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# ELASTIC-WAVE INVERSE SCATTERING WITH ACTIVE AND PASSIVE SOURCE REFLECTION DATA

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**Abstract.** We develop a comprehensive theory and microlocal analysis of reverse-time imaging – also referred to as reverse-time migration or RTM – for the anisotropic elastic wave equation based on the single scattering approximation. We consider a configuration reminiscent of the seismic inverse scattering problem. In this configuration, we have an interior (point) body-force source which generates elastic waves, which scatter off discontinuities in the properties of earth's materials (anisotropic stiffness, density), and are observed at receivers on the earth's surface. The receivers detect all the components of displacement. We introduce (i) an anisotropic elastic-wave RTM inverse scattering transform, and for the case of mode conversions (ii) a microlocally equivalent formulation avoiding knowledge of the source via the introduction of so-called array receiver functions. These allow a seamless integration of passive source and active source approaches to inverse scattering.

**Key words.** elastic wave equation, inverse scattering, receiver functions, microlocal analysis

**AMS subject classifications.** 86A15, 35R30

**1. Introduction.** We develop a program and analysis for elastic wave-equation inverse scattering, based on the single scattering approximation, from two interrelated points of view, known in the seismic imaging literature as "receiver functions" (passive source) and "reverse-time migration" (active source).

We consider an interior (point) body-force source which generates elastic waves, which scatter off discontinuities in the properties of earth's materials (anisotropic stiffness, density), and are observed at receivers on the earth's surface. The receivers detect all the components of displacement. We decompose the medium into a smooth background model and a singular contrast and assume the single scattering or Born approximation. The inverse scattering problem concerns the reconstruction of the constrast given a background model.

In this paper, we extend the original reverse-time imaging or migration (RTM) procedure for scalar waves [39, 24, 1] to elastic waves. We make use of the integral formulation of Schneider [31] and the inverse scattering integral equation of Bojarski [2]. Elastic-wave RTM has recently become a subject of considerable interest. The current developments, however, have been limited to approaches based on ad hoc scalar-wave approximations [34, 46, 20, 22, 9, 10, 12]. In this framework, the RTM imaging condition is tied to a decomposition into polarizations (for such a decomposition in quasi-homogeneous media, see [42, 43]).

We develop a comprehensive theory and microlocal analysis of reverse-time imaging for the anisotropic elastic wave equation. We construct the corresponding normal operator and an inverse-scattering transform which naturally removes the "smooth artifacts" discussed in [44, 27, 11, 41, 19]. Our work is based on results presented in [4, 33] while assuming a common-source data acquisition. The three main results are: (i) the development of an anisotropic polarized-wave equation formulation, (ii) the introduction of a new (anisotropic) elastic-wave RTM inverse scattering transform, and (iii) the reformulation of (ii) using mode-converted wave constituents removing the knowledge of the source by introducing the notion of array receiver functions which generalize the notion of receiver functions.

Under the assumption of absence of source caustics (the generation of caustics between the source and scattering points), the RTM inverse scattering transform (cf. (i)) defines a Fourier integral operator the propagation of singularities of which is decscribed by a canonical graph. Thus this transform is amenable to expansions into wave packets or curvelets and partial reconstruction

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as developed in [5]. Moreover, if a set of sources is available, the RTM imaging condition lends itself to a (linearized) reflection tomography formulation in terms of (linearized) transmission tomography. The polarized-wave equation formulation (cf. (ii)) is well-suited for a frequency-domain implementation of the type presented in [38]. The array receiver functions (cf. (iii)) provide a seamless integration of passive source and active source approaches to inverse scattering.

Over the past decades, converted seismic waves have been extensively used in global seismology to identify discontinuities in earth's crust, lithosphere-asthenosphere boundary, and mantle transition zone. The method commonly used has been the one of receiver functions, which were introduced and developed by Vinnik [37] and Langston [21]. In this method, essentially, the converted (scattered) $S$-wave observation is deconvolved (in time) with the corresponding incident $P$-wave observation at each available receiver, and assumes a planarly layered earth model. Various refinements have been developed for arrays of receivers. We mention binning according to common-conversion points (Dueker and Sheehan [7]) and diffraction stacking (Revenaugh [29]); an analysis of (imaging with) receiver functions starting from plane-wave single scattering has been given by Rydberg and Weber [30]. (Plane-wave) Kirchhoff migration for mode-converted waves was considered by Bostock [3] and Poppeliers and Pavlis [28], while its extension to wave-form inversion was developed by Frederiksen and Revenaugh [13]. Receiver functions, however, being bilinear in the data, do not fit a description directly in terms of Kirchhoff migration, being linear in the data. We resolve this issue by making precise under which limiting assumptions receiver function imaging is equivalent with (Kirchhoff-style) RTM via the synthesis of source plane waves.

The outline of the paper is as follows. In the next section, we summarize the construction of the polarized-wave equation, the construction of the elastic-wave parameterix, the WKBJ approximation, and introduce the single scattering approximation. In Section 3, we develop the (anisotropic) elastic-wave RTM inverse scattering transform. In Section 4, we introduce array receiver functions and reformulate this inverse scattering transform by back extrapolating the observed incident field. In Section 5, we discuss how the receiver functions used in global seismology can be recovered from array receiver functions in flat, planarly layered earth models using the WKBJ approximation. In Section 6 we conclude with some final remarks.

**2. Single scattering: Common source acquisition geometry.** The propagation and scattering of seismic waves is governed by the elastic wave equation, which is written in the form

$$P_{il} u_l = f_i, \tag{2.1}$$

where

$$u_l = \sqrt{\rho(x)}(\text{displacement})_l, \qquad f_i = \frac{1}{\sqrt{\rho(x)}}(\text{volume force density})_i, \tag{2.2}$$

and

$$P_{il} = \delta_{il}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x_j}\frac{c_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k} + \text{l.o.t.}\ , \tag{2.3}$$

where l.o.t. stands for lower order terms. Here, $x \in \mathbb{R}^n$ and the subscripts $i, j, k, l \in \{1, \ldots, n\}$. The system of partial differential equations is assumed to be of principal type. It supports different wave types (modes). System (2.1) is real, time reversal invariant, and its solutions satisfy reciprocity.

**2.1. Polarizations.** First, we consider a smoothly varying medium. Decoupling of the modes is then accomplished by diagonalizing the system. We describe how the system (2.1) can be decoupled by transforming it with appropriate matrix-valued pseudodifferential operators, $Q(x, D_x)_{iM}$, $D_x = -\mathrm{i}\frac{\partial}{\partial x}$, see Taylor [35], Ivrii [45] and Dencker [6]. Since the time derivative in $P_{il}$ is already in diagonal form, it remains only to diagonalize its spatial part,

$$A_{il} = -\frac{\partial}{\partial x_j}\frac{c_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k} + \text{l.o.t.}\ .$$

The goal becomes finding $Q_{iM}$ and $A_M$ such that

$$Q(x, D_x)^{-1}_{Mi}\, A_{il}(x, D_x)\, Q(x, D_x)_{lN} = \operatorname{diag}(A_M(x, D_x)\,;\, M = 1, \ldots, n)_{MN}. \tag{2.4}$$

The indices $M, N$ denote the mode of propagation. Then

$$u_M = Q(x, D_x)^{-1}_{Mi} u_i, \quad f_M = Q(x, D_x)^{-1}_{Mi} f_i \tag{2.5}$$

satisfy the uncoupled equations

$$P_M(x, D_x, D_t) u_M = f_M, \tag{2.6}$$

where $D_t = -\mathrm{i}\frac{\partial}{\partial t}$, and in which

$$P_M(x, D_x, D_t) = \frac{\partial^2}{\partial t^2} + A_M(x, D_x).$$

(Pseudodifferential equation (2.6) can be used in implementing elastic-wave inverse scattering in the reverse-time migration (RTM) approach.)

Because of the properties of stiffness related to (i) the conservation of angular momentum, (ii) the properties of the strain-energy function, and (iii) the positivity of strain energy, subject to the adiabatic and isothermal conditions, the principal symbol $A^{\text{prin}}_{il}(x, \xi)$ of $A_{il}(x, D_x)$ is a positive, symmetric matrix. Hence, it can be diagonalized by an orthogonal matrix. On the level of principal symbols, composition of pseudodifferential operators reduces to multiplication. Therefore, we let $Q^{\text{prin}}_{iM}(x, \xi)$ be this orthogonal matrix, and we let $A^{\text{prin}}_M(x, \xi)$ be the eigenvalues of $A^{\text{prin}}_{il}(x, \xi)$, so that

$$Q^{\text{prin}}_{Mi}(x, \xi)^{-1} A^{\text{prin}}_{il}(x, \xi) Q^{\text{prin}}_{lN}(x, \xi) = \operatorname{diag}(A^{\text{prin}}_M(x, \xi))_{MN}. \tag{2.7}$$

The principal symbol $Q^{\text{prin}}_{iM}(x, \xi)$ is the matrix that has as its columns the orthonormalized polarization vectors associated with the modes of propagation. If the $A^{\text{prin}}_M(x, \xi)$ are all different, $A_{il}$ can be diagonalized with a unitary operator, that is, $Q(x, D_x)^{-1} = Q(x, D_x)^*$; see Appendix A.

Remark 2.1. *In the isotropic case, for $n = 3$, the symbol matrix $A^{\text{prin}}_{il}(x, \xi)$ attains the form,*

$$\rho A^{\text{prin}}_{il}(x, \xi) = \begin{pmatrix} (\lambda + \mu)\, \xi_1^2 + \mu\, |\xi|^2 & (\lambda + \mu)\, \xi_1 \xi_2 & (\lambda + \mu)\, \xi_1 \xi_3 \\ (\lambda + \mu)\, \xi_1 \xi_2 & (\lambda + \mu)\, \xi_2^2 + \mu\, |\xi|^2 & (\lambda + \mu)\, \xi_2 \xi_3 \\ (\lambda + \mu)\, \xi_1 \xi_3 & (\lambda + \mu)\, \xi_2 \xi_3 & (\lambda + \mu)\, \xi_3^2 + \mu\, |\xi|^2 \end{pmatrix},$$

*where $\lambda = \lambda(x)$ and $\mu = \mu(x)$ denote the Lamé parameters. We find that*

$$\widetilde{Q}^{\text{prin}} = \widetilde{Q}^{\text{prin}}(\xi) = \begin{pmatrix} | & | & | \\ \widetilde{Q}_{\text{P}} & \widetilde{Q}_{\text{SV}} & \widetilde{Q}_{\text{SH}} \\ | & | & | \end{pmatrix},$$

*which is independent of $x$ and where*

$$\widetilde{Q}_{\text{P}} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \quad \widetilde{Q}_{\text{SH}} = n \times \widetilde{Q}_{\text{P}} = \begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{pmatrix}, \quad \widetilde{Q}_{\text{SV}} = \widetilde{Q}_{\text{P}} \times \widetilde{Q}_{\text{SH}} = \begin{pmatrix} -\xi_1 \xi_3 \\ -\xi_2 \xi_3 \\ \xi_1^2 + \xi_2^2 \end{pmatrix},$$

*with $n = (0, 0, 1)^t$, diagonalizes $A^{\text{prin}}_{il}(x, \xi)$:*

$$\operatorname{diag}(\rho A^{\text{prin}}_M(x, \xi)\,;\, M = 1, \ldots, n) = \begin{pmatrix} (\lambda + 2\mu)\, |\xi|^2 & 0 & 0 \\ 0 & \mu\, |\xi|^2 & 0 \\ 0 & 0 & \mu\, |\xi|^2 \end{pmatrix}.$$

