Proceedings of the Project Review, Geo-Mathematical Imaging Group (Purdue University, West Lafayette IN), Vol. 1 (2010) pp. 51-60.

THE GOUY PHASE ANOMALY FOR HARMONIC AND TIME-DOMAIN PARAXIAL GAUSSIAN BEAMS

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Abstract. In this study, the Gouy phase anomaly resulting from the focusing of wave solutions is illustrated using 2D paraxial Gaussian beams. For harmonic Gaussian beams, this gives rise to a continuous variation of the Gouy phase as a function of propagation distance. This is in contrast to the discontinuous phase anomaly at caustics for ray solutions. However, as the beam-width of a Gaussian beam at a focus gets smaller, the Gouy phase anomaly becomes more concentrated near the focus and approaches that of the ray solution. The Gouy phase for a harmonic Gaussian beam is first illustrated in a homogeneous medium, and then in a quadratic velocity waveguide where the beam can pass through multiple focus points. However for multiple focus points, care must be taken to ensure that the phase remains continuous by phase unwrapping. Finally, an example is shown of the Gouy phase for a time-domain signal using a Gabor wavelet. This illustrates the phase advance of a time-domain signal which can affect the travel-times for a large distance range along the beam. Intuitively, as a wave solution gets "squeezed" at a focus, it "squirts" forward by slightly increasing its apparent speed in the propagation direction.

Key words. the Gouy phase, paraxial Gaussian beams

1. Introduction.

Paraxial Gaussian beams are bounded solutions to the wave equation and can be used to model different types of beamed signals, for example from transducers in the laboratory. Gaussian beams can also be used for the superposition and synthesis of other types of wavefields (e.g. Popov, 1982; Cerveny et al., 1982; Nowack and Aki, 1984). Here we will investigate the so-called Gouy phase anomaly for harmonic and time-domain, paraxial Gaussian beams.

For all wave solutions there is an interplay between the spatial extent of the solution with the range of wavenumbers required to model it. A spatial restriction or "squeeze" of a wave solution will result in a change in the average wavenumber in the direction of propagation, and a corresponding phase anomaly. The phase anomaly for a spherically converging wave solution was first observed by Gouy(1890a,b), where it was found be equal to $-\pi$.

In ray theory, there is an abrupt change of phase at a caustic where the ray amplitude is singular. This phase shift can be determined by matching the ray solutions on either side of a caustic (Cerveny, 2001; Chapman, 2004). The so-called "index of the ray trajectory" (Kravstov and Orlov, 1990) counts the number and order of the caustics encountered by the ray in order to compute the cumulative phase shift. This is now referred to as the KMAH index after the work of Keller (1958), Maslov (1965), Arnold (1967) and Hormander (1971). This can be related to the continuous Gouy phase for general wave and beam solutions. Paraxial Gaussian beams are approximate wave solutions along rays which have complex travel times and are non-singular at caustics (Popov, 2002).

2. Gaussian beams and the Gouy phase.

Exact Gaussian beams can be derived in several ways (Siegman, 1986) including the complex source point approach in which an analytic continuation of a point source from a real source location is performed (Deschamps, 1971; Felsen, 1976). Other approaches to derive Gaussian beams are the differential equation approach based on the "paraxial" wave equation, the Huygens-Fresnel integral with an initial Gaussian amplitude profile, a plane wave expansion approach, and solutions to the Helmholtz equation in oblate spheroidal coordinate systems (Siegman, 1986).

Although Gaussian beam solutions don't focus to a point, they do narrow to a beam-waist with a beam-width of $L(z_0)$, where the transverse amplitude decays to $1/e$ of the peak value. For a Gaussian beam in a homogeneous medium, the beam-width as a function of distance z can be

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expressed as

$$L(z) = L(z_0)\left(1 + \left(\frac{z}{Z_R}\right)^2\right)^{1/2} \tag{2.1}$$

where $Z_R = \pi L^2(z_0)/\lambda$ is called the Rayleigh distance and λ is the wavelength (Siegman, 1986). At the Rayleigh distance the beam-width equals $\sqrt{2}L(z_0)$ after which the beam begins to diverge. The Gouy phase anomaly $\psi(z)$ for a Gaussian beam in a homogeneous medium can be written as $\psi(z) = -\frac{n}{2}\tan^{-1}(z/Z_R)$, where $n = 1$ in 2D and $n = 2$ for 3D. For heterogeneous media, the Gouy phase anomaly is often more complicated. However, the Gouy phase for beams and other wave solutions is a continuous function of propagation distance, in contrast to ray solutions.

There have been a number of explanations for the Gouy phase anomaly, including those related to analogies with quantum physics (Subharo, 1995; Simon and Mukunda, 1993). However, this requires a specialized background in quantum physics, which is useful but not essential. Boyd (1980) presented a geometric construction for the Gouy phase in terms of lateral beam spread.

