Proceedings of the Project Review, Geo-Mathematical Imaging Group (Purdue University, West Lafayette IN), Vol. 1 (2010) pp. 61-68.

SEISMIC INTERFEROMETRY AND THE ESTIMATION OF THE GREEN'S FUNCTION USING GAUSSIAN BEAMS

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Abstract. In this study, seismic interferometry is investigated in which the Green's function is estimated between two receivers by cross-correlation and integration over sources. For smoothly varying source strengths, the dominant contributions for the correlation integral come from the stationary phase directions in the forward and backward directions from the alignment of the two receivers. Gaussian beams can be used to evaluate the correlation integral and concentrates the amplitudes in a vicinity of the stationary phase regions instead of relying on phase interference. Several numerical examples are shown to illustrate how this process works. The use of Gaussian beams for the evaluation of the correlation integral results in stable estimates, and also provides physical insight into the estimation of the Green's function using seismic interferometry.

Key words. Seismic Interferometry, Gaussian beams

1. Introduction. Seismic interferometry can be used to determine the Green's function between two receivers as if a source were located at one receiver and recorded at the other receiver using distant seismic energy. The development and many applications of seismic interferometry are described in the recent books by Wapenaar, Draganov and Robertsson (2008) and Schuster (2009) (see also Lobkis and Weaver, 2001; Derode et al., 2003; Wapanaar, 2004; and Wappenaar et al., 2005). Pioneering work dates back to Aki (1957) who extracted the properties of the shallow sub-surface from microseismic noise and also to Claerbout (1968) who showed that the autocorrelation of the transmission response is equal to the reflection response and its time-reversed version in a layered medium. More recent applications of seismic interferometry in the areas of exploration seismology include Bakulin and Calvert (2004) and Schuster et al. (2004), in ultrasound by Weaver and Lobkis (2001), in crustal seismology by Campillo and Paul (2003), Sabra et al. (2005a,b), Roux et al. (2005) and Shapiro et al. (2005), and in helioseismology by Rickett and Claerbout (1999).

Here we investigate the extraction of the Green's function based on the cross-correlation of signals at two receivers using seismic energy from a distribution of surrounding sources with a uniform angular spectrum of incident seismic energy. For simplicity here we only investigate the acoustic case. Gaussian beams are then used to evaluate the resulting interference integral in order to concentrate the amplitudes of the contributions to near the stationary phase directions. This results in a stable estimate, and also provides physical insight into where the dominant contributions of the seismic energy come from in the evaluation of the Green's function.

2. Reciprocity of the Convolution and Correlation Types. A derivation of the acoustic reciprocity relations for the convolution and correlation types is now given (see also, Schuster, 2009). In the frequency domain, one can write

$$G(B,A) - G_0(A,B) = \int dS(x) \left[G_0(x,B)\frac{\partial G(x,A)}{\partial n} - G(x,A)\frac{\partial G_0(x,B)}{\partial n} \right] \tag{2.1}$$

where $G(B,A)$ is the Green function with a source at A and a receiver at B, and G_0 is a second Green's function for a possibly different medium in the same volume. The integral is along the boundary $dS(x)$ of the volume. If the boundary conditions are homogeneous, the boundary integral vanishes. This could result, for example, if the integration surface is at infinity with Sommerfeld outgoing radiation conditions. If the boundary integral vanishes, and for the same medium for G and G_0, then formal reciprocity of the Green's function results with $G(B,A) = G(A,B)$.

In Eqn. (2.1), if $G_0(A,B) = G^*(A,B)$, the adjoint Green's function, then

$$G(B,A) - G^*(A,B) = \int dS(x) \left[G^*(x,B)\frac{\partial G(x,A)}{\partial n} - G(x,A)\frac{\partial G^*(x,B)}{\partial n} \right] \tag{2.2}$$

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For this case the boundary integral generally does not vanish at infinity. Applying reciprocity of the Green's function then

$$G(B,A) - G^*(B,A) = \int dS(x) \left[G^*(B,x)\frac{\partial G(A,x)}{\partial n} - G(A,x)\frac{\partial G^*(B,x)}{\partial n} \right] \tag{2.3}$$