*Upon normalizing the columns of* $\widetilde{Q}^{\rm prin}$*, we obtain the unitary symbol matrix,* $Q^{\rm prin}$*, with*

$$(Q^{\rm prin})^{-1} = (Q^{\rm prin})^* = \begin{pmatrix} \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{\xi_3}{|\xi|} \\ \frac{-\xi_1\xi_3}{(\xi_1^2+\xi_2^2)^{1/2}|\xi|} & \frac{-\xi_2\xi_3}{(\xi_1^2+\xi_2^2)^{1/2}|\xi|} & \frac{(\xi_1^2+\xi_2^2)^{1/2}}{|\xi|} \\ \frac{-\xi_2}{(\xi_1^2+\xi_2^2)^{1/2}} & \frac{\xi_1}{(\xi_1^2+\xi_2^2)^{1/2}} & 0 \end{pmatrix}.$$

*We note that* $\widetilde{Q}_{\rm SV}$ *and* $\widetilde{Q}_{\rm SH}$ *are zero if* $\xi \parallel n$*. This reflects the fact that it is not possible to construct a non-vanishing continuous tangent vector field on* $S^2$*, which is the consequence of the non-vanishing of the Euler characteristic of* $S^2$ *(which is* 2*).*

*With the projections onto* P *and* S*, it follows that*

$$Q_{i1}^{\rm prin}(Q^{\rm prin})^*_{1j} u_j = \left( -\nabla(-\Delta^{-1}(\nabla \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix})) \right)_i$$

*and*

$$[Q_{i2}^{\rm prin}(Q^{\rm prin})^*_{2j} + Q_{i3}^{\rm prin}(Q^{\rm prin})^*_{3j}]\, u_j = \left( \nabla \times (-\Delta^{-1}(\nabla \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix})) \right)_i$$

*in accordance with the Helmholtz decomposition of* $u$*.*

Having assumed that $W_{il}$ is of principal type, the multiplicities of the eigenvalues $A_M^{\rm prin}(x,\xi)$ are constant, whence the principal symbol $Q_{iM}^{\rm prin}(x,\xi)$ depends smoothly on $(x,\xi)$ and microlocally equation (2.7) carries over to an operator equation. Taylor [35] has shown that if this condition is satisfied, then decoupling can be accomplished to all orders.

The second-order equations (2.6) inherit the symmetries of the original system, such as time-reversal invariance and reciprocity. Time-reversal invariance follows because the operators $Q_{iM}(x,D_x), A_M(x,D_x)$ can be chosen in such a way that $Q_{iM}(x,\xi) = -\overline{Q_{iM}(x,-\xi)}$, $A_M(x,\xi) = \overline{A_M(x,\xi)}$. Then $Q_{iM}, A_M$ are real-valued. Reciprocity for the causal Green's function $G_{ij}(x,x_0,t-t_0)$ means that $G_{ij}(x,x_0,t-t_0) = G_{ji}(x_0,x,t-t_0)$. Such a relationship also holds (modulo smoothing operators) for the Green's function $G_M(x,x_0,t-t_0)$ associated with (2.6).

**2.2. The Green's function.** To evaluate the Green's function, we use the first-order system for $u_M$ that is equivalent to (2.6),

$$\frac{\partial}{\partial t}\begin{pmatrix} u_M \\ \frac{\partial u_M}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -A_M(x,D_x) & 0 \end{pmatrix}\begin{pmatrix} u_M \\ \frac{\partial u_M}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 \\ f_M \end{pmatrix}. \tag{2.8}$$

This system can be decoupled also, namely by the matrix-valued pseudodifferential operator

$$\begin{pmatrix} 1 & 1 \\ -\mathrm{i}B_M(x,D_x) & \mathrm{i}B_M(x,D_x) \end{pmatrix},$$

where $B_M(x,D_x) = \sqrt{A_M(x,D_x)}$, which is a pseudodifferential operator of order 1 that exists because $A_M(x,D_x)$ is positive definite. The principal symbol of $B_M(x,D_x)$ is given by $B_M^{\rm prin}(x,\xi) = \sqrt{A_M^{\rm prin}(x,\xi)}$. Then

$$u_{M,\pm} = \tfrac{1}{2}u_M \pm \tfrac{1}{2}\mathrm{i}B_M(x,D_x)^{-1}\frac{\partial u_M}{\partial t}, \quad f_{M,\pm} = \pm\tfrac{1}{2}\mathrm{i}B_M(x,D_x)^{-1} f_M \tag{2.9}$$

satisfy the two first-order ("half wave") equations

$$P_{M,\pm}(x,D_x,D_t)u_{M,\pm} = f_{M,\pm}, \tag{2.10}$$

where

$$P_{M,\pm}(x, D_x, D_t) = \frac{\partial}{\partial t} \pm \mathrm{i}B_M(x, D_x), \quad P_{M,+}P_{M,-} = P_M. \tag{2.11}$$

We construct operators $G_{M,\pm}$ with Lagrangian distribution kernels $G_{M,\pm}(x, x_0, t)$ that solve the initial value problem for (2.10). The operators $G_{M,\pm}$ are Fourier integral operators. Their construction is well known, see for example Duistermaat [8], Chapter 5. Singularities are propagated along the bicharacteristics, that are determined by Hamilton's equations generated by the principal symbol $\omega \pm B_M^{\text{prin}}(x, \xi)$ of (2.10),

$$\begin{aligned} \frac{\partial x}{\partial \lambda} &= \pm \frac{\partial}{\partial \xi} B_M^{\text{prin}}(x, \xi) \quad , \quad & \frac{\partial t}{\partial \lambda} &= 1, \\ \frac{\partial \xi}{\partial \lambda} &= \mp \frac{\partial}{\partial x} B_M^{\text{prin}}(x, \xi) \quad , \quad & \frac{\partial \omega}{\partial \lambda} &= 0. \end{aligned} \tag{2.12}$$

Naturally, the solution may be parameterized by $t$. We denote the solution of (2.12) with the $+$ sign and initial values $(x_0, \xi_0)$ at $t = 0$ by $(x_M(x_0, \xi_0, t), \xi_M(x_0, \xi_0, t))$. The solution with the $-$ sign is found upon reversing the time direction and is given by $(x_M(x_0, \xi_0, -t), \xi_M(x_0, \xi_0, -t))$.

A complete view of the propagation of singularities is provided by the canonical relation of the operator $G_{M,\pm}$, given by

$$C_{M,\pm} = \{(x_M(x_0, \xi_0, \pm t), t, \xi_M(x_0, \xi_0, \pm t), \omega = \mp B_{M,\pm}(x_0, \xi_0); x_0, \xi_0)\}. \tag{2.13}$$

A convenient choice of phase function for the oscillatory integral representation for the kernel of $G_{M,\pm}$ is described in Maslov and Fedoriuk [23]. They state that one can always use a subset of the cotangent vector components as phase variables. Let us assume coordinates for $C_{M,+}$ of the form

$$(x_I, x_0, \xi_J, \omega), \tag{2.14}$$

where $I \cup J$ is a partition of $\{1, \ldots, n\}$. It follows from Theorem 4.21 in Maslov and Fedoriuk [23] that there is a function $S_M(x_I, x_0, \xi_J, \omega)$, such that locally $C_{M,+}$ is given by

$$\begin{aligned} x_J &= -\frac{\partial S_M}{\partial \xi_J} \quad , \quad & t &= -\frac{\partial S_M}{\partial \omega}, \\ \xi_I &= \frac{\partial S_M}{\partial x_I} \quad , \quad & \xi_0 &= -\frac{\partial S_M}{\partial x_0}. \end{aligned} \tag{2.15}$$

Here we take into account the fact that $C_{M,+}$ is a canonical relation, which introduces a minus sign for $\xi_0$. A nondegenerate phase function for $C_{M,+}$ is then found to be

$$\phi_{M,+}(x, x_0, t, \xi_J, \omega) = S_M(x_I, x_0, \xi_J, \omega) + \langle \xi_J, x_J \rangle + \omega t. \tag{2.16}$$

The canonical relation $C_{M,-}$ can be obtained from $C_{M,+}$, viz.

$$C_{M,-} = \{(x, t, -\xi, -\omega; x_0, -\xi_0) \,|\, (x, t, \xi, \omega; x_0, \xi_0) \in C_{M,+}\}.$$

Thus a phase function for $C_{M,-}$ is $\phi_{M,-}(x, x_0, t, \xi_J, \omega) = -\phi_{M,+}(x, x_0, t, -\xi_J, -\omega)$.

A theorem of Hörmander (see also Theorem 5.1.2 of Duistermaat [8]) implies that the operator $G_{M,\pm}$ is microlocally a Fourier integral operator. Hence, we have an expression for its kernel, $G_{M,\pm}(x, x_0, t)$, in the form of an oscillatory integral,

$$\begin{aligned} &G_{M,\pm}(x, x_0, t) \\ &\quad = (2\pi)^{-\frac{|J|+1}{2} - \frac{2n+1}{4}} \int a_{M,\pm}(x_I, x_0, \xi_J, \omega) \exp[\mathrm{i}\phi_{M,\pm}(x, x_0, t, \xi_J, \omega)] \,\mathrm{d}\xi_J \,\mathrm{d}\omega. \end{aligned} \tag{2.17}$$

The amplitude $a_{M,\pm}(x_I, x_0, \xi_J, \omega)$ satisfies a transport equation along the bicharacteristics $(x_M(x_0, \xi_0, \pm t), \xi_M(x_0, \xi_0, \pm t))$.

We may define the canonical relation for $G_M$ as $C_M = C_{M,+} \cup C_{M,-}$ and a phase function

$$\phi_M = \begin{cases} \phi_{M,-} & \text{if } \omega > 0, \\ \phi_{M,+} & \text{if } \omega < 0. \end{cases}$$

It follows from (2.9) that the Green's function for the second-order decoupled equation is given by

$$\begin{aligned}(2.18)\quad & \int G_M(x, x_0, t - t_0) f_M(x_0, t_0)\, \mathrm{d}x_0 \mathrm{d}t_0 \\ &= \tfrac{1}{2}\mathrm{i} \int \left[G_{M,+}(x, x_0, t - t_0) - G_{M,-}(x, x_0, t - t_0)\right] B_M(x_0, D_{x_0})^{-1} f_M(x_0, t_0)\, \mathrm{d}x_0 \mathrm{d}t_0,\end{aligned}$$

where $B_M(x_0, D_{x_0})^{-1}$ is of order $-1$. We have

$$(2.19)\qquad G_M(x, x_0, t) = (2\pi)^{-\frac{|J|+1}{2} - \frac{2n+1}{4}} \int a_M(x_I, x_0, \xi_J, \omega) \exp[\mathrm{i}\phi_M(x, x_0, t, \xi_J, \omega)]\, \mathrm{d}\xi_J\, \mathrm{d}\omega,$$

in which, to highest order,

$$(2.20)\qquad |a_M(x_I, x_0, \xi_J, \omega)| = (2\pi)^{1/4} \left|\det \frac{\partial(x_0, \xi_0, t)}{\partial(x_I, x_0, \xi_J, \omega)}\right|^{1/2} \frac{1}{2|\omega|}.$$

The determinant can be obtained by solving the perturbed Hamilton system following from (2.12).

The amplitude is an element of $M_{C_M} \otimes \Omega^{1/2}(C_M)$, the tensor product of the Keller-Maslov bundle $M_{C_M}$ and the half-densities on the canonical relation $C_M$. The Keller-Maslov bundle gives a factor $\mathrm{i}^k$, where $k$ is an index, which we will absorb in the amplitude.

In case $J = \emptyset$, the generating function $S_M$ reduces to frequency, $\omega$, times the negative of travel time, which we denote by $T_M = T_M(x, x_0)$.

