$ vanishes outside a compact domain or a ball of finite radius, R_s.

THEOREM 2.5 (Sommerfeld radiation condition). *Let $d \ge 2$ and Assumptions 1-5 hold true. For $\lambda \gg \|p\|_{C^2(|x|>R_s)}$ there exist a unique solution to the Helmholtz equation (2.1), and constants C_a, which satisfy*

$$\int_{\mathbb{R}^d} \left|\nabla u(x) - \mathrm{i} n(x)^{1/2} u(x) \frac{x}{|x|}\right|^2 \frac{\mathrm{d}x}{1+|x|} \le C_a \int_{\mathbb{R}^d} |f(x)|^2 (1+|x|)^a \mathrm{d}x \tag{2.23}$$

for all $a > 1$.

For the proof, see [24]. Applying this theorem, we can introduce the solution operator, S,

$$S: \ L^2(\mathbb{R}^d, \|\cdot\|_{\mathcal{B}}) \to L^2(\mathbb{R}^d, \|\!|\cdot|\!\|), \quad f \mapsto u, \tag{2.24}$$

where u is the unique solution to the Helmholtz equation $-(\Delta + n(x))u = f$ (cf. (2.1)) satisfying the Sommerfeld radiation condition (cf. (2.23)).

3. Scattering series. We decompose $n(x) = \omega^2 c(x)^{-2}$ in (2.1) into a reference component and a contrast according to

$$c(x)^{-2} = c_0(x)^{-2}(1 + \alpha(x)), \tag{3.1}$$

and assume that $n_0(x) = \omega^2 c_0(x)^{-2}$ satisfies Assumptions 1-5 introduced in the previous section. Bounds for $n_0(x)$ follow from

$$0 < c_1 \le c_0(x) \le c_2 < \infty. \tag{3.2}$$

The relative contrast function, $\alpha(x)$, satisfies

ASSUMPTION 6. *The relative contrast, α is in $L^\infty(\mathbb{R}^d)$ and has compact support.*

We have

$$P: \ L^2(\mathbb{R}^d, \|\!|\cdot|\!\|) \cap H^2_{loc}(\mathbb{R}^d) \to L^2(\mathbb{R}^d, \|\cdot\|_{\mathcal{B}}), \quad u \mapsto -(\Delta + \omega^2 c_0(x)^{-2})u. \tag{3.3}$$

Lemma 3.1. *Let $\alpha(x)$ satisfy Assumption 6, and $E = \operatorname{supp}\alpha$. Then the multiplication by α satisfies the estimate*

$$\|\alpha u\|_{\mathcal{B}} \leq R_\alpha \|\alpha\|_{L^\infty} |||u|||, \tag{3.4}$$

for all $u \in L^2(\mathbb{R}^d, |||\cdot|||)$, where $R_\alpha = \inf\{\sum R_k \mid \{B_k\}$ is a covering of E by balls B_k with radii $R_k\}$.

Proof. For any u with $|||u||| = 1$, it suffices to prove that $\|\chi_E u\|_{\mathcal{B}} \leq R_\alpha$. Assume that $\{B_k\}_{k=1}^M$ is a covering of E by balls and R_k is the radius of B_k. Let $\{\phi_k\}$ be the partition of unity corresponding to $\{B_k\}$ and $\tilde{u} = \sum \phi_k u$. Then $\tilde{u} = \chi_{\cup B_k} u$ and $|||\tilde{u}||| \leq |||u||| = 1$. Note that $|||\phi_k u||| \leq 1$ implies that $\|\phi_k u\|_{L^2} \leq R_k^{1/2}$. Let $b_k(x) = R_k^{-1}\phi_k u$; we have

$$\|b_k(x)\|_{L^2} = R_k^{-1}\|\phi_k u\|_{L^2} \leq R_k^{-1/2},$$

whence b_k is a block. Then

$$\|\chi_E u\|_{\mathcal{B}} \leq \|\chi_{\cup B_k} u\|_{\mathcal{B}} \leq \sum R_k.$$

Taking the infimum over all coverings by balls yields $\|\chi_E u\|_{\mathcal{B}} \leq R_\alpha$. □

We then introduce the multiplication operator, T, according to

$$T:\ L^2(\mathbb{R}^d, |||\cdot|||) \to L^2(\mathbb{R}^d, \|\cdot\|_{\mathcal{B}}), \quad u \mapsto \omega^2\alpha(x)c_0(x)^{-2}u. \tag{3.5}$$

Indeed, with Assumption 6 and Lemma 3.1,

$$\|\omega^2\alpha c_0^{-2}u\|_{\mathcal{B}} \leq \omega^2 c_1^{-2} R_\alpha \|\alpha\|_{L^\infty} |||u|||. \tag{3.6}$$

Remark 3.2. *R_α as defined Lemma 3.1 has the following properties: R_α is less than or equal to half of the diameter of $\operatorname{supp}\alpha$. If $\alpha = \sum \alpha_k$ then $R_\alpha \leq \sum R_{\alpha_k}$. In particular, R_α is less than or equal to half of the sum of the diameters of the connected components of $\operatorname{supp}\alpha$.*

The Helmholtz equation for u can be written in the contrast form,

$$Pu = f + Tu; \tag{3.7}$$

then u solves the Lippmann-Schwinger equation,

$$(I - ST)\,u = u_0, \quad \text{with} \quad Pu_0 = f. \tag{3.8}$$

This integral equation is equivalent to the scattering problem described by (3.7) subject to the Sommerfeld radiation condition.