This is the acoustic reciprocity relation of the correlation type. If the sources at x along the boundary are in the far-field from the receivers, then $\frac{\partial G(B,x)}{\partial n} \sim ikG(B,x)$, where $k = \omega/v$ is the wavenumber, and this can be written

$$2\mathrm{i}\ \mathrm{Im}\ G(B,A) = G(B,A) - G^*(B,A) = 2\mathrm{i}k \int dS(x) G^*(B,x) G(A,x) \tag{2.4}$$

For this case, we can obtain the imaginary part of the Green's function from B to A from the cross-correlation at the receivers and then integrating over all sources along the boundary. The complete Green's function can then be obtained from this based on the causality properties of the Green's function.

If the sources along the boundary are variable in source strength, a bias in the results could occur, for example from noise sources from distant storms at sea that are dominantly located only at a small number of directions from the receivers. However, this can be corrected for if the source strengths along the boundary are known. More generally, the sources along the boundary can have source spectra that are not flat with frequency. In this case, the correlation integral can be written

$$2\mathrm{i}\ \mathrm{Im}\ G(B,A) = G(B,A) - G^*(B,A) = 2\mathrm{i}k \int dS(x) \frac{1}{s_0^2(x,\omega)} P^*(B,x,\omega) P(A,x,\omega) \tag{2.5}$$

where $P^*(B,x,\omega)$ and $P(A,x,\omega)$ are the general signals from sources on the surrounding boundary each with source spectra $s_0(x,\omega)$ where the ω dependence is explicitly shown. Thus, the source spectra, including variable source amplitudes, can be corrected for by dividing out the power spectra of the sources for each frequency and position along the boundary. However, only the power spectra are required, or the autocorrelation of the source wavelets, and not the complete source spectra which is a much weaker constraint. For any zeroes in the source spectra, appropriate damping would need to be used.

3. The correlation integral in 2D homogeneous media. In 2D homogeneous media, the Green's function can be written

$$G(x,x_0) = \frac{\mathrm{i}}{4} H_0^{(1)}(kr) \tag{3.1}$$

where $r = |x - x_0|$ and k is the wavenumber. In the far-field, this can be approximately evaluated as

$$G(x,x_0) \sim \sqrt{\frac{1}{8\pi kr}} e^{\mathrm{i}(kr+\pi/4)} \tag{3.2}$$

Inserting Eqn. (3.2) into Eqn. (2.4) results in (Fan and Snieder, 2000)

$$2\mathrm{i}\ \mathrm{Im}\ G(B,A) = G(B,A) - G^*(B,A) = \frac{\mathrm{i}}{4\pi} \int dS(x) \sqrt{\frac{1}{r_{Ax} r_{Bx}}} e^{\mathrm{i}k(r_{Ax} - r_{Bx})} \tag{3.3}$$

where r_{Ax} and r_{Bx} are the distances from x to the receivers A and B, respectively.

If the distance R in Figure 1 is much larger than the distance L between the receivers A and B, then as a first approximation, $r_{Ax} - r_{Bx} \sim L\cos\varphi$ in the exponential and $r_{Ax} = r_{Bx} \sim R$ in the amplitude term. Letting the arc-length on the circular boundary be $dS = Rd\varphi$, then

$$2\mathrm{i}\ \mathrm{Im}\ G(B,A) = G(B,A) - G^*(B,A) = \frac{\mathrm{i}}{4\pi} \int d\varphi e^{\mathrm{i}kL\cos\varphi} \tag{3.4}$$