**Flat, smoothly layered media**. Here, we make use of results in [40, 16, 17, 14, 15, 32]. We introduce coordinates $x = (x', z)$ if $x_n = z$ is the (depth) coordinate normal to the surface, and write $c_{jk;il} = (c_{jk})_{il} = c_{ijkl}$. We consider the displacement, $\rho^{-1/2} u_i$, and the traction, $\sum_{k,l=1}^n c_{nk;il} \frac{\partial(\rho^{-1/2} u_l)}{\partial x_k}$, and form

$$(2.21)\qquad W = \begin{pmatrix} \rho^{-1/2} u_i \\ \sum_{k,l=1}^n c_{nk;il} \frac{\partial(\rho^{-1/2} u_l)}{\partial x_k} \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f_i \end{pmatrix}, \ i = 1, \ldots, n.$$

The elastic wave equation (cf. (2.1)-(2.3)) can then be rewritten as the system of equations,

$$(2.22)\qquad \frac{\partial W_a}{\partial z} = \mathrm{i} \sum_{b=1}^{2n} C_{ab}(x', z, D_{x'}, D_t) W_b + F_a,$$

with

$$\begin{aligned}(2.23)\quad & C_{ab}(x', z, D_{x'}, D_t) \\ &= -\mathrm{i} \begin{pmatrix} -\sum_{q=1}^{n-1} \sum_{j=1}^n (c_{nn})^{-1}_{ij} c_{nq;jl} \frac{\partial}{\partial x_q} & (c_{nn})^{-1}_{il} \\ -\sum_{p,q=1}^{n-1} \frac{\partial}{\partial x_p} b_{pq;il} \frac{\partial}{\partial x_q} + \rho \delta_{il} \frac{\partial^2}{\partial t^2} & -\sum_{p=1}^{n-1} \frac{\partial}{\partial x_p} c_{pn;ij} (c_{nn})^{-1}_{jl} \end{pmatrix}_{ab}, \ i, l = 1, \ldots, n,\end{aligned}$$

where $b_{pq;il} = c_{pq;il} - \sum_{j,k=1}^{n} c_{pn;ij}(c_{nn})^{-1}_{jk} c_{nq;kl}$. Diagonalizing the system, microlocally, involves

$$(2.24)\quad C_{ab}(x', z, D_{x'}, D_t) = \sum_{\mu,\nu=1}^{2n} L(x', z, D_{x'}, D_t)_{a\mu} \operatorname{diag}(C_\mu(x', z, D_{x'}, D_t); \mu = 1, \ldots, 2n)_{\mu\nu} L(x', z, D_{x'}, D_t)^{-1}_{\nu b}.$$

The principal parts of the symbols $C_\mu(x', z, \xi', \omega)$ are the solutions for $\zeta$ of

$$A_M^{\mathrm{prin}}(x', z, \xi', \zeta) = \omega^2.$$

In smoothly layered media one can Fourier transform (2.22) with respect to $x'$ and $t$ and obtain a system of ordinary differential equations for $\widetilde{W}(z) = \widetilde{W}(\xi', z, \omega) = \int W(x', z, t) \exp[-\mathrm{i}(\sum_{j=1}^{n-1} \xi_j x_j + \omega t)] \,\mathrm{d}x'\mathrm{d}t$,

$$(2.25)\qquad \frac{\partial \widetilde{W}_a}{\partial z} = \mathrm{i} \sum_{b=1}^{2n} C_{ab}(z, \xi', \omega) \widetilde{W}_b + \widetilde{F}_a.$$

We choose the $C_\mu$ such that the homogeneity property, $C_\mu(z, \xi', \omega) = \omega C_\mu(z, \omega^{-1}\xi', 1)$, extends to $\omega < 0$. We have

$$(2.26)\qquad L_{a\mu}(z, \xi', \omega) = \begin{pmatrix} Q_{iM(\mu)}(z, (\xi', C_\mu(z, \xi', \omega))) \\ \sum_{k,l=1}^{n} c_{nk;il}(-\mathrm{i})(\xi', C_\mu(z, \xi', \omega))_k Q_{lM(\mu)}(z, (\xi', C_\mu(z, \xi', \omega))) \end{pmatrix}_{a\mu},$$

with inverse

$$(2.27)\qquad L^{-1}(z, \xi', \omega) = N(z, \xi', \omega) L^t(z, \xi', \omega)\, J, \quad \text{where } J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Here, $N(z, \xi', \omega)$ is a diagonal normalization matrix, $\operatorname{diag}(N_\mu(z, \xi', \omega))_{\mu\nu}$. It follows that

$$(2.28)\quad N_\mu(z, \xi', \omega)^{-1} = \sum_{i=1}^{n} Q_{iM(\mu)}(z, (\xi', C_\mu(z, \xi', \omega))) \sum_{k,l=1}^{n} [c_{nk;il} + c_{nk;li}]\,(-\mathrm{i})(\xi', C_\mu(z, \xi', \omega))_k Q_{lM(\mu)}(z, (\xi', C_\mu(z, \xi', \omega))).$$

The index mapping $\mu \to M(\mu)$ assigns the appropriate mode to the depth component of the wave vector.

We cast (2.25) into an equivalent initial value problem. Let $\widetilde{W}_{ab}(z, z_0)$ be the solution to

$$\frac{\partial \widetilde{W}_a}{\partial z} = \mathrm{i} \sum_{b=1}^{2n} C_{ab}(z, \xi', \omega) \widetilde{W}_b, \quad \widetilde{W}(z_0) = I_{2n}.$$

Then $\widetilde{W}_a(z) = \int_{z_0}^{z} \sum_{b=1}^{2n} \widetilde{W}_{ab}(z, z_0) \widetilde{F}_b(z_0)\,\mathrm{d}z_0$ solves (2.25). We introduce

$$(2.29)\qquad \dot{W} = \begin{pmatrix} \rho^{-1/2} u_i \\ \sum_{k,l=1}^{n} c_{nk;il} D_t^{-1} \frac{\partial(\rho^{-1/2} u_l)}{\partial x_k} \end{pmatrix}, \quad \dot{F} = \begin{pmatrix} 0 \\ D_t^{-1} f_i \end{pmatrix};$$

with $\xi' = \omega p'$, we identify $\widetilde{\dot{W}}(p', z, \omega) = \widetilde{W}(\omega p', z, \omega)$ whereas

$$\frac{\partial \widetilde{\dot{W}}_a}{\partial z} = \mathrm{i}\omega \sum_{b=1}^{2n} C_{ab}(z, p', 1) \widetilde{\dot{W}}_b + \widetilde{\dot{F}}_b.$$

In the WKBJ approximation, in the absence of turning rays (the characteristics are nowhere horizontal), we have

$$
\begin{aligned}
(2.30)\quad \widetilde{\dot{W}}_{ab}(z,z_0) &\approx \sum_{\mu=1}^{2n} L_{a\mu}(z,p',1)Y_\mu(z,p',1) \\
&\qquad \exp\left[\mathrm{i}\omega\int_{z_0}^{z} C_\mu(\bar{z},p',1)\,\mathrm{d}\bar{z}\right] Y_\mu(z_0,p',1)^{-1}L^{-1}_{\mu b}(z_0,p',1) \\
&= \sum_{\mu=1}^{2n} L_{a\mu}(z,p',1)Y_\mu(z,p',1)\exp\left[\mathrm{i}\omega\int_{z_0}^{z} C_\mu(\bar{z},p',1)\,\mathrm{d}\bar{z}\right] Y_\mu(z_0,p',1)(L^t(z_0,p',1)J)_{\mu b}.
\end{aligned}
$$

Here, $Y_\mu(z,p',1) = [N_\mu(z,p',1)]^{1/2}$. We identify the "vertical" travel time

$$
\tau_\mu(z,z_0,p') = -\int_{z_0}^{z} C_\mu(\bar{z},p',1)\,\mathrm{d}\bar{z}.
$$

To obtain the tensor $G_{ij}$, we substitute a $\delta$ source for $f_i$, yielding $J\widetilde{F} = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \delta(.-z_0)$:

$$
\begin{aligned}
(2.31)\quad G_{ij}(x',z,x_0',z_0,t-t_0) &\approx \sum_{\mu=1}^{2n}{}' \frac{1}{(2\pi)^n}\iint Q_{iM(\mu)}(z,(p',C_\mu(z,p',1)))Y_\mu(z,p',1) \\
&\qquad \exp\left[\mathrm{i}\omega\left(-\tau_\mu(z,z_0,p') + \sum_{l=1}^{n-1} p_l'(x'-x_0')_l + t - t_0\right)\right] \\
&\qquad\qquad Y_\mu(z_0,p',1)Q^t_{M(\mu)j}(z_0,(p',C_\mu(z,p',1)))\,\mathrm{d}p'|\omega|^{n-1}\mathrm{d}\omega;
\end{aligned}
$$

which values of $\mu$ contribute depends on whether $z > z_0$ ("downgoing") or $z < z_0$ ("upgoing"). The (negative) values of the components of $p'$ associated with the ray connecting $(z_0, x_0')$ with $(z, x')$ is the solution of the equation

$$
\partial_{p'}\tau_\mu(z,z_0,p') = x' - x_0'.
$$

**2.3. The single scattering approximation.** In the contrast formulation the total value of the medium parameters $\rho, c_{ijkl}$ is written as the sum of a smooth background constituent $\rho(x), c_{ijkl}(x)$ and a singular perturbation $\delta\rho(x), \delta c_{ijkl}(x)$, viz. $\rho + \delta\rho$, $c_{ijkl} + \delta c_{ijkl}$; we assume that $\delta\rho, \delta c_{ijkl} \in \mathcal{E}'(X)$ with $X$ a compact subset of $\mathbb{R}^n$. This decomposition induces a perturbation of $P_{il}$ (cf. (2.3)),

$$
\delta P_{il} = \delta_{il}\frac{\delta\rho(x)}{\rho(x)}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x_j}\frac{\delta c_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k}.
$$

We denote the causal solution operator associated with (2.1) by $G_{il}$ and its distribution kernel by $G_{il}(x, x_0, t-t_0)$. The first-order perturbation, $\delta G_{il}$, of $G_{il}$ admits the representation

$$
(2.32)\qquad \delta G_{jk}(\hat{x},\tilde{x},t) = -\int_0^t\int_X G_{ji}(\hat{x},x_0,t-t_0)\,\delta P_{il}(x_0,D_{x_0},D_{t_0})\,G_{lk}(x_0,\tilde{x},t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0,
$$

which is the Born approximation. Here, $\tilde{x}$ denotes a source location, $\hat{x}$ a receiver location, and $x_0$ a scattering point location. We restrict our time window (of observation) to $(0,T)$ for some $0 < T < \infty$. Because the background model is smooth the operator $\delta G_{il}$ contains only the single scattered field.

We introduce the $MN$ contribution, $\delta G_{MN}$, to $\delta G_{jk}$ as follows,

$$\begin{aligned}(2.33)\quad & \int \delta G_{jk}(\hat{x},\tilde{x},t-\tilde{t}) f_k(\tilde{x},\tilde{t})\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{t} \\ &= Q(\hat{x},D_{\hat{x}})_{jM} \int \delta G_{MN}(\hat{x},\tilde{x},t-\tilde{t})\,(Q(\tilde{x},D_{\tilde{x}})^{-1}_{Nk} f_k)(\tilde{x},\tilde{t})\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{t}.\end{aligned}$$

We apply reciprocity in $(\hat{x},x_0)$ to the integrand of the right-hand side and obtain

$$\begin{aligned}(2.34)\quad \delta G_{MN}(\hat{x},\tilde{x},t) &= -\int_0^t \int_X (Q(x_0,D_{x_0})^{-1})^*_{iM} G_M(x_0,\hat{x},t-t_0) \\ &\qquad \delta P_{il}(x_0,D_{x_0},D_{t_0})\,Q(x_0,D_{x_0})_{lN} G_N(x_0,\tilde{x},t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0 \\ &= -\int_0^t \int_X (Q(x_0,D_{x_0})^{-1})^*_{iM} G_M(x_0,\hat{x},t-t_0) \frac{\partial}{\partial(t_0,x_{0,j})} \left(\delta_{il}\frac{\delta\rho(x_0)}{\rho(x_0)}, -\frac{\delta c_{ijkl}(x_0)}{\rho(x_0)}\right) \\ &\qquad \frac{\partial}{\partial(t_0,x_{0,k})}\,Q(x_0,D_{x_0})_{lN} G_N(x_0,\tilde{x},t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0 \\ &= \int_0^t \int_X \frac{\partial}{\partial(t_0,x_{0,j})}\,(Q(x_0,D_{x_0})^{-1})^*_{iM} G_M(x_0,\hat{x},t-t_0) \left(\delta_{il}\frac{\delta\rho(x_0)}{\rho(x_0)}, -\frac{\delta c_{ijkl}(x_0)}{\rho(x_0)}\right) \\ &\qquad \frac{\partial}{\partial(t_0,x_{0,k})}\,Q(x_0,D_{x_0})_{lN} G_N(x_0,\tilde{x},t_0)\,\mathrm{d}x_0\,\mathrm{d}t_0 \\ &= \int_X \left\{ \int_0^t \frac{\partial}{\partial(t_0,x_{0,j})}\,Q(x_0,D_{x_0})_{iM} G_M(x_0,\hat{x},t-t_0) \right. \\ &\qquad \left. \frac{\partial}{\partial(t_0,x_{0,k})}\,Q(x_0,D_{x_0})_{lN} G_N(x_0,\tilde{x},t_0)\,\mathrm{d}t_0 \right\} \\ &\qquad\qquad \left(\delta_{il}\frac{\delta\rho(x_0)}{\rho(x_0)}, -\frac{\delta c_{ijkl}(x_0)}{\rho(x_0)}\right)\,\mathrm{d}x_0\end{aligned}$$

upon integration by parts; see Appendix A. Reciprocity implies that $G_M(x_0,\hat{x},t-t_0) = G_M(\hat{x},x_0,t-t_0)$ while $G_N(x_0,\tilde{x},t_0) = G_N(\tilde{x},x_0,t_0)$.