The so-called scattering or distorted Born series corresponds with the Neumann series solution of (3.8):

$$u = u_0 + u_1 + u_2 + \cdots, \quad u_0 = Sf,\ u_n = STu_{n-1},\ n = 1, 2, \ldots \tag{3.9}$$

(In Colton & Kress [6], Sf is identified as a volume potential.)

3.1. Convergence, $d \geq 3$. Here, we study the convergence of the distorted Born series for $d \geq 3$.

THEOREM 3.3. *Let $n_0(x) = \omega^2 c_0(x)^{-2}$ satisfy Assumptions 1-5, and let $\alpha(x)$ satisfy Assumption 6. If*

$$\omega < \Omega(c_1, c_2, \beta_1, \beta_2, \alpha) = \frac{c_1^2}{C^{1/2} R_\alpha \|\alpha\|_{L^\infty} c_2}, \tag{3.10}$$

where $C = C(d, \beta_1, \beta_2, \gamma)$ is the constant appearing in Corollary 2.4 and R_α is defined in Lemma 3.1, then the scattering series (cf. (3.9)) converges in $L^2(\mathbb{R}^d, \|\!|\cdot|\!\|)$. Furthermore, we have the remainder estimate,

$$\|\!|u - \sum_{n=0}^{N} u_n|\!\| \leq C \frac{c_2^2}{c_1^2} R_\alpha \|\alpha\|_{L^\infty} \frac{(\omega/\Omega(c_1, c_2, \beta_1, \beta_2, \alpha))^N}{1 - (\omega/\Omega(c_1, c_2, \beta_1, \beta_2, \alpha))} \|f\|_{\mathcal{B}}. \tag{3.11}$$

Proof. We begin with establishing an estimate for $ST: L^2(\mathbb{R}^d, \|\!|\cdot|\!\|) \to L^2(\mathbb{R}^d, \|\!|\cdot|\!\|),\ u \mapsto STu$. Using the energy-type estimate in Corollary 2.4, we obtain

$$C\|f\|_{\mathcal{B}}^2 \geq \|\!|\nabla u|\!\|^2 + \|\!|\omega c_0^{-1} u|\!\|^2 \geq \|\!|\omega c_0^{-1} u|\!\|^2 \geq \frac{\omega^2}{c_2^2} \|\!|u|\!\|^2. \tag{3.12}$$

Combining this estimate with (3.6) gives

$$\|\!|S(Tu)|\!\| \leq C^{1/2} \frac{c_2}{\omega} \|\omega^2 \alpha c_0^{-2} u\|_{\mathcal{B}} \leq C^{1/2} R_\alpha \omega \|\alpha\|_{L^\infty} \frac{c_2}{c_1^2} \|\!|u|\!\|. \tag{3.13}$$

Thus $\|\!|ST|\!\| \leq \omega\, \Omega(c_1, c_2, \beta_1, \beta_2, \alpha)^{-1}$, whence the scattering series converges if $\omega \Omega(c_1, c_2, \beta_1, \beta_2, \alpha)^{-1} < 1$.

The estimate for the remainder follows straightforwardly,

$$\begin{aligned} \|\!|u - \sum_{n=0}^{N} u_n|\!\| = \|\!|\sum_{n=N+1}^{\infty} (ST)^n u_0|\!\| &\leq \|\!|u_0|\!\| \sum_{n=N+1}^{\infty} \|\!|ST|\!\|^n \\ &\leq C^{1/2} c_2 \omega^{-1} \|f\|_{\mathcal{B}} \frac{(\omega/\Omega(c_1, c_2, \beta_1, \beta_2, \alpha))^{N+1}}{1 - (\omega/\Omega(c_1, c_2, \beta_1, \beta_2, \alpha))}, \end{aligned}$$

from which (3.11) follows. □

For $N = 1$, (3.11) implies an estimate for the distorted Born approximation in the $\|\!|\cdot|\!\|$ norm. Condition (3.10) can also be viewed, for given frequency or wavelength $2\pi c_1 \omega^{-1}$, as a bound on $R_\alpha \|\alpha\|_{L^\infty}$.

3.2. Special case: Constant background wavespeed. To establish comparisons with results in the literature, in this section, we consider the case in which c_0 is a constant and $f \in L^2(\mathbb{R}^d, \|\cdot\|_B)$. We will apply the (energy) estimates given by Agmon and Hörmander [1] and recover a condition for convergence of the Born series given by Sylvester [20].