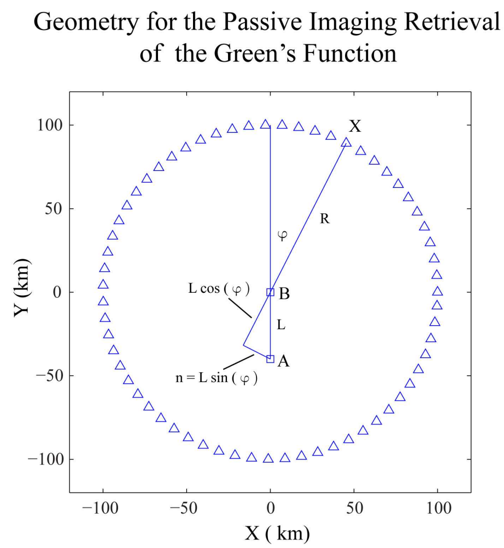

FIG. 1. *This shows the geometry for the passive estimation of the Green's function between two receivers at A and B with sources on the surrounding boundary shown by triangles. The distance between the two sensors is L. The angle φ measures the location along the surrounding circular boundary at a radius R from the receiver at B.*

This results in a plane-wave decomposition of the correlation field from the distant sources at the two receiver positions, and can be represented as a Bessel function where

$$\frac{\mathrm{i}}{2\pi}\int d\varphi e^{\mathrm{i}kL\cos\varphi} = J_0(kL) = \frac{1}{2}\left[H_0^{(1)}(kL) - H_0^{(1)}(-kL)\right] \tag{3.5}$$

This shows that the correlation Green's function can be written as a sum of plane waves incident on the receivers from distant sources at all angles (Fan and Snieder, 2009).

4. The correlation integral evaluated with Gaussian beams. A paraxial approximation for the phase term in the correlation integral Eqn. (3.3) can be written

$$r_{Ax} - r_{Bx} \sim L\cos\varphi + \frac{1}{2}Mn^2 \tag{4.1}$$

where $L\cos\varphi$ is shown in Figure 1, $n = L\sin\varphi$, and M is the second derivative of the travel-times at the A receiver position. M is proportional to the wavefront curvature and can be evaluated using dynamic ray tracing, but in a homogeneous medium is known in closed form.

M can now be extrapolated to a complex value where now $M = \frac{p}{q} = \frac{v^{-1}}{L\cos\varphi+\varepsilon}$. ε is called the beam parameter and can be written as $\varepsilon = -\mathrm{i}L_0^2$. The parameter L_0 can be related to the beam-width as $L(s_0) = \sqrt{\frac{2v}{\omega}}L_0$, where ω is the frequency and v is the medium velocity (see Cerveny, et al.,1982; Popov, 1982; Nowack and Aki, 1984) For more recent overviews of Gaussian beam summations, see Cerveny (2001), Popov (2002), Nowack(2003) and Cerveny and Psencik (2007). The correlation integral with complex beam parameters can then be written

$$G(B,A) - G^*(B,A) = \frac{\mathrm{i}}{4\pi}\int d\varphi\sqrt{\frac{R}{(R+L\cos\varphi+\varepsilon)}}e^{\mathrm{i}\omega(\frac{L\cos\varphi}{v}+\frac{1}{2}\mathrm{Real}(M)n^2)}e^{-\frac{\omega}{2}\mathrm{Imag}(M)n^2} \tag{4.2}$$

This results in an amplitude decay instead of just phase interference of the plane wave components in Eqn. (3.4).

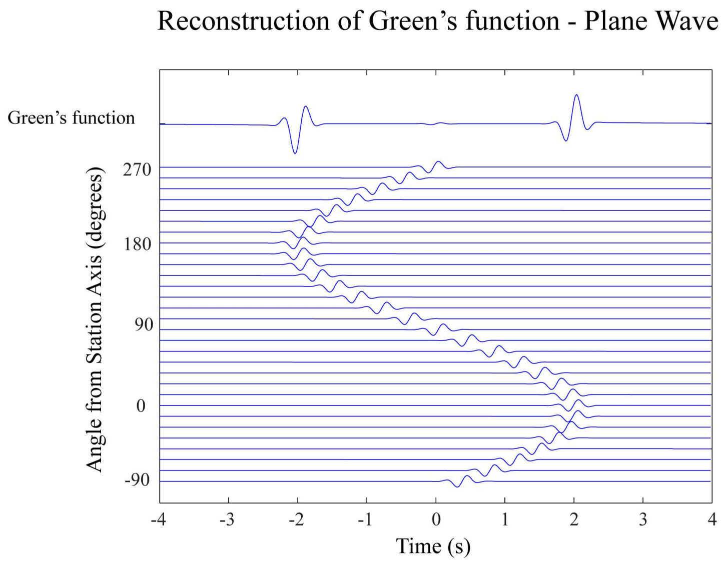

FIG. 2. *This shows the reconstruction of the Green function on the top trace from the summation of the integrand components in the correlation integral in Eqn. (2.4) at the receivers A and B from the sources as a function of angle on the surrounding circle in Figure 1. A Gabor wavelet is assumed for the source wavelet.*