Microlocally, we can substitute (2.19) for $G_M$, with appropriate substitutions for its arguments: $(\hat{x},\hat{\xi}_{\hat{J}})$ for $(x,\xi_J)$ while $\hat{I}\cup\hat{J} = \{1,\ldots,n\}$; for $G_N$ we have the substitutions $(\tilde{x},\tilde{\xi}_{\tilde{J}})$ for $(x,\xi_J)$ while $\tilde{I}\cup\tilde{J} = \{1,\ldots,n\}$. It follows that the composition

$$\frac{\partial}{\partial(t_0,x_{0,j})}\,(Q(x_0,D_{x_0})^{-1})^*_{iM} G_M$$

is a Fourier integral operator with the same phase as $G_M$, and amplitude that to highest order equals the product

$$a_M(\hat{x}_{\hat{I}},x_0,\hat{\xi}_{\hat{J}},\omega) Q(x_0,\hat{\xi}_0)_{iM}\,\mathrm{i}\,(\omega,\hat{\xi}_{0,j}),$$

subject to the substitution $\hat{\xi}_0 = \xi_0(\hat{x}_{\hat{I}},x_0,\hat{\xi}_{\hat{J}},\omega)$ (cf. (2.15)).

We write (2.34) in the form of an oscillatory integral. To this end, we introduce the radiation patterns $(w_{MN;0}, w_{MN;ijkl})$ as

$$w_{MN;0}(\hat{x}_{\hat{I}},\tilde{x}_{\tilde{I}},x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega) = -Q_{iM}(x_0,\hat{\xi}_0) Q_{iN}(x_0,\tilde{\xi}_0)\,\omega^2, \tag{2.35}$$

$$w_{MN;ijkl}(\hat{x}_{\hat{I}},\tilde{x}_{\tilde{I}},x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega) = Q_{iM}(x_0,\hat{\xi}_0) Q_{lN}(x_0,\tilde{\xi}_0)\,\hat{\xi}_{0,j}\tilde{\xi}_{0,k}, \tag{2.36}$$

subject to the substitutions $\hat{\xi}_0 = \xi_0(\hat{x}_{\hat{I}},x_0,\hat{\xi}_{\hat{J}},\omega)$, $\tilde{\xi}_0 = \xi_0(\tilde{x}_{\tilde{I}},x_0,\tilde{\xi}_{\tilde{J}},\omega)$ (cf. (2.15)). We introduce the amplitude,

$$b_{MN}(\hat{x}_{\hat{I}},\tilde{x}_{\tilde{I}},x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega) = (2\pi)^{-\frac{n-1}{4}}\,a_M(\hat{x}_{\hat{I}},x_0,\hat{\xi}_{\hat{J}},\omega)\,a_N(\tilde{x}_{\tilde{I}},x_0,\tilde{\xi}_{\tilde{J}},\omega) \tag{2.37}$$

and the phase function

$$\Phi_{MN}(\hat{x},\tilde{x},t,x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega) = \phi_M(\hat{x},x_0,t,\hat{\xi}_{\hat{J}},\omega) + \phi_N(\tilde{x},x_0,t,\tilde{\xi}_{\tilde{J}},\omega) - \omega t. \tag{2.38}$$

Then

$$\delta G_{MN}(\hat{x},\tilde{x},t) = (2\pi)^{-\frac{|\hat{J}|+|\tilde{J}|+1}{2}-\frac{3n+1}{4}} \int\int_X b_{MN}(\hat{x}_{\hat{I}},\tilde{x}_{\tilde{I}},x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega) \\ \left(w_{MN;0}(\hat{x}_{\hat{I}},\tilde{x}_{\tilde{I}},x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega)\frac{\delta\rho(x_0)}{\rho(x_0)} + w_{MN;ijkl}(\hat{x}_{\hat{I}},\tilde{x}_{\tilde{I}},x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega)\frac{\delta c_{ijkl}(x_0)}{\rho(x_0)}\right) \\ \exp[\mathrm{i}\,\Phi_{MN}(\hat{x},\tilde{x},t,x_0,\hat{\xi}_{\hat{J}},\tilde{\xi}_{\tilde{J}},\omega)]\,\mathrm{d}x_0\,\mathrm{d}\hat{\xi}_{\hat{J}}\mathrm{d}\tilde{\xi}_{\tilde{J}}\mathrm{d}\omega \tag{2.39}$$

modulo lower-order terms in amplitude.

Data are measurements of the scattered wave field which we relate here to the Green's function perturbation in (2.34): They are assumed to be representable by $\delta G_{MN}(\hat{x},\tilde{x},t)$ for $(\hat{x},\tilde{x},t)$ in some acquisition manifold, which contains the receiver points and time. Let

$$\hat{y} \mapsto \hat{x}(\hat{y}),\ \tilde{y} \mapsto \tilde{x}(\tilde{y})$$

be coordinate transformations, such that $y = (\hat{y},\tilde{y},t)$, $y' = (\hat{y}',t)$, $y'' = (\hat{y}'',\tilde{y})$ with $\hat{y} = (\hat{y}',\hat{y}'')$, and the acquisition manifold is given by $Y = \Sigma_s \times \{0\} \times (0,T)$, that is, $\tilde{y} = 0$ while $\Sigma_s \subset \{\hat{y}'' = 0\}$. This is the case in which the source is fixed; the dimension of $\hat{y}''$ is 1, whence the codimension of this acquisition manifold is $c = n+1$. We sometimes write $\tilde{x}(0) = s$ and $\hat{y}' = r$.

REMARK 2.2. *The operator $Q(\hat{x},D_{\hat{x}})_{jM}$ in (2.33) can be removed from the data. In the further analysis, we will consider the case of a (locally) flat acquisition hypersurface, and write $\hat{x} = (\hat{x}',\hat{z})$ or $\hat{y}' = r = \hat{x}'$ and $\hat{y}'' = \hat{z}$. Assuming upgoing waves at the receivers, we can replace $Q(\hat{x},D_{\hat{x}})_{jM}$ by an operator, $\bar{Q}(\hat{x}',\hat{z},D_{\hat{x}'},D_t)_{jM(\mu)}$, with principal symbol $Q_{jM(\mu)}(\hat{x}',\hat{z},(\hat{\xi}',C_\mu(\hat{x}',\hat{z},\hat{\xi}',\omega)))$. The action of this operator can be restricted naturally to $\hat{z} = 0$. It also has a parametrix, whence, microlocally, the polarizations in the data can be separated.*

In this framework, with $f_k = e_k\delta_{\tilde{x}}\delta_0$, the scattered-field data, $d_{MN}(r,t;s)$, are modeled by the operator,

$$F_{MN} : \left(\delta_{il}\frac{\delta\rho(x)}{\rho(x)}, \frac{\delta c_{ijkl}(x)}{\rho(x)}\right) \to \\ \left.\int \delta G_{MN}(\hat{x},\tilde{x}',t)\,\mathcal{Q}^{-1}_{Nk'}(\tilde{x}',\tilde{x})e_{k'}\,\mathrm{d}\tilde{x}'\right|_{\hat{x}=\hat{x}(r,0),\tilde{x}=s} \\ \text{(no sum over } N); \tag{2.40}$$

here $\mathcal{Q}^{-1}$ denotes the kernel of $Q^{-1}$. *Incident* field data, $d_N(r,t;s)$, are modeled by

$$\left.\int G_N(\hat{x},\tilde{x}',t)\,\mathcal{Q}^{-1}_{Nk'}(\tilde{x}',\tilde{x})e_{k'}\,\mathrm{d}\tilde{x}'\right|_{\hat{x}=\hat{x}(r,0),\tilde{x}=s} \qquad \text{(no sum over } N).$$

To simplify the analysis, we will consider a polarized source, $f_k = \mathcal{Q}_{kN}(.,s)\delta_0$, leading to the introduction of $\bar{F}_{MN} : (\delta_{il}\frac{\delta\rho(x)}{\rho(x)}, \frac{\delta c_{ijkl}(x)}{\rho(x)}) \to \delta G_{MN}(\hat{x},\tilde{x},t)|_{\hat{x}=\hat{x}(r,0),\tilde{x}=s}$, modelling data $\bar{d}_{MN}(r,t;s)$, and the incident field data $\bar{d}_N(r,t;s)$.

We investigate the propagation of singularities by the mapping in (2.40). Let $\omega = \mp B_M(x_0,\hat{\xi}_0)$, and

$$\hat{x} = x_M(x_0,\hat{\xi}_0,\pm\hat{t}),\ \tilde{x} = x_N(x_0,\tilde{\xi}_0,\pm\tilde{t}),\ t = \hat{t}+\tilde{t}, \\ \hat{\xi} = \xi_M(x_0,\hat{\xi}_0,\pm\hat{t}),\ \tilde{\xi} = \xi_N(x_0,\tilde{\xi}_0,\pm\tilde{t})$$

denote the bicharacteristics in modes $N$ and $M$, originating at $x_0$. We then obtain $(y(x_0,\hat{\xi}_0,\tilde{\xi}_0,\hat{t},\tilde{t}),\eta(x_0,\hat{\xi}_0,\tilde{\xi}_0,\hat{t},\tilde{t}))$ by transforming $(\hat{x},\tilde{x},\hat{t}+\tilde{t},\hat{\xi},\tilde{\xi},\omega)$ to $(y,\eta)$ coordinates. We invoke the following assumptions

Assumption 1. *There are no elements $(y',0,\eta',\eta'')$ with $(y',\eta')\in T^*Y\backslash 0$ such that, for some $\hat{\xi}_0$*

$$y' = y'(x_0,\hat{\xi}_0,-\hat{\xi}_0,\hat{t},\tilde{t}),\ \eta' = \eta'(x_0,\hat{\xi}_0,-\hat{\xi}_0,\hat{t},\tilde{t}),\ \textit{and}\ \hat{y}''(x_0,\hat{\xi}_0,\hat{t}) = 0,\ \tilde{y}(x_0,-\hat{\xi}_0,\tilde{t}) = 0.$$

This assumption is related to the condition under which the pull back of a distribution is again a distribution (Gel'fand and Shilov, 1958). If $N\neq M$ this assumption is generically satisfied; if $N=M$, this excludes scattering over $\pi$. In the further analysis we will focus on the conversion where $N$ corresponds with $qP$ and $M$ corresponds with $qSV$, in particular with a view to developing array receiver functions.