We have

$$P_0 :\ L^2(\mathbb{R}^d, \|\cdot\|_{B^*}) \cap H^2_{loc}(\mathbb{R}^d) \to L^2(\mathbb{R}^d, \|\cdot\|_B), \quad u \mapsto -\Delta u - \omega^2 c_0^{-2} u. \tag{3.14}$$

Applying Theorem 2.5, establishing uniqueness of the solution to $P_0 u = f$, we obtain the resolvent

$$S_0 :\ L^2(\mathbb{R}^d, \|\cdot\|_B) \to L^2(\mathbb{R}^d, \|\cdot\|_{B^*}), \quad f \mapsto \int_{\mathbb{R}^d} G_0(x - y) f(y) \mathrm{d}y, \tag{3.15}$$

where $G_0(.-y)$ denotes the fundamental solution satisfying $P_0 G_0(.-y) = \delta_y$. Now the multiplication operator is

$$T_0: \; L^2(\mathbb{R}^d, \|\cdot\|_{B^*}) \to L^2(\mathbb{R}^d, \|\cdot\|_B), \quad u \mapsto \omega^2 \alpha(x) c_0^{-2} u; \tag{3.16}$$

indeed

$$\begin{aligned} \|\omega^2 \alpha c_0^{-2} u\|_B &\leq \omega^2 c_0^{-2} \|\alpha\|_{L^\infty} \left(\|u\|_{L^2(B_0)} + \sum_{j=0}^{M-1} 2^{j/2} \|u\|_{L^2(A_j)} \right) \\ &\leq \omega^2 c_0^{-2} \|\alpha\|_{L^\infty} \left(\|u\|_{B^*} + \sum_{j=0}^{M-1} 2^j \|u\|_{B^*} \right) = R\omega^2 c_0^{-2} \|\alpha\|_{L^\infty} \|u\|_{B^*}. \end{aligned} \tag{3.17}$$

The Born series

$$u = u_0 + u_1 + u_2 + \cdots, \quad u_0 = S_0 f, \; u_n = S_0 T_0 u_{n-1}, \; n = 1, 2, \ldots \tag{3.18}$$

converges in $L^2(\mathbb{R}^d, \|\cdot\|_{B^*})$ if

$$\omega < \Omega_0(c_0, \|\alpha\|_{L^\infty}) = \frac{c_0}{C_0 R \|\alpha\|_{L^\infty}}; \tag{3.19}$$

here, $C_0 = C(d)$ in (2.4). This condition corresponds with condition [20, (43)] with R replaced by $R(D) = \sup_{\Theta \in S^{d-1}} R(D, \Theta)$ in which $D = \operatorname{supp} \alpha$ and

$$R(D, \Theta) = \sup_{x \in D} \mu(\{t \,|\, x + t\Theta \in D\});$$

μ denotes the one-dimensional Lebesgue measure. Condition (3.19) follows from estimating the operator norm of $S_0 T_0$:

$$\begin{aligned} | \, S_0(T_0 u)\|_{B^*} &\leq \frac{C_0}{(1 + \omega^2 c_0^{-2})^{1/2}} \|T_0 u\|_B \\ &\leq C_0 \frac{\omega^2 c_0^{-2}}{(1 + \omega^2 c_0^{-2})^{1/2}} R \|\alpha\|_{L^\infty} \|u\|_{B^*} \leq C_0 c_0^{-1} \omega R \|\alpha\|_{L^\infty} \|u\|_{B^*}. \end{aligned} \tag{3.20}$$

Convergence of the Born series in the maximum norm was analyzed and discussed by Cheney and Rose [5], De Hoop [7], and Colton and Kress [6]. Natterer [14] discusses the convergence in $L^2(|x| < R)$ (for each $R > 0$) while Moskow and Schotland [13] discuss the L^∞ convergence. These results make use of an estimate for the fundamental solution G_0, that is, for $d = 3$,

$$\sup_{|x| \leq R} \int_{|y| < R} \left| \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \right| \mathrm{d}y = \int_{|y| < R} \frac{1}{|y|} \mathrm{d}y = 2\pi R^2, \quad k = \omega c_0^{-1},$$

which is uniform in frequency. The condition for convergence follows to be

$$\omega < \Omega_\infty(c_0, \|\alpha\|_{L^\infty}) = \frac{2c_0}{R \|\alpha\|_{L^\infty}^{1/2}} \tag{3.21}$$

[6, Theorem 8.4]. As compared with our estimate, we note the difference in the power of $\|\alpha\|_{L^\infty}$.

4. Convergence of the scattering series, $d = 2$. Here, we impose more restrictive assumptions on the background. The Morrey-Campanato norm for $d = 2$ is defined as [24]

$$\|\!|u|\!\|_{R_0}^2 = \sup_{y\in\mathbb{R}^2, R>R_0} \frac{1}{R}\int_{|x-y|<R} |u(x)|^2 \mathrm{d}x, \tag{4.1}$$

and its dual as

$$N_{R_0}(f) = \sum_{j\in\mathbb{Z}, j>J} 2^{(j+1)/2}\|f\|_{L^2(A_j)} + \left(R_0 \int_{|x|<R_0} |f(x)|^2 \mathrm{d}x \right)^{1/2}, \tag{4.2}$$

where J is defined by $2^J \le R_0 < 2^{J+1}$.

Assumption 7. *The function $n(x)$ is in $W^{1,\infty}_{loc}(\mathbb{R}^2)$, $n(x) \ge n_2 > 0$, and admits the decomposition*

$$n(x) = n_a(x) + n_b(x), \tag{4.3}$$

where $n_b \in L^\infty(\mathbb{R}^2)$ and n_a satisfies the estimate

$$\|n_a v\|_2 \le (1 - C_n)\,\|\nabla v\|_2 \tag{4.4}$$

for all $v \in C_0^\infty(\mathbb{R}^2)$ and some constant $C_n > 0$.