5. Examples. Figure 2 shows results of the correlation integral in Eqn. (3.4) for the case of the sources on a distant circular boundary where $R = 400km$ with the distance between the two receivers of $L = 10km$ with the geometry shown in Figure 1. The waves are assumed to be planar at the receiver locations resulting in the plane-wave decomposition of the correlation integral in Eqn. (3.4). In order to investigate a time-domain beam solution for the correlation integral, a Gabor wavelet is used for the sources on the surrounding boundary, and this can be written as

$$f(t) = \mathrm{e}^{-(2\pi f_M(t-t_i)/\gamma)^2} \cos(2\pi f_M(t-t_i)+\nu) \tag{5.1}$$

where f_M is the center frequency and γ determines the number of wave cycles under the Gaussian envelope (Cerveny, 2001). For the examples shown, $f_M = 3Hz$, $\gamma = 3.5$ and $\nu = 0$. Thus, the results would be as in Eqn. (3.4) but convolved with a Gabor wavelet.

For the example shown in Figure 2, the source strengths and source waveforms are the same for all positions on the boundary. The individual components of the integrand are seen to form an inverted S shape as a function of angle. The summation of all the integrand components results in an estimate of the correlation Green's function filtered by the Gabor wavelet and is shown in the top trace. For this case, each of the angular components of the integrand are roughly the same in amplitude and pulse shape, but are located at different times. Nonetheless, the dominant components of the integrand are from angles 0 degrees and 180 degrees aligned with the orientation of the two receivers. For these angles, the nearby integrand components are in-phase and provide the dominant stationary phase contributions to the integrand. Components at other angles destructively interfere resulting in no contributions to the correlation integral, with the exception of small end effects at angles of -90 degrees and 270 degrees where the integration terminates.

The regions of stationary phase in the correlation integral are illustrated in Figure 3. The dashed lines show the zones of dominant contributions to the integral based on the stationary phase of the components oriented in the forward and backward directions of the aligned receiver pair. The positive contribution to the correlation Green's function comes from the 0 degree direction, and the negative component from the 180 degree direction. Nonetheless, all components are needed to avoid truncation effects and allow these components to sum to zero by destructive interference. For this case in Figure 2, the source strength is the same for all angular directions on the boundary and

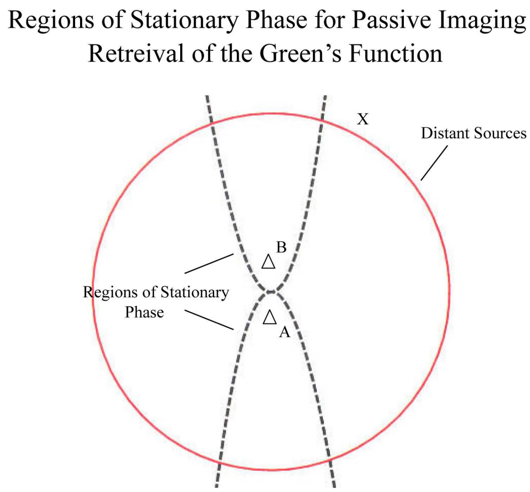

FIG. 3. *The regions of stationary phase of the oscillatory correlation integral in Eqn. (2.4) are shown by the dashed lines. A and B are the receiver positions and the sources are located on the circular boundary (modified from Fan and Snieder, 2009).*