Assumption 2. *The matrix*

$$\frac{\partial \hat{y}''}{\partial(x_0,\hat{\xi}_0,\hat{t})}\ \textit{has rank}\ 1. \tag{2.41}$$

This assumption essentially states the absence of grazing receiver rays. With Assumptions 1 and 2 it follows that $F_{MN}$ in (2.40) defines a Fourier integral operator. The propagation of singularities by (2.40) is governed by the canonical relation

$$\begin{aligned}\Lambda^F_{MN} = \{&(\hat{y}'(x_0,\hat{\xi}_0,\hat{t}),\hat{t}+\tilde{t},\hat{\eta}'(x_0,\hat{\xi}_0,\hat{t}),\omega;x_0,\hat{\xi}_0+\tilde{\xi}_0)\mid\\ &B_M(x_0,\hat{\xi}_0)=B_N(x_0,\tilde{\xi}_0)=\mp\omega,\tilde{y}(x_0,\tilde{\xi}_0,\tilde{t})=0,\hat{y}''(x_0,\hat{\xi}_0,\hat{t})=0\}\\ &\subset T^*Y\backslash 0\times T^*X\backslash 0.\end{aligned} \tag{2.42}$$

**3. Imaging converted waves: Common source.** An image is obtained by appling the adjoint, $F^*_{MN}$, to the data, $d_{MN}$:

$$\begin{aligned}I^{MN}_{ijkl}(x_0;s) &= (F^*_{MN}d_{MN})_{ijkl}(x_0)\\ &= \int\int\int_{\Sigma_s}\frac{\partial}{\partial x_{0,j}}Q(x_0,D_{x_0})_{iM}G_M(x_0,\hat{x}(r,0),t-t_0)\,d_{MN}(r,t;s)\mathrm{d}r\mathrm{d}t\\ &\qquad \frac{\partial}{\partial x_{0,k}}Q(x_0,D_{x_0})_{lN}\int G_N(x_0,\tilde{x}',t_0)\,\mathcal{Q}^{-1}_{Nk'}(\tilde{x}',s)e_{k'}\,\mathrm{d}\tilde{x}'\,\mathrm{d}t_0.\end{aligned} \tag{3.1}$$

and similarly for the density contrast upon replacing $\frac{\partial}{\partial x_{0,j}}$ and $\frac{\partial}{\partial x_{0,k}}$ by $\frac{\partial}{\partial t_0}$.

Assumption 3. *(Bolker condition) No caustics form between the source and scattering points in mode $N$.*

Let $S_M(\hat{y}'_I,x_0,\hat{\eta}'_J,\omega)$ denote the generating function associated with the canonical relation of the parametrix in mode $M$ (cf. (2.19)) subjected to the restriction to the surface $\hat{y}''=0$. We write $\hat{x_0}=\hat{x_0}(x_0,\hat{\xi}_0)=x_M(x_0,\hat{\xi}_0,\hat{t}_0)$ where $t_0$ is determined by $\hat{y}''(x_0,\hat{\xi}_0,\hat{t}_0)=0$.

THEOREM 3.1. *The common-source normal operator, $F^*_{MN}F_{MN}$, acting on density and stiffness tensors, is pseudodifferential of order $n-1$. Its principal symbol is given by*

$$(2\pi)^{-n-\frac{1}{2}} \left|\det \frac{\partial(\hat{x})}{\partial(\hat{y})}\right|^{-1}_{\hat{y}''=0} \frac{1}{4\omega^2}\, a_N(s,x_0,\omega)^2 \left| \frac{\mathrm{d}\hat{y}''}{\mathrm{d}\hat{t}} \right|^{-1}$$
$$\left[1 - \langle \frac{\partial T_N(x_0,s)}{\partial x_0}, \frac{\partial B^{\mathrm{prin}}_M}{\partial \xi} \Big|_{x_0,\xi=\xi_0+\omega\frac{\partial T_N(x_0,s)}{\partial x_0}} \rangle\right]^{-1},$$
$$\omega = \omega(x_0,\xi_0,s), \qquad B^{\mathrm{prin}}_M\left(x_0, \omega(x_0,\xi_0,s)^{-1}\xi_0 + \frac{\partial T_N(x_0,s)}{\partial x_0}\right) = 1,$$

*up to radiation pattern factors.*

*Proof.* We evaluate the symbol of the normal operator associated with $\bar{F}_{MN}$. We use (2.39); in the absence of "source-ray" caustics, we have $\tilde{J}=\emptyset$. We find that (cf. (2.20))

$$(3.2)\quad |b_{MN}(\hat{y}'_I,s,x_0,\hat{\eta}'_J,\omega)| = (2\pi)^{-\frac{n+1}{2}} \left|\det \frac{\partial(\hat{x})}{\partial(\hat{y})}\right|^{-1/2}_{\hat{y}''=0}$$
$$\left|\det \frac{\partial(\hat{y}'_I,\hat{y}'',\hat{\eta}'_J,\omega)}{\partial(\hat{x}_{\hat{I}},\hat{\xi}_{\hat{J}},\omega)}\right|^{-1/2}_{\hat{y}''=0} (2\pi)^{1/4} \left|\det \frac{\partial(x_0,\hat{\xi}_0,\hat{t})}{\partial(\hat{x}_{\hat{I}},x_0,\hat{\xi}_{\hat{J}},\omega)}\right|^{1/2} \frac{1}{2|\omega|}\, a_N(s,x_0,\omega)$$

($a_N(s,x_0,\omega)$ contains a factor $(2\pi)^{1/4}$), while $\Phi_{MN} = \phi_M + \phi_N - \omega t$ with

$$(3.3)\quad \phi_N(s,x_0,t,\omega) = -\omega(T_N(x_0,s)-t),$$
$$\phi_M(\hat{y}',x_0,t,\hat{\eta}'_J,\omega) = S_M(\hat{y}'_I,x_0,\hat{\eta}'_J,\omega) + \langle\hat{\eta}'_J,\hat{y}'_J\rangle + \omega t$$

($I\cup J = \{1,\dots,n-1\}$). We can express $(\hat{\xi}_0,\tilde{\xi}_0)$ as functions of $(\hat{y}'_I,s,x_0,\hat{\eta}'_J,\omega)$:

$$\tilde{\xi}_0 = \omega\frac{\partial T_N(x_0,s)}{\partial x_0}, \quad \hat{\xi}_0 = -\frac{\partial S_M(\hat{y}'_I,x_0,\hat{\eta}'_J,\omega)}{\partial x_0}.$$

Subject to such a substitution, we transform the radiation patterns using (2.35)-(2.36),

$$w_{MN;0}(\hat{y}'_I,s,x_0,\hat{\eta}'_J,\omega) = -Q_{iM}(x_0,\hat{\xi}_0)Q_{iN}(x_0,\tilde{\xi}_0)\,\omega^2,$$
$$w_{MN;ijkl}(\hat{y}'_I,s,x_0,\hat{\eta}'_J,\omega) = Q_{iM}(x_0,\hat{\xi}_0)Q_{lN}(x_0,\tilde{\xi}_0)\,\hat{\xi}_{0,j}\tilde{\xi}_{0,k}.$$

We expand the phase of the oscillatory integral representation of $\bar{F}^*_{MN}\bar{F}_{MN}$, and carry out the integration over $\hat{y}_J$. We obtain an oscillatory integral with phase function,

$$\langle\omega\left[\frac{1}{\omega}\frac{\partial S_M(\hat{y}'_I,x_0,\hat{\eta}'_J,\omega)}{\partial x_0} - \frac{\partial T_N(x_0,s)}{\partial x_0}\right],(\bar{x}_0-x_0)\rangle + \text{l.o.t.},$$

and integration variables $(\hat{y}'_I,\hat{\eta}'_J,\omega)$; here, we exploit the fact that this operator is pseudodifferential.

The oscillatory integral representation of the normal operator (up to the principal symbol) contains the factor and measure,

$$(3.4)\quad \left|\det \frac{\partial(x_0,\hat{\xi}_0,\hat{t})}{\partial(\hat{x}_{\hat{I}},x_0,\hat{\xi}_{\hat{J}},\omega)}\right| \left|\det \frac{\partial(\hat{y}'_I,\hat{y}'',\hat{\eta}'_J,\omega)}{\partial(\hat{x}_{\hat{I}},\hat{\xi}_{\hat{J}},\omega)}\right|^{-1}_{\hat{y}''=0} \mathrm{d}\hat{y}'_I\mathrm{d}\hat{\eta}'_J\mathrm{d}\omega$$
$$= \left|\frac{\mathrm{d}\hat{y}''}{\mathrm{d}\hat{t}}\right|^{-1} \left|\frac{\partial B^{\mathrm{prin}}_M}{\partial\xi}\right|^{-1}_{x_0,\xi=\hat{\xi}_0} \mathrm{d}\Sigma_{M,x_0}(\hat{\xi}_0)\mathrm{d}\omega,$$

where the integration in $\hat{\xi}_0$ is over (part of) the surface $\Sigma_{M,x_0}$ : $B_M^{\text{prin}}(x_0, \hat{\xi}_0) = \omega$. We have $\mathrm{d}\Sigma_{M,x_0}(\hat{\xi}_0)\mathrm{d}\omega = |\omega|^{n-1}\mathrm{d}\Sigma_{M,x_0}(\hat{p}_0)\mathrm{d}\omega$ if $\hat{\xi}_0 = \omega\hat{p}_0$; $\hat{p}_0 = \frac{1}{\omega}\frac{\partial S_M(\hat{y}'_I, x_0, \hat{\eta}'_J, \omega)}{\partial x_0}$. Furthermore,

$$\left| \frac{\mathrm{d}\hat{y}''}{\mathrm{d}\hat{t}} \right| = \left| \left\langle \frac{\partial \hat{y}''}{\partial \hat{x}}, \frac{\partial B_M^{\text{prin}}}{\partial \xi} \right|_{\hat{x}(\hat{y}),(\frac{\partial(\hat{y})}{\partial(\hat{x})})^t\hat{\eta}} \right\rangle \right|.$$

We then consider a change of variables of integration, $(\omega, \hat{p}_0) \to \xi_0$, with

$$\xi_0 = \omega \left[ \hat{p}_0 - \frac{\partial T_N(x_0, s)}{\partial x_0} \right]$$

for given $x_0$; see Fig. 1. Let $\nu(x_0, .)$ denote the normal to $\Sigma_{M,x_0}$, that is,

$$\nu(x_0, \hat{p}_0) = \left| \frac{\partial B_M^{\text{prin}}}{\partial \xi} \right|^{-1} \frac{\partial B_M^{\text{prin}}}{\partial \xi} \Bigg|_{x_0, \xi = \hat{p}_0}, \quad \hat{p}_0 \in \Sigma_{M,x_0}.$$

Then

$$\mathrm{d}\xi_0 = \langle \hat{p}_0 - \frac{\partial T_N(x_0, s)}{\partial x_0}, \nu \rangle \, |\omega|^{n-1} \mathrm{d}\omega \mathrm{d}\Sigma_{M,x_0}(\hat{p}_0).$$

We note that

$$\langle \hat{p}_0, \nu \rangle = \left| \frac{\partial B_M^{\text{prin}}}{\partial \xi} \right|^{-1}_{x_0, \xi = \hat{p}_0}, \quad \hat{p}_0 \in \Sigma_{M,x_0}.$$

Hence,

$$\mathrm{d}\xi_0 = \left| \frac{\partial B_M^{\text{prin}}}{\partial \xi} \right|^{-1}_{x_0, \xi = \hat{p}_0} \left[ 1 - \langle \frac{\partial T_N(x_0, s)}{\partial x_0}, \frac{\partial B_M^{\text{prin}}}{\partial \xi} \Big|_{x_0, \xi = \hat{p}_0} \rangle \right] |\omega|^{n-1} \mathrm{d}\omega \mathrm{d}\Sigma_{M,x_0}(\hat{p}_0). \tag{3.5}$$