Assumption 8 (virial condition). *We have*

$$2\sum_{j\in\mathbb{Z}} \textit{ess sup}_{A_j} \frac{(x\cdot\nabla n(x))_-}{n(x)} = \beta < 1; \tag{4.5}$$

here, $(a)_-$ denotes the negative part of $a \in \mathbb{R}$.

Theorem 2.2 is replaced by

Theorem 4.1 (Morrey-Campanato estimate [24]). *Let $R_0 = n_2^{-1/2}$. If Assumptions 7 and 8 hold true then there exists a constant $\tilde{C} = \tilde{C}(\beta)$ such that the solution of (2.2) satisfies, for all $\varepsilon > 0$,*

$$\|\!|\nabla u_\varepsilon|\!\|_{R_0}^2 + \|\!|n^{1/2}u_\varepsilon|\!\|_{R_0}^2 \le \tilde{C}(\beta)(\varepsilon + \|n_b\|_{L^\infty}) N_{R_0}(n^{-1/2}f)^2. \tag{4.6}$$

With this estimate, the limiting absorption principle can be applied. The limiting function, u, satisfies the following Morrey-Campanato energy estimate replacing (2.16):

Corollary 4.2. *Let $R_0 = n_2^{-1/2}$. If Assumptions 7 and 8 hold true then the solution of (2.1) satisfies the estimate,*

$$\|\!|\nabla u|\!\|_{R_0}^2 + \|\!|n^{1/2}u|\!\|_{R_0}^2 \le C(n_2, \beta) N_{R_0}(f)^2. \tag{4.7}$$

Invoking Assumption 5, and using Theorem 2.5, the existence and uniqueness of the solution to the 2-D Helmholtz equation then follows. Thus we can introduce the solution operator, S,

$$S:\ L^2(\mathbb{R}^2, N_{R_0}(\cdot)) \to L^2(\mathbb{R}^2, \|\!|\cdot|\!\|_{R_0}), \quad f \mapsto u, \tag{4.8}$$

where u is the unique solution to the Helmholtz equation $-(\Delta + n(x))u = f$ (cf. (2.1)) satisfying the Sommerfeld radiation condition (cf. (2.23)).

We introduce a background and a contrast as in (3.1), and invoke Assumption 6. We have

$$P:\ L^2(\mathbb{R}^2, \|\!|\cdot\|\!|_{R_0}) \cap H^2_{loc}(\mathbb{R}^2) \to L^2(\mathbb{R}^2, N_{R_0}(\cdot)), \quad u \mapsto -(\Delta + \omega^2 c_0(x)^{-2})u. \tag{4.9}$$

and introduce the multiplication operator, T, according to

$$T:\ L^2(\mathbb{R}^2, \|\!|\cdot\|\!|_{R_0}) \to L^2(\mathbb{R}^2, N_{R_0}(\cdot)), \quad u \mapsto \omega^2\alpha(x)c_0(x)^{-2}u. \tag{4.10}$$

Indeed, with $R_0 < R$,

$$\begin{aligned} N_{R_0}(\omega^2\alpha c_0^{-2}u) = \sum_{j\in\mathbb{Z}, j>J} &\left(2^{j+1}\int_{A_j} |\alpha(x)\omega^2 c_0(x)^{-2}u(x)|^2 \mathrm{d}x \right)^{1/2} \\ &+ \left(R_0 \int_{|x|<R_0} |\alpha(x)\omega^2 c_0(x)^{-2}u(x)|^2 \mathrm{d}x \right)^{1/2} \\ &\le \sum_{j=J-1}^{M-1} 2^{j+1} \left(\frac{1}{2^{j+1}} \int_{|x|<R} |\alpha(x)\omega^2 c_0(x)^{-2}u(x)|^2 \mathrm{d}x \right)^{1/2} \\ &+ \omega^2 c_1^{-2}\|\alpha\|_{L^\infty}(R_0+\delta) \left(\frac{1}{R_0+\delta} \int_{|x|<R_0+\delta} |u(x)|^2 \mathrm{d}x \right)^{1/2} \\ \le \omega^2 c_1^{-2}\|\alpha\|_{L^\infty}\|\!|u\|\!|_{R_0} \sum_{j=J-1}^{M-1} 2^{j+1} &+ \omega^2 c_1^{-2}(R_0+\delta)\|\alpha\|_{L^\infty}\|\!|u\|\!|_{R_0} = \omega^2 c_1^{-2}(2R+\delta)\|\alpha\|_{L^\infty}\|\!|u\|\!|_{R_0} \end{aligned}$$

for all $\delta > 0$. Hence,

$$N_{R_0}(\omega^2\alpha c_0^{-2}u) \le 2\omega^2 c_1^{-2}R\|\alpha\|_{L^\infty}\|\!|u\|\!|_{R_0}. \tag{4.11}$$