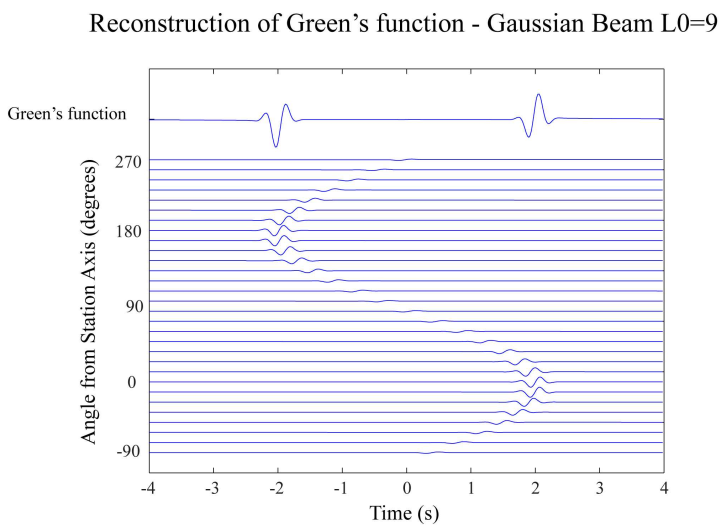

FIG. 4. *This shows the reconstruction of the Green function on the top trace from the summation of the integrand Gaussian beam components in the approximate Gaussian beam correlation integral in Eqn. (4.2). The value L_0 in the complex beam parameter was chosen as 9. A Gabor wavelet is assumed for the source wavelet.*

results in equal amplitudes of the negative and positive times for the correlation Green's function. However, in the general case, the source strengths will be different, and if not corrected for will result in different positive and negative time amplitudes for the correlation Green's function.

We would now like to incorporate amplitude weighting of the integrand components using Gaussian beams as described above to assist in concentrating the integrand contributions to near the stationary directions. This also has the advantage of not requiring integration contributions for all angular directions which otherwise would require phase interference, but rather only include those contributions only near the stationary phase directions.

In Figure 4, the estimate of the correlation Green's function is shown at the top for the case of summation of Gaussian beam correlation components. The geometry of the sources and the two receivers A and B is again shown in Figure 1. Similar to the plane-wave case, a Gabor wavelet with

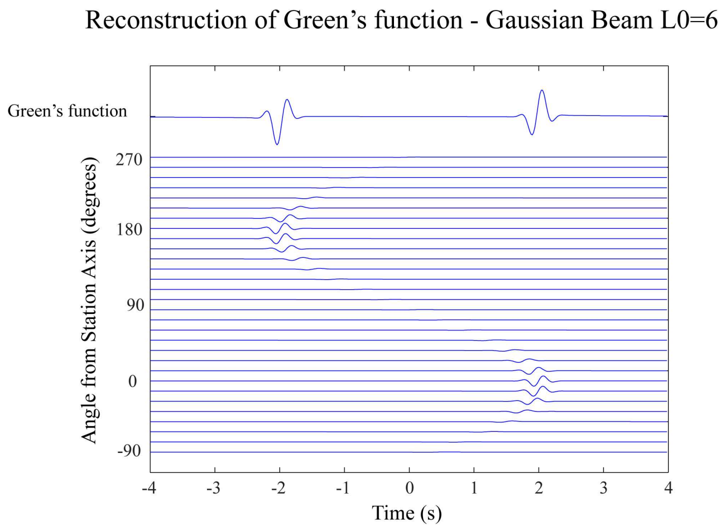

FIG. 5. *This shows the reconstruction of the Green function on the top trace from the summation of the integrand Gaussian beam components in the approximate Gaussian beam correlation integral in Eqn. (4.2). The value L_0 in the complex beam parameter was chosen as 6. A Gabor wavelet is assumed for the source wavelet.*

a reference frequency of 3 Hz is used with equal source amplitudes for all directions on the boundary. For this case, the beam parameter is chosen with $L_0 = 9$ which corresponds to a beam-width at the reference frequency of 6.6 km at the receiver B location. The use of Gaussian beams can be seen in Figure 4 to restrict the contributions to the vicinity of the stationary phase directions. Also, endpoint errors are removed by the amplitude decay of the beams away from the stationary phase directions. This provides a more stable evaluation of the correlation integral than purely phase interference while also restricting the contributions to just those in the forward and back azimuth directions from the direction of the receiver pair.