Combining (3.4) with (3.5) yields the final Jacobian. The principal symbol of the normal operator follows from multiplying this Jacobian with the square of the amplitude in (3.2) and the dyadic product of (3.4) with itself. □

We can compensate for the factor

$$\left| \frac{\mathrm{d}\hat{y}''}{\mathrm{d}\hat{t}} \right|^{-1} \left[ 1 - \langle \frac{\partial T_N(x_0, s)}{\partial x_0}, \frac{\partial B_M^{\text{prin}}}{\partial \xi} \Big|_{x_0, \xi = \hat{p}_0} \rangle \right]^{-1},$$

upon replacing $\bar{F}^*_{MN}$ by an imaging operator, $\bar{H}_{MN}$, say, making use of $\omega\hat{p}_0 = \frac{\partial S_M}{\partial x_0}$, the homogeneity of $B_M^{\text{prin}}$, and Egorov's theorem. We define pseudodifferential operators $\Xi$ and $\Theta$ with principal symbols

$$\Xi_0(x_0, \xi_0, \omega) = \omega, \quad \Xi_j(x_0, \xi_0, \omega) = \xi_{0j}$$

and

$$\Theta_0(x_0, \xi_0, \omega) = \omega, \quad \Theta_j(x_0, \xi_0, \omega) = \frac{\partial B_M^{\text{prin}}}{\partial \xi_j}(x_0, \xi_0)\,\omega.$$

Let $\mathcal{S}(\omega) = \frac{1}{\omega^2}$ and

$$\Psi(\hat{y}', 0, \hat{\eta}', \omega) = \left| \left\langle \frac{\partial \hat{y}''}{\partial \hat{x}}, \frac{\partial B_M^{\text{prin}}}{\partial \xi} \right|_{\hat{x}(\hat{y}',0),(\frac{\partial(\hat{y})}{\partial(\hat{x})})^t(\hat{\eta}', C_\mu(\hat{y}',0,\hat{\eta}',\omega))} \right\rangle \right|,$$

then

$$(\bar{H}_{MN} d_{MN})_{ijkl}(x_0) = \int \mathcal{S}(D_{t_0}) \sum_{p=0}^{n} \left[ \Theta_p(x_0, D_{x_0}, D_t) \frac{\partial}{\partial x_{0,j}} Q(x_0, D_{x_0})_{iM} \int\int_{\Sigma_s} (\Psi(r, 0, D_r, D_t) G_M)(x_0, \hat{x}(r, 0), t - t_0)\, d_{MN}(r, t; s)\, \mathrm{d}r \mathrm{d}t \right] \Xi_p(x_0, D_{x_0}, D_t) \frac{\partial}{\partial x_{0,k}} Q(x_0, D_{x_0})_{lN} G_N(x_0, s, t_0)\, \mathrm{d}t_0$$

and similarly for the density contrast upon replacing $\frac{\partial}{\partial x_{0,j}}$ and $\frac{\partial}{\partial x_{0,k}}$ by $\frac{\partial}{\partial t_0}$.

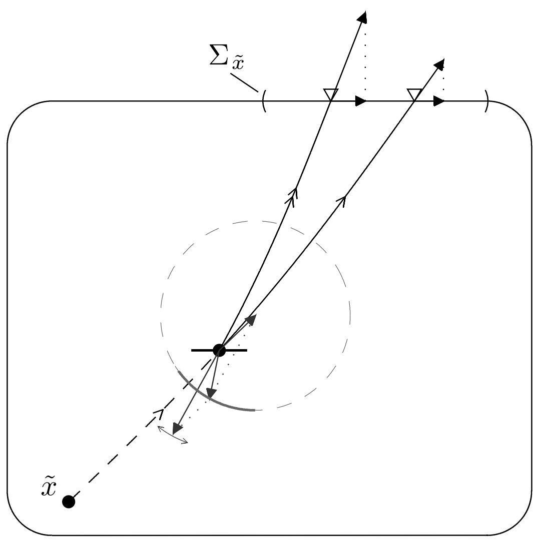


FIG. 1. *Integration over receivers, the construction of* $\xi_0$ *(isotropic case). The ray with single arrow corresponds with the incident field, which is also observed at the boundary; the ray with double arrows corresponds with the scattered field.* $\Sigma_s$ *indicates the array.*

REMARK 3.2. *Adjoint state formulation: (i) In the decoupled formulation, using (2.6), we obtain* $(F^*_{MN} d_{MN})_{ijkl}$ *upon successively solving*

$$P_N(x, D_x, D_t) w_{N;s} = \delta(t) \mathcal{Q}^{-1}_{Nk'}(x, s) e_{k'}, \quad w_{N;s}|_{t=0} = 0,\ \partial_t w_{N;s}|_{t=0} = 0,$$

*and*

$$P_M(x, D_x, D_t) u^*_M = \int \mathrm{d}r\, d_{MN}(r, s, T - t) \delta(x - \hat{x}(r, 0)), \quad u^*_M|_{t=0} = 0,\ \partial_t u^*_M|_{t=0} = 0,$$

*and evaluating the cross correlation*

$$\int_0^T \frac{\partial}{\partial x_{0,j}} Q(x_0, D_{x_0})_{iM} u^*_M(x_0, T - t; s) \frac{\partial}{\partial x_{0,k}} Q(x_0, D_{x_0})_{lN} w_{N;s}(x_0, t; s) \mathrm{d}t = I^{MN}_{ijkl}(x_0; s).$$

*(ii) The corresponding adjoint state formulation associated with the (full) elastic wave equation (2.1) is given by the forward equation*

$$P_{iq}(x, D_x, D_t) w_{q;s} = \delta(t)\delta(x - s) e_k, \quad w_{q;s}|_{t=0} = 0, \ \partial_t w_{q;s}|_{t=0} = 0,$$

*the adjoint state equation*

$$P_{ip}(x, D_x, D_t) u_p^* = \int \mathrm{d}r \, d_i(r, s, T - t)\delta(x - \hat{x}(r, 0)), \quad u_p^*|_{t=0} = 0, \ \partial_t u_p^*|_{t=0} = 0,$$

*and cross correlation*

$$\int_0^T \frac{\partial}{\partial x_{0,j}} \mathcal{P}_{ip}^M(x_0, D_{x_0}) u_p^*(x_0, T - t; s) \frac{\partial}{\partial x_{0,k}} \mathcal{P}_{lq}^N(x_0, D_{x_0}) w_{q;s}(x_0, t; s) \mathrm{d}t = I_{ijkl}^{MN}(x_0; s),$$

*where $\mathcal{P}_{ji}^M = Q_{jM} Q_{Mi}^{-1}$ (no sum) is the projection onto mode $M$ and $\mathcal{P}_{ji}^N = Q_{jN} Q_{Ni}^{-1}$ (no sum) is the projection onto mode $N$ [25, 26, 36]. The adjoint state formulation of / RTM approach to imaging has been used to compensate for the normal operator via Least-Squares optimization; see, for example, [18].*

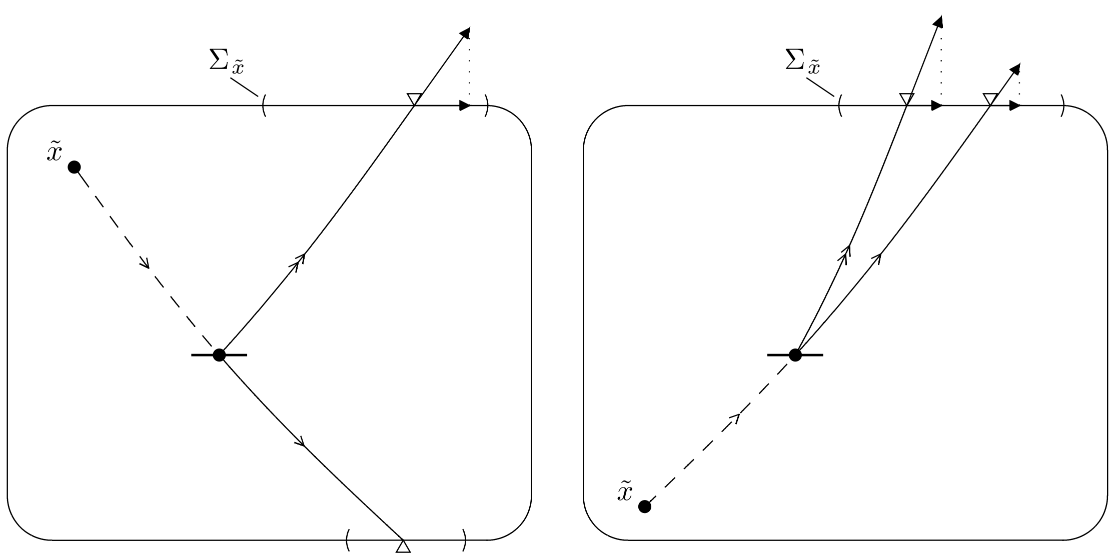


FIG. 2. *Array receiver functions: Detecting the incident field, and scattered field. Top: distinct arrays; bottom: single array (teleseismic situation).*

## 4. Array receiver functions.

**4.1. Backpropagating the incident field.** In this section, we will remove the knowledge of the source by obtaining the incident field, $Q(x_0, D_{x_0})_{lN} \int G_N(x_0, \tilde{x}', t_0)\, \mathcal{Q}_{Nk'}^{-1}(\tilde{x}', s) e_{k'} \, \mathrm{d}\tilde{x}'$ in (3.1), which corresponds with $w_{l;s}(x_0, t_0; s)$, from the data.

Backward extrapolation yields ($\Sigma_s$ is contained in a flat surface)

$$w_{q;s}(x_0, t_0; s) = -2 \int\int_{\Sigma_s} c_{nk;il}(\hat{x}(r', 0)) \frac{\partial G_{ql}}{\partial \hat{x}_k}(x_0, \hat{x}(r', 0), t' - t_0)\, d_i(r', t'; s)\, \mathrm{d}r' \mathrm{d}t',$$

restricting the wave field to a particularly polarized upgoing constituent (cf. (2.26); the integrand is related to $N_\nu$ (cf. (2.28)) via $N = N(\nu)$); upon decomposition into polarizations, we obtain

$$w_{q;s}(x_0, t_0; s) = -2Q(x_0, D_{x_0})_{qN} \iint_{\Sigma_s} c_{nk;il}(\hat{x}(r', 0)) \frac{\partial}{\partial \hat{x}_k} G_N(x_0, \hat{x}(r', 0), t' - t_0)(Q(\hat{x}, D_{\hat{x}})^{-1}_{Nl} d_i)(r', t'; s)\, \mathrm{d}r'\mathrm{d}t'$$

so that

$$w_{N;s}(x_0, t_0; s) = \iint_{\Sigma_s} (-2) \frac{\partial}{\partial \hat{x}_k} G_N(x_0, \hat{x}(r', 0), t' - t_0) (c_{nk;il}(\hat{x}(r', 0)) Q(\hat{x}, D_{\hat{x}})^{-1}_{Nl} Q(\hat{x}, D_{\hat{x}})_{iN} d_N)(r', t'; s)\, \mathrm{d}r'\mathrm{d}t' \tag{4.1}$$

(no sum over $N$). Following the propagation of singularities, it becomes clear that the singularities in the wave field are recovered at $x_0$ if the ray connecting $s$ with $x_0$ intersects the boundary at a point in $\Sigma_s$, see Fig. 3.