Combining this estimate with estimate (4.7) yields

$$\|\!|S(Tu)\|\!|_{R_0} \le \frac{C(n_2,\beta)^{1/2}}{\omega c_2^{-1}} N_{R_0}(Tu) \le 2C(n_2,\beta)^{1/2}c_2c_1^{-2}\omega R\|\alpha\|_{L^\infty}\|\!|u\|\!|_{R_0}. \tag{4.12}$$

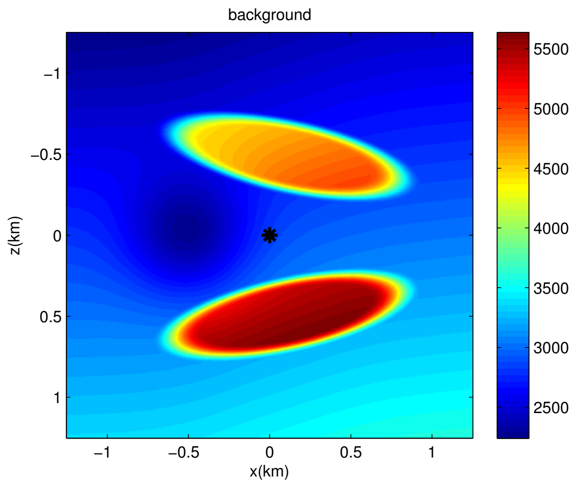
It follows that the scattering series converges if

$$\omega < \Omega_2(c_1, c_2, \beta, \|\alpha\|_{L^\infty}) = \frac{c_1^2}{2C(n_2,\beta)^{1/2}R\|\alpha\|_{L^\infty}c_2}. \tag{4.13}$$

5. Numerical examples. We illustrate the convergence of the (adaptive) scattering series by computing $u_0, u_1, \ldots, u_9$ (the first 10 terms) where $u_n = STu_{n-1},\ n = 1, 2, \ldots, 9$ and $u_0 = Sf$ is the incident field, cf. (3.9). The parameters that characterize the model are chosen to be: $\lambda = 2\pi c_1\omega^{-1}$, $\rho = R\lambda^{-1}$ and $\eta = c_2c_1^{-1}\|\alpha\|_{L^\infty}$. As the source, we take a Gaussian function: $f_s(x) = e^{-|x|^2/8}$. We consider $d = 2$.

The background wavespeed model, $c_0(x)$, is shown in Fig. 2; the relative contrast, $\alpha(x)$, is concentrated on the boundaries of the inclusions and is shown in Fig. 4. The complete model ($c(x)$) is illustrated in Fig. 5, and is motivated by salt intrusions in a sedimentary environment. The 'star' in the pictures indicates the position of the source.

We consider a set of ρ values by fixing R and varying $\omega = 2\pi f$: $f = 5, 15, 25, 30$ and 40 Hz; $\eta = 2.25$ is kept fixed. Fig. 6 shows the real parts of the truncated series expansions for the scattered fields. In Fig. 7 we illustrate the real part of the incident field, and In Fig. 8 we illustrate the real part of the series truncation error (which is about 0.2% here), for $\rho = 11.16$. The Morrey-Campanato norms of the successive terms in Born series are given in Figure 9.



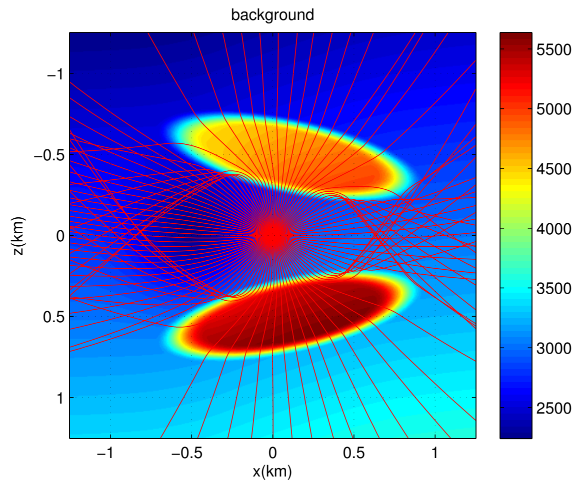
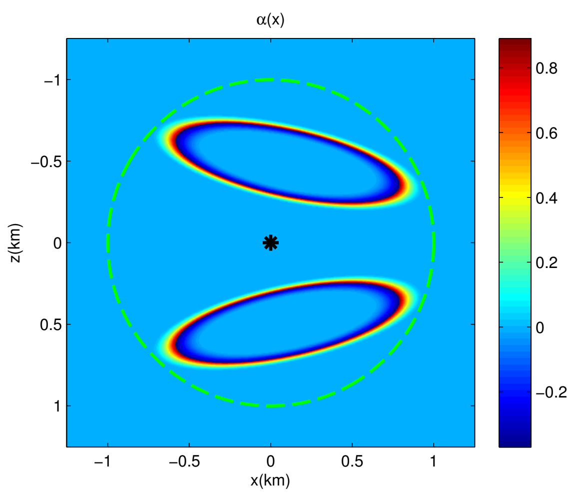


FIG. 2. *Background wavespeed model $c_0(x)$; $c_2 c_1^{-1} = 2.52$.*

FIG. 3. *Ray geometry – geodesics in the background model originating at the source indicated by a star.*