The energy in the integrand of the correlation integral can be further concentrated near the stationary phase directions by choosing a smaller beam parameter with $L_0 = 6$ which corresponds to a beam-width at the receiver B location of 4.3 km at the reference frequency of the Gabor wavelet. This is shown in Figure 5 where now the concentration of the integrand components of the correlation integral is made smaller still. However, this process can't be taken too far since after a certain point, the beams start to diverge again. An optimally narrow beam-width at the receiver point for planar beams at the initial point was given by Cerveny et al. (1982). Nonetheless, the beam contributions to the correlation integral can be restricted to the first Fresnel zone of the in-phase contributions near the stationary phase directions by a suitable choice of the beam parameter. The sources from other directions are then not required to evaluate the correlation integral using Gaussian beams.

If the source strengths are different but smoothly varying as a function of azimuth, then the stationary phase directions will still be the dominant contributions. However, now there may be differences in the amplitudes for the positive and negative time components of the correlation Green's function. As an example, Figure 6 shows a case where the source strengths vary as $(1 + \rho \cos \varphi)$ where $\rho = .3$. In this case, the source strengths in the forward and backward directions vary from 1.3 to .7, respectively. This results in an amplitude difference for the positive and negative time components of the correlation Green's function a shown in Figure 6. However, using Gaussian beams for the evaluation of the Green's function clearly identifies where the differences in source strength are coming from.

6. Conclusion. In this study, seismic interferometry is investigated for two receivers with sources along the surrounding boundary. The Green's function can then be estimated between the

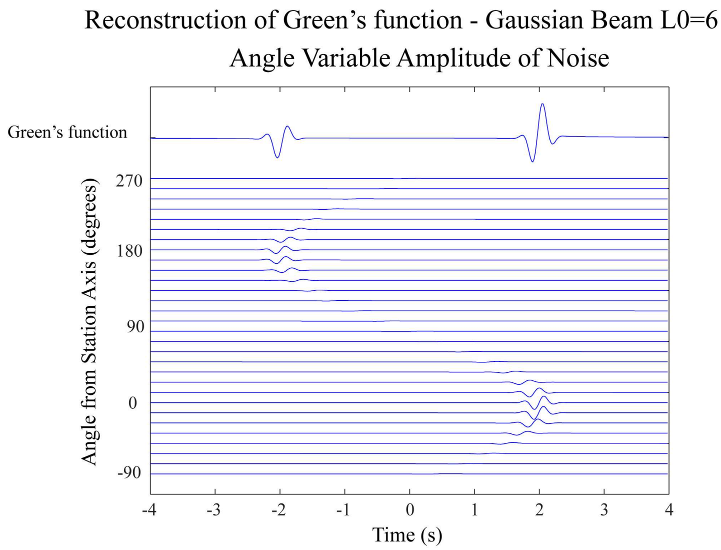

FIG. 6. *This shows the reconstruction of the Green function on the top trace from the summation of the integrand Gaussian beam components in the approximate Gaussian beam correlation integral in Eqn. (4.2). The value L_0 in the complex beam parameter was chosen as 6. A Gabor wavelet is assumed for the source wavelet. For this case, the source strengths were variable as a function of azimuthal angle from the sources and then results in a different amplitude of the summation for the positive times and the negative times.*

two receivers by the cross-correlation of the signals recorded at both receivers and integrated for all sources. For smoothly varying source strengths with direction, then the dominant contribution for the correlation integral is from the stationary phase directions in the forward and back directions from the alignment of the two receivers. The use of Gaussian beams can be used to concentrate the amplitudes of the source contributions to a vicinity of the stationary phase regions, resulting in stable estimates of the Green's function. Several numerical examples are shown to illustrate this process. The application of Gaussian beams to the evaluation of the correlation Green's function provides further insight into how seismic interferometry works.

Acknowledgments. This work was supported in part by the National Science Foundation and the Air Force Geophysics Laboratory and partly by the members of the Geo-Mathematical Imaging Group (GMIG) at Purdue University.

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