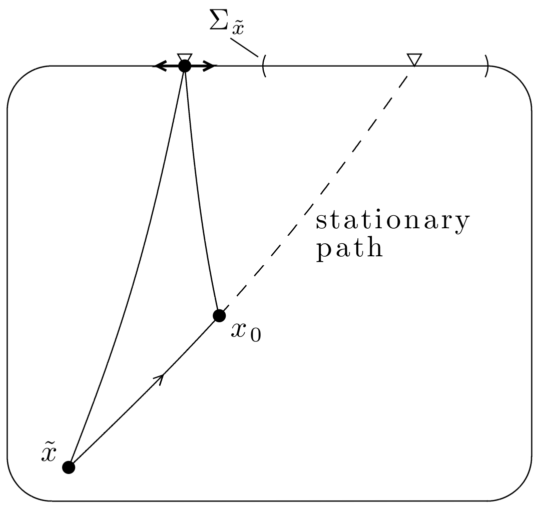


FIG. 3. *Backward extrapolation and propagation of singularities.*

**4.2. Cross correlation formulation.** Upon substituting (4.1) into (3.1), we obtain

$$\begin{aligned} I^{MN}_{ijkl}(x_0; s) = \int \frac{\partial}{\partial x_{0,j}} Q(x_0, D_{x_0})_{iM} \int_{t_0}^{T} \int_{\Sigma_s} G_M(x_0, \hat{x}(r, 0), t - t_0) \\ d_{MN}(r, t; s)\mathrm{d}r\mathrm{d}t \\ \frac{\partial}{\partial x_{0,k}} Q(x_0, D_{x_0})_{lN} \int_{t_0}^{T} \int_{\Sigma_s} (-2) \frac{\partial}{\partial \hat{x}_{k'}} G_N(x_0, \hat{x}(r', 0), t' - t_0) \\ (c_{nk';i'l'}(\hat{x}(r', 0)) Q(\hat{x}, D_{\hat{x}})^{-1}_{Nl'} Q(\hat{x}, D_{\hat{x}})_{i'N} d_N)(r', t'; s)\, \mathrm{d}r'\mathrm{d}t'\ \mathrm{d}t_0. \end{aligned} \tag{4.2}$$

Interchanging orders of integration leads to the introduction of

$$\begin{aligned} \mathcal{K}_{ijkl;MNk'}(x_0; r, t, r') = \int_0^t \frac{\partial}{\partial x_{0,j}} Q(x_0, D_{x_0})_{iM} G_M(x_0, \hat{x}(r, 0), t - t_0) \\ \frac{\partial}{\partial x_{0,k}}\, Q(x_0, D_{x_0})_{lN} (-2) \frac{\partial}{\partial \hat{x}_{k'}} G_N(x_0, \hat{x}(r', 0), -t_0)\, \mathrm{d}t_0 \end{aligned} \tag{4.3}$$

defining an integral operator $K_{MN}$, say. The propagation of singularities by $K_{MN}$ is described by the Lagrangian submanifold,

$$\Lambda^K_{MN} = \{(x_0, \hat{\xi}_0 - \tilde{\xi}_0; \hat{y}'(x_0, \hat{\xi}_0, \hat{t}), \hat{y}'(x_0, -\tilde{\xi}_0, \tilde{t}), \hat{t} - \tilde{t}, \\ \hat{\eta}'(x_0, \hat{\xi}_0, \hat{t}), -\hat{\eta}'(x_0, -\tilde{\xi}_0, \tilde{t}), \omega) \,|B_M(x_0, \hat{\xi}_0) = B_N(x_0, \tilde{\xi}_0) = \mp\omega, \\ \hat{y}''(x_0, \hat{\xi}_0, \hat{t}) = 0, \hat{y}''(x_0, -\tilde{\xi}_0, \tilde{t}) = 0\} \subset T^*X\backslash 0 \times T^*Z\backslash 0,$$

where $Z = \Sigma_s \times \Sigma_s \times (0, T)$, see Fig. 4.

ASSUMPTION 4. *The matrix*

$$\frac{\partial(\hat{y}''(x_0, \hat{\xi}_0, \hat{t}), \hat{y}''(x_0, -\tilde{\xi}_0, \tilde{t}))}{\partial(x_0, \hat{\xi}_0, \tilde{\xi}_0, \hat{t}, \tilde{t})} \text{ has rank } 2. \tag{4.4}$$

With this assumption, following the proof of [33, Theorem 4.2], we find

LEMMA 4.1. *Let $M \neq N$. $K_{MN} : \mathcal{E}'(Z) \to \mathcal{D}'(X)$ is a Fourier integral operator with canonical relation $\Lambda^K_{MN}$.*

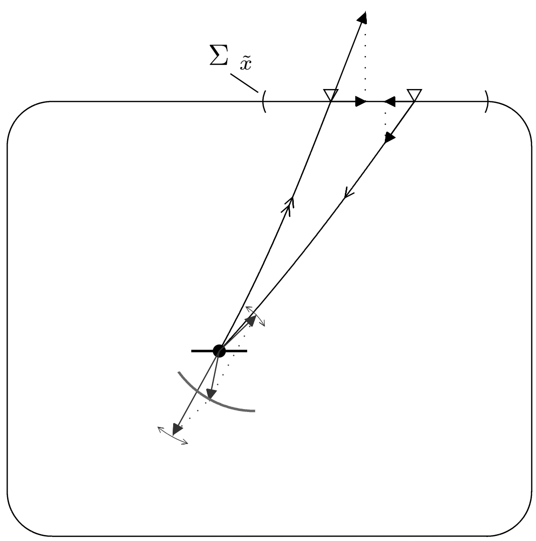


FIG. 4. *The canonical relation, $\Lambda^K_{MN}$, of $K_{MN}$ (isotropic case).*

DEFINITION 4.2. *For $M \neq N$, the array receiver function (ARF), $\mathsf{R}_{MN}$, is defined as the bilinear map, $d \to \mathsf{R}_{MNk'}$, with*

$$(\mathsf{R}_{MNk'}(d(.,.;s)))(r,t,r') = \int d_{MN}(r, t+t'; s) \\ (c_{nk';i'l'}(\hat{x}(r',0))Q(\hat{x}, D_{\hat{x}})^{-1}_{Nl'}Q(\hat{x}, D_{\hat{x}})_{i'N}d_N)(r',t';s)\,\mathrm{d}t' \quad \textit{(no sum over } N\textit{)}. \tag{4.5}$$

The observational assumption is that $d_{MN} = d_M$, whence $\mathsf{R}_{MNk'}(d(.,.;s))$ can be obtained from the multi-component data. Using (4.2) it now follows that

LEMMA 4.3. *(Equivalence) It holds true that, microlocally,*

$$(K_{MN}(\mathsf{R}_{MN}(d(.,.;s)))(.,.,.))_{ijkl}(x_0) = (F^*_{MN} d_{MN}(.,.;s))_{ijkl}(x_0). \tag{4.6}$$

Following the propagation of singularities, $\Lambda^K_{MN} \circ \mathrm{WF}(\mathsf{R}_{MN}(d(.,.,;s)))$, substituting $F_{MN}\delta c_{ijkl}$ for $d_{MN}$, reveals how the geometry of imaging $\delta c_{ijkl}$ reduces to the geometry depicted in Fig. 1 by combining Fig. 4 with Fig. 3.

**5. Flat, translationally invariant models: Propagation of singularities, receiver functions.** In view of translational invariance, (2.34) attains the form,

$$\begin{aligned}\delta G_{MN}(\hat{x},\tilde{x},t) = \int_{[0,Z]} \Bigg\{ \int_0^t \int \frac{\partial}{\partial(t_0,x_{0,j})}\, Q(x_0,D_{x_0})_{iM} G_M(x_0,\hat{x},t-t_0) \\ \frac{\partial}{\partial(t_0,x_{0,k})}\, Q(x_0,D_{x_0})_{lN} G_N(x_0,\tilde{x},t_0)\,\mathrm{d}x_0'\mathrm{d}t_0 \Bigg\} \\ \left(\delta_{il}\frac{\delta\rho(z_0)}{\rho(z_0)}, -\frac{\delta c_{ijkl}(z_0)}{\rho(z_0)}\right)\mathrm{d}z_0,\end{aligned} \tag{5.1}$$

writing $x_0 = (x_0', z_0)$. Upon restriction to $\hat{x} = \hat{x}(r,0), \tilde{x} = s$, the expression in between braces on the right-hand side defines the kernel, $\mathcal{F}^0_{MN;ijkl}(r,t;z_0)$ say, of a single scattering operator $F^0_{MN}$:

$$\begin{aligned}\mathcal{F}^0_{MN;ijkl}(r,t;z_0) = \int \int_0^t \int \frac{\partial}{\partial(t_0,x_{0,j})}\, Q(x_0,D_{x_0})_{iM} G_M(x_0,\hat{x}(r,0),t-t_0) \\ \frac{\partial}{\partial(t_0,x_{0,k})}\, Q(x_0,D_{x_0})_{lN} G_N(x_0,\tilde{x}',t_0)\,\mathrm{d}x_0'\mathrm{d}t_0\ \mathcal{Q}^{-1}_{Nk'}(\tilde{x}',s)e_{k'}\,\mathrm{d}\tilde{x}',\end{aligned} \tag{5.2}$$

cf. (2.40). The associated imaging operator, $(F^0_{MN})^*$, maps the (conversion) data to an image as a function of $z_0$ (and $s$): $I^{MN}_{ijkl}(z_0;s) = ((F^0_{MN})^* d_{MN})_{ijkl}(z_0)$, cf. (3.1).

We introduce so-called midpoint-offset coordinates $r' = m - h$ and $r = m + h$ ($\mathrm{d}r\mathrm{d}r' = 2\mathrm{d}m\mathrm{d}h$). Following the construction that leads to Lemma 4.3, we find that

$$(F^0_{MN})^* d_{MN}(.,.;s) = K^0_{MN}(\mathsf{R}_{MN}(d(.,.;s))(.,.,.)),$$

where $K^0_{MN}$ is an operator with kernel

$$\begin{aligned}&\mathcal{K}^0_{ijkl;MNk'}(z_0;m+h,t,m-h) \\ &\quad = 2\int_0^t \int \frac{\partial}{\partial x_{0,j}} Q(x_0,D_{x_0})_{iM} G_M(x_0',z_0,\hat{x}(m+h,0),t-t_0) \\ &\qquad \frac{\partial}{\partial x_{0,k}}\, Q(x_0,D_{x_0})_{lN}(-2)\frac{\partial}{\partial \hat{x}_{k'}} G_N(x_0',z_0,\hat{x}(m-h,0),-t_0)\,\mathrm{d}x_0'\mathrm{d}t_0,\end{aligned} \tag{5.3}$$

cf. (4.3). To study the propagation of singularities by this operator, we substitute the WKBJ approximations for $G_M$ and $G_N$ (cf. (2.31)) in this expression.