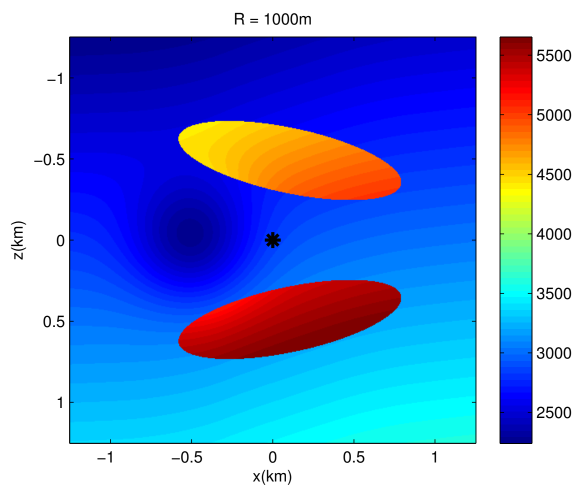


FIG. 4. *The relative contrast model $\alpha(x)$. The dashed circle has radius R. Note that the relative contrast is concentrated near the boundaries of the scatterers.*

FIG. 5. *The wavespeed model, $c(x)$.*

Figure 12 shows the Morrey-Campanato norms for the recursion when the contrast function represents the entire obstacles (Figure 11) and corresponding background (Figure 10) to form the same model $c(x)(\eta = 4.62$ while the other parameters are the same). The series starts to diverge at a much lower frequency (15 Hz).

6. Discussion. We studied the modelling of seismic data in terms of the trace of solutions to the Helmholtz equation in $\mathbb{R}^d, d \geq 2$, subjected to the Sommerfeld radiation condition. Modelling with the Helmholtz equation has become an important component in what seismologists call 'full waveform inversion', that is, the reconstruction of $n(x)$ through minimizing a data misfit; see, for example, [16], and [15] for the method of adjoint states.

While using real-valued angular frequencies, we have restrictions on the regularity of the wavespeed describing the model. Using a model satisfying these restictions as a backgound, via a contrast source formulation, the Lippmann-Schwinger equation, and the corresponding scattering series, bounded

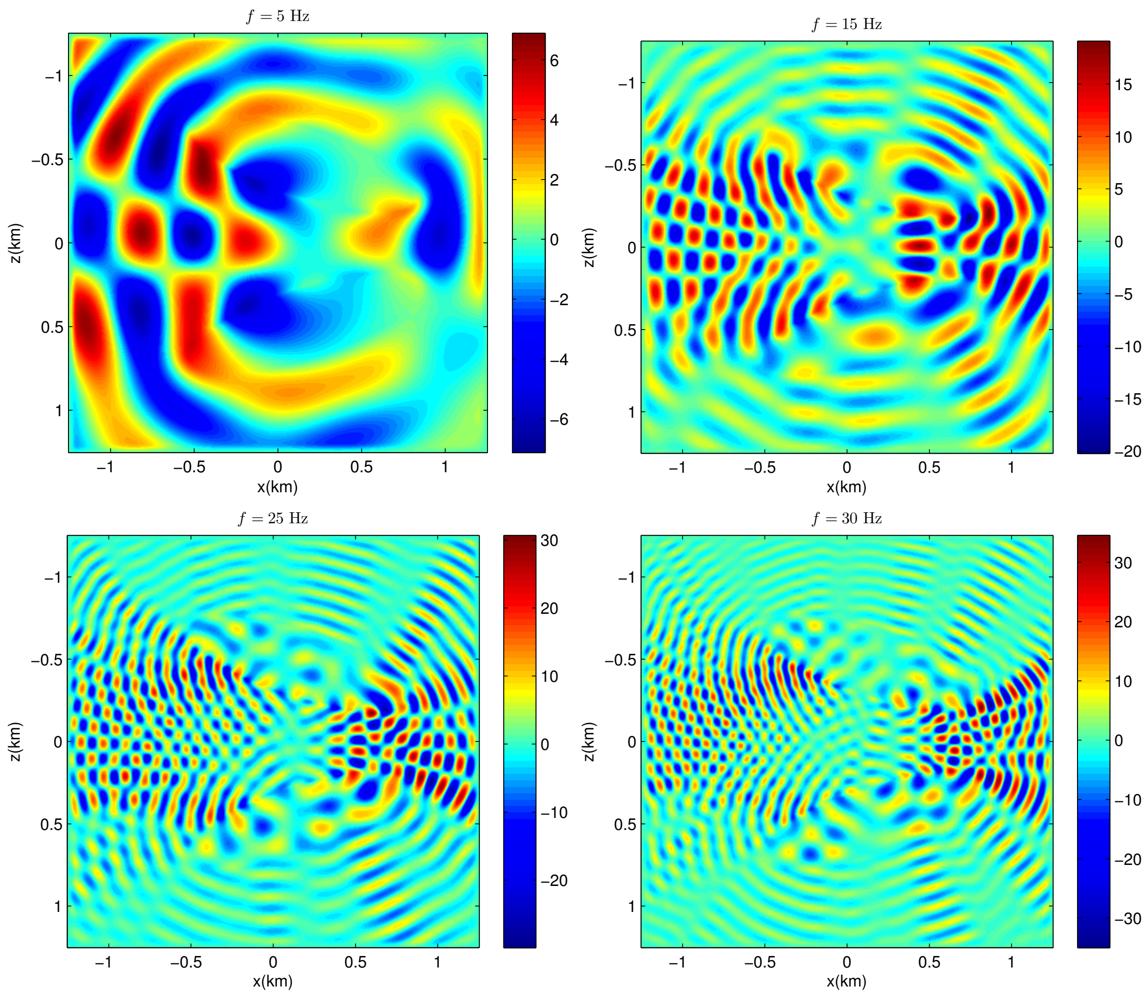


FIG. 6. *Real parts of the truncated scattering series – scattered field; top left:* $\rho = 2.23$*, top right:* $\rho = 6.70$*, bottom left:* $\rho = 11.16$*, bottom right:* $\rho = 13.40$*.*

and measureable variations in the wavespeed are introduced. In our formulation, in the context of seismic applications, the background model is assumed to be consistent with tomographic reconstructions.

We recover conditions for convergence of the scattering series in the framework of heterogeneous background models of limited smoothness. We allow the background models to contain an isolated interface or discontinuity (Γ in Fig. 1). Such an interface can aid in the illumination of scatterers in (seismic) inverse scattering problems with partial boundary data; see, for example, [9]. Our conditions are similar to the one for homogeneous background models, though convergence is established in a particular, Morrey-Campanato norm. However, we show numerical evidence that our conditions are conservative in as far as how they measure the support of the contrast.

The scattering series provides an expansion of the data, and hence of the Neumann-to-Dirichlet map, in multiple sattered waves. This expansion can be exploited in further developing techniques for analyzing the seismic wave field [25]. We also obtain an estimate of the distorted Born approximation in the mentioned Morrey-Campanato norm. For continuous contrasts, this estimate can be applied to develop an understanding of for which frequencies the spectral content of the contrast can be sensed using the distorted Born approximation or the scattering series expansion. This can be viewed

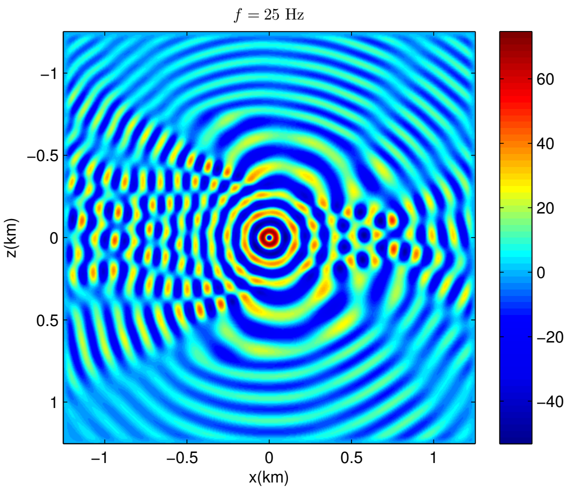


FIG. 7. *The incident field,* $u_0(x)$ *for* $\rho = 11.16$.

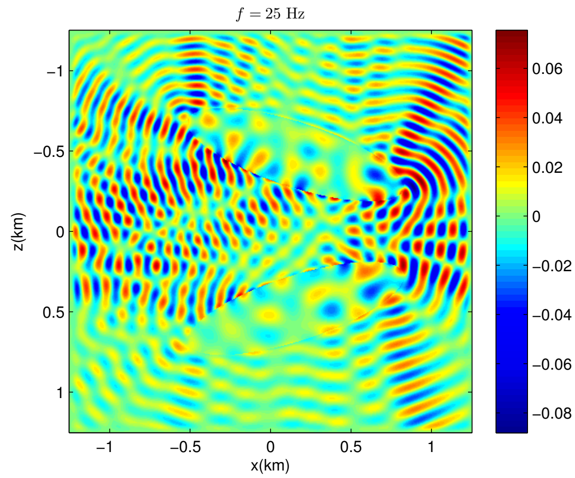


FIG. 8. *The error,* $u(x) - u_1(x) - \cdots - u_9(x)$*, for* $\rho = 11.16$.

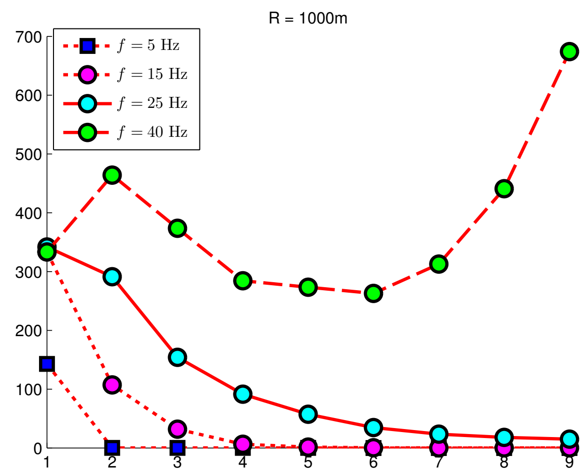


FIG. 9. *Morrey-Campanato norms of* $u_n (n = 1, 2, \ldots 9)$

as a modelling counterpart to Pratt's [16] strategy for 'full waveform inversion', and is elaborated below.