$K^0_{MN}$**, imaging**. We focus on the propagation of singularities and, hence, the relevant phase functions; the amplitudes follow from standard stationary phase arguments. The WKBJ phase function associated with $\mathcal{K}^0_{ijkl;MNk'}$ becomes

$$\omega\left[\, -\tau_\mu(0,z_0,\hat{p}) + \tau_\nu(0,z_0,\hat{p}') + \sum_{j=1}^{n-1}(\hat{p}-\hat{p}')_j(m-x_0')_j - \sum_{j=1}^{n-1}(\hat{p}+\hat{p}')_j h_j + t\,\right].$$

Carrying out the integrations over $x_0'$ and $\hat{p}'$ leads to $\mathcal{K}^0_{ijkl;MNk'}(z_0; m+h, t, m-h) \approx \dot{\mathcal{K}}^0_{ijkl;MNk'}(z_0; h, t)$, which admits an integral representation with WKBJ phase function

$$\omega\left[ -\tau_\mu(0, z_0, \hat{p}) + \tau_\nu(0, z_0, \hat{p}) - 2\sum_{j=1}^{n-1} \hat{p}_j h_j + t \right].$$

We get

$$K^0_{MN}(\mathsf{R}_{MN}(d(.,.;s))(.,.,.)) \approx \dot{K}^0_{MN}(\mathsf{R}^0_{MN}(d(.,.;s))(.,.,.)),$$

where

$$(\mathsf{R}^0_{MNk'}(d(.,.;s)))(h,t) = \int (\mathsf{R}_{MNk'}(d(.,.;s)))(m+h, t, m-h)\,\mathrm{d}m. \tag{5.4}$$

Applying the method of stationary phase to the integral representation for $\dot{\mathcal{K}}^0_{ijkl;MNk'}(z_0; h, t)$ in $\hat{p}$, yields stationary points $\hat{p} = \hat{p}^0(z_0, h)$ satisfying

$$-\left[ \frac{\partial \tau_\mu(0, z_0, \hat{p})}{\partial \hat{p}} - \frac{\partial \tau_\nu(0, z_0, \hat{p})}{\partial \hat{p}} \right] = 2h, \tag{5.5}$$

revealing the propagation of singularities: This equation defines a pair of rays sharing the same horizontal slowness $\hat{p}$, originating at (image) depth $z_0$, and reaching the acquisiton surface at $r = \frac{\partial \tau_\mu(0,z_0,\hat{p})}{\partial \hat{p}}$, $r' = \frac{\partial \tau_\nu(0,z_0,\hat{p})}{\partial \hat{p}}$, respectively; in the imaging point of view, the rays intersect at depth $z_0$, whence $r' - r = 2h$. The corresponding differential travel time is given by $\tau_\mu(0, z_0, \hat{p}^0(z_0, h)) - \tau_\nu(0, z_0, \hat{p}^0(z_0, h)) + 2\sum_{j=1}^{n-1} \hat{p}^0_j(z_0, h) h_j$ (we note that $\hat{p}^0_j(z_0, h)$ is the negative of the usual geometric ray parameter in view of our Fourier transform convention). The geometry is illustrated in Fig. 5 (pair of solid rays).

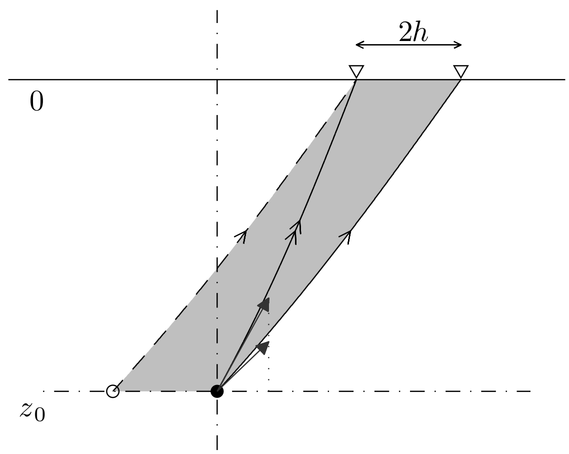


FIG. 5. *Propagation of singularities by $K^0_{MN}$. The double arrows relate to (scattered) mode $\mu$, while the single arrow relates to (incident) mode $\nu$. Translational invariance yields alternative ray pairs (dashed) for which the phase of $\mathcal{K}^0_{ijkl;MNk'}$ is stationary.*

$R^0_{MN}$**, modelling**. Using (5.1), substituting the WKBJ approximations for $G_M$ and $G_N$ in $(\mathsf{R}^0_{MNk'}(d(.,.;s)))(h,t)$, and carrying out the integrations over $x_0'$ and $\tilde{p}'$, we obtain a linear integral operator representation (acting on $(\delta_{il}\frac{\delta\rho(z_0')}{\rho(z_0')}, -\frac{\delta c_{ijkl}(z_0')}{\rho(z_0')})$) for the integrand in (5.4): For fixed $s =$

$(s', z_s)$, $z_s > z_0$, the WKBJ phase function of its kernel representation is

$$\omega \left[ -\tau_\mu(0, z_0', \hat{p}) - \tau_\nu(z_0', z_s, \hat{p}) + \tau_\nu(0, z_s, \tilde{p}) + \sum_{j=1}^{n-1} \hat{p}_j (m - h - s')_j - \sum_{j=1}^{n-1} \tilde{p}_j (m + h - s')_j + t \right].$$

Carrying out the integrations over $m$ and then $\hat{p}$ leads to $\int (\mathsf{R}_{MNk'}(d(.,.;s)))(m+h, t, m-h)\,\mathrm{d}m \approx (\dot{\mathsf{R}}^0_{MNk'}(d(.,.;s)))(h,t)$, which admits an integral representation with WKBJ phase function

$$\omega \left[ -\tau_\mu(0, z_0', \tilde{p}) + \tau_\nu(0, z_0', \tilde{p}) - 2 \sum_{j=1}^{n-1} \tilde{p}_j h_j + t \right];$$

the integration over $\tilde{p}$ signifies a "plane-wave" decomposition of the source, while the explicit dependence on $(s', z_s)$ has disappeared. Thus

$$\dot{K}^0_{MN}(\mathsf{R}^0_{MN}(d(.,.;s))(.,.)) \approx \dot{K}^0_{MN}(\dot{\mathsf{R}}^0_{MN}(d(.,.;s))(.,.)),$$

yielding a resolution analysis in depth. The stationary phase analysis in, and following (5.5) applies, leading to the introduction of $\tilde{p}^0(z_0', h)$ and associated ray geometry if there is a non-vanishing contrast (horizontal reflector) at depth $z_0'$.

For the singularities to appear in $(\dot{\mathsf{R}}^0_{MNk'}(d(.,.;s)))(h,t)$, it is necessary that $\tilde{p}^0(z_0', h)$ coincides with the stationary value, $\tilde{p}_s$ say, of $\tilde{p}$ associated with the WKBJ approximation of the *incident* field ($d_N$), determined by $s$ and $m+h$.

**Receiver Functions, plane-wave synthesis**. Let $\tilde{d}(.,.;\tilde{p}_s)$ denote the frequency-domain data obtained by synthesizing a source *plane wave* with parameter $\tilde{p}_s$ (as in plane-wave Kirchhoff migration) including an appropriate amplitude weighting function derived from the WKBJ approximation (here, we depart from the single source acquisition). We correlate these data using (4.5) subjected to a Fourier transform in time, with $d(.,.;s)$ replaced by $\tilde{d}(.,.;\tilde{p}_s)$. We obtain $\widehat{\mathsf{R}}_{MNk'}(\tilde{d}(.,.;\tilde{p}_s))(m+h, \omega, m-h)$, with the property

$$\dot{\mathsf{R}}^0_{MNk'}(d(.,.;s))(h,t) \sim \int \widehat{\mathsf{R}}_{MNk'}(\tilde{d}(.,.;\tilde{p}_s))(m_0, \omega, m_0) \exp\left( \mathrm{i}\omega \left[ -2 \sum_{j=1}^{n-1} \tilde{p}_{sj} h_j + t \right] \right) \mathrm{d}\tilde{p}_s \mathrm{d}\omega \tag{5.6}$$

for any $(m_0, 0)$ contained in $\Sigma_s$. The quantity $\widehat{\mathsf{R}}_{MNk'}(\tilde{d}(.,.;\tilde{p}_s))(m_0, \omega, m_0)$ is what seismologists call a receiver function. The phase shift and receiver function are illustrated by the dashed ray (single array) paired with the ray indicated by double arrows in Fig. 5.

**6. Discussion.** We presented a program for elastic-wave inverse scattering of reflection seismic data generated by an active (known) source or a passive (unknown) source, focussed on mode conversions. We derived the normal operator for, and an adaption of the elastic-wave reverse-time migration (RTM) approach to partially incorporate the parametix of the normal operator and suppress smooth artifacts. We introduced array receiver functions (ARFs), an extension of the notion of receiver functions, which can be used for inverse scattering of passive source data with a resolution comparable to RTM. In principle, the ARFs can be used also for imaging with certain (mode-converted) multiple scattered waves. Microlocally, the RTM and the generalized Radon transform (GRT) based inverse scattering are the same, the key restriction being the absence of source caustics. We established how ARFs reduce to receiver functions in the case of flat, planarly layered earth models.

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**Appendix A. Diagonalization of $A_{il}$ with a unitary operator.**

If the $A_M^{\text{prin}}(x,\xi)$ are all different, $A_{il}$ can be diagonalized with a unitary operator, that is, $Q(x,D_x)^{-1} = Q(x,D_x)^*$. We write $\widetilde{Q}^0_{lN}(x,D_x) = Q^{\text{prin}}_{lN}(x,D_x)$; then $(\widetilde{Q}^0)^*_{Mi}(x,D_x)$ has pricipal symbol $(Q^{\text{prin}})^t_{Mi}(x,\xi)$. We have

$$(\widetilde{Q}^0)^*_{Mi}(x,D_x)\widetilde{Q}^0_{iN}(x,D_x) = \delta_{MN} + R^0_{MN}(x,D_x)$$

where $R^0_{MN}(x,D_x)$ is self adjoint and of order $-1$. Then $Q^0_{iN}(x,D_x) = (\widetilde{Q}^0(I+R^0)^{-1/2})_{iN}(x,D_x)$ is, microlocally, unitary. We write

$$A_1^0(x,D_x) = \tfrac{1}{2}[A_1^{\text{prin}}(x,D_x) + (A_1^{\text{prin}})^*(x,D_x)],$$
$$A_2^0(x,D_x) = \operatorname{diag}(\tfrac{1}{2}[A_M^{\text{prin}}(x,D_x) + (A_M^{\text{prin}})^*(x,D_x)]; M = 2,\dots,n),$$

so that

$$(Q^0)^*AQ^0 = \begin{pmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $B_{11}$ and the elements of $B_{12}$ (a $1\times(n-1)$ matrix), $B_{21}$ (a $(n-1)\times 1$ matrix of pseudodifferential operators), and $B_{22}$ (a $(n-1)\times(n_1)$ matrix of pseudodifferential operators), are of order 1; $B$ must be self adjoint, whence $B_{12} = B_{21}^*$.

Next, we seek an operator, $\widetilde{Q}^1_{lN}(x,D_x) = \delta_{lN} + r^1_{lN}(x,D_x)$, assuming that

$$r^1 = \begin{pmatrix} 0 & -(r^1_{21})^* \\ r^1_{21} & 0 \end{pmatrix},$$

whence $(r^1)^* = -r^1$, such that

$$((\widetilde{Q}^1)^* \left( \begin{pmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \widetilde{Q}^1)_{21} = 0,$$
$$((\widetilde{Q}^1)^* \left( \begin{pmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \widetilde{Q}^1)_{12} = 0,$$

modulo terms of order 0. This holds true if

$$r^1_{21}A_1^0 - A_2^0 r^1_{21} = B_{21};$$

$r^1_{21}$ must be of order $-1$. Up to principal parts, this is a matrix equation for the symbol of $r^1_{21}$, given the principal symbols of $A_1^0$ and $A_2^0$; we note that the principals of $A_1^{\text{prin}}$ and $A_1^0$, and of $\operatorname{diag}(A_M^{\text{prin}}; M = 2,\dots,n)$ and $A_2^0$, coincide. Because the eigenvalues of $A_2^0(x,\xi)$ all differ from $A_1^0(x,\xi)$, it follows that this system of algebraic equations has a solution. With the solution we form the unitary operator $Q^1_{iN}(x,D_x) = ((I+r^1)(I-(r^1)^2)^{-1/2})_{iN}(x,D_x)$. Then

$$(Q^1)^*(Q^0)^*AQ^0Q^1 = \begin{pmatrix} A_1^1 & 0 \\ 0 & A_2^1 \end{pmatrix} + \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where $A_1^1$ and $A_2^1$ are self adjoint. $C_{11}$ and the elements of $C_{12}$ (a $1\times(n-1)$ matrix of pseudodifferential operators), $C_{21}$ (a $(n-1)\times 1$ matrix of pseudodifferential operators), and $C_{22}$ (a $(n-1)\times(n_1)$ matrix of pseudodifferential operators), are of order 0; $C$ must be self adjoint, whence $C_{12} = C_{21}^*$. This procedure is continued to find $Q^0Q^1\cdots Q^k$ which is microlocally unitary and brings $A$ in block diagonal form modulo terms of order $1-k$. Next, we repeat the procedure for $A_2^k$.

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