Recursive scattering series. We carry out a multi-scale or frequency decomposition of $\alpha(x)$. We restrict ourselves here to $\alpha \in C^0$. Let φ be a low-pass filter with $\hat{\varphi}(\xi) = 1$ for $|\xi| \le 1$ and $= 0$ for $|\xi| > 2$. We introduce

$$\widetilde{\alpha}_k(x) = (\varphi(2^{-k/2} D_x)\alpha)(x),$$

with the property that

$$\|\alpha - \widetilde{\alpha}_k\|_{L^\infty} \le C_f 2^{-k/2}$$

for some constant C_f, uniform in k. We form

$$\alpha(x) = \widetilde{\alpha}_0(x) + \sum_{k=1}^{\infty} \alpha_k(x), \quad \alpha_k(x) = \widetilde{\alpha}_k(x) - \widetilde{\alpha}_{k-1}(x).$$

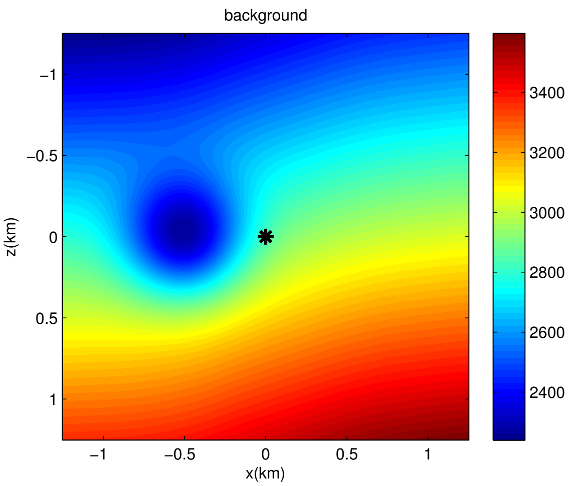


FIG. 10. *Background wavespeed model* $c_0(x)$*;* $c_2 c_1^{-1} =$ 1.61.

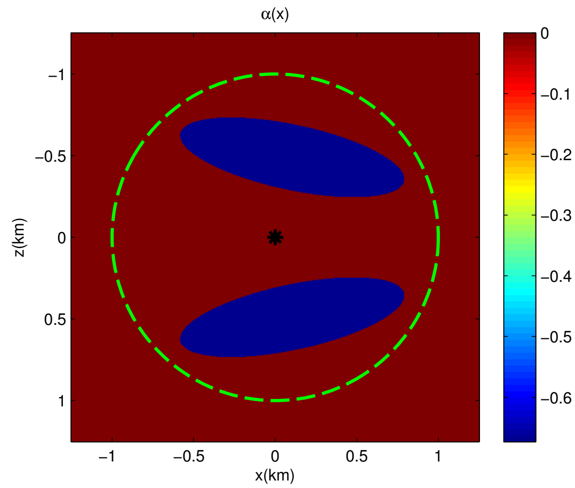


FIG. 11. *The relative contrast model* $\alpha(x)$*. Note that the relative contrast fulfills the scatterers.*

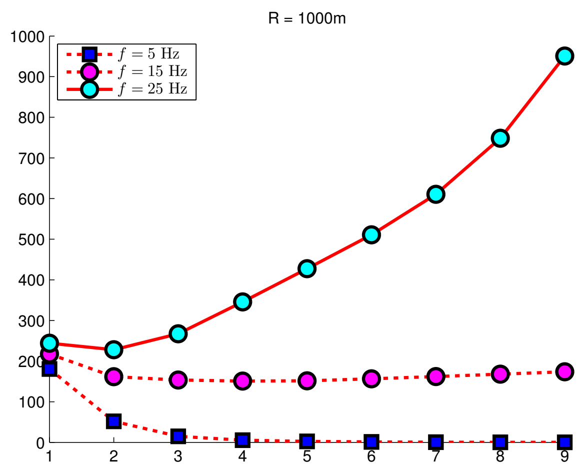


FIG. 12. *Morrey-Campanato norms of* $u_n (n = 1, 2, \ldots 9)$

We introduce the sequence of scattering series, $u^{(k)}$, $k = 1, 2, \ldots$, where, $u^{(k)}$ is obtained using the background

$$c^{(k-1)}(x)^{-2} = c_0(x)^{-2}(1 + \widetilde{\alpha}_{k-1}(x)),$$

and relative contrast $\alpha_k(x)$, with $\|\alpha_k\|_{L^\infty} = \mathcal{O}(2^{-k/2})$. In the process, we assume that $n^{(k-1)}(x) = \omega^2 c^{(k-1)}(x)^{-2}$ satisfies (2.7) and Assumption 5. The convergence conditions (cf. (3.10)) become

$$\omega < \Omega_k(c_1, c_2, \beta_1, \beta_2, \|\alpha\|_{L^\infty}) = \frac{c_1^2}{C^{1/2} R_\alpha \|\alpha_k\|_{L^\infty} c_2} = \mathcal{O}(2^{k/2}).$$

Thus, if coarser scales in the contrast are included in the background leaving finer scale variations, one can admit higher frequencies.

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