Feng and Winful (2001) gave an explanation of the Gouy phase in terms of a shift in the expectation value of the axial wavenumber resulting from an increase in the transverse wavenumber when a beam is focused or confined. Consider a plane wave of frequency ω, speed v, and wavenumber k with a magnitude of $\frac{\omega}{v}$. k has three components that are related by

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{2.2}$$

Although the magnitude of k is a constant, the presence of the transverse components reduce the magnitude of the axial component k_z. The effective axial wavenumber across the face of the beam can be defined as

$$\bar{k}_z = k - \frac{\langle k_x^2 \rangle}{k} - \frac{\langle k_y^2 \rangle}{k} \tag{2.3}$$

where the expectations are taken across the beam in the x and y lateral directions. The Gouy phase anomaly can then be written (Feng and Winful, 2001) as

$$\psi(z) = -\frac{1}{k}\int_0^z \left\{\langle k_x^2 \rangle + \langle k_y^2 \rangle\right\} dz \tag{2.4}$$

For a harmonic Gaussian beam in a homogeneous medium, the Gouy phase is $\psi(z) = -(\frac{1}{2} + \frac{1}{2})\tan^{-1}(z/Z_R)$ where Z_R is the Rayleigh range. For $z \to \infty$, then $\psi(z) = -\pi/2$ (with a $-\pi/4$ for each lateral dimension). For a 3D Gaussian beam, the Gouy phase is progressive from 0 to $-\pi/2$ from the beam waist for $0 < z < \infty$. For 2-D ribbon beams, the phase shift goes from 0 to $-\pi/4$ for $0 < z < \infty$. In Huygens-Fresnel integrals, a $-\pi/2$ phase shift is also required between an incident 3D wavefront and the diverging secondary wavelets. For $-\infty < z < +\infty$, the Gouy phase results in a phase shift of $-\pi$ for a 3D wave going through a focus, and for a Gaussian beam this phase shift is progressive.

The Gouy phase described above is for the fundamental mode Gaussian beam. For higher modes in an Hermite-Gaussian beam expansion, the Gouy phase is

$$\psi(z) = -(m + n + 1)\tan^{-1}\left(\frac{z}{Z_R}\right) \tag{2.5}$$

where the indices m and n are the mode numbers for the higher order Gaussian beams (Feng and Winful, 2001).

Wave solutions, including beams, exhibit a spatial and wavenumber duality where an uncertainty principle results with

$$\Delta k_\rho \Delta\rho \geq \text{constant} \tag{2.6}$$

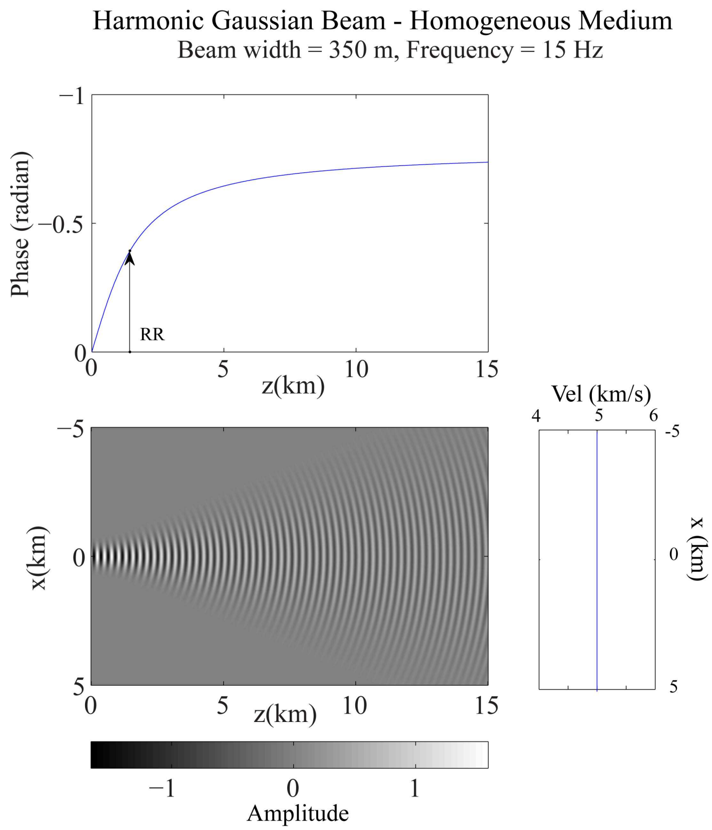

Fig. 1. *Harmonic Gaussian beam for a frequency of 15 Hz in a homogeneous medium. The beam-waist is at* $z = 0$ *with a beam-width of 350 m. The beam-field is shown at the bottom and the Gouy phase anomaly at the top. The Rayleigh distance is indicated by RR.*

where ρ is a general transverse coordinate and k_ρ is the corresponding transverse wavenumber. As $\Delta\rho$ gets smaller from wave focusing, then Δk_ρ gets gets bigger resulting in a larger Gouy phase anomaly. However, the phase space taken up by the higher order Gaussian beams is generally larger than for zeroth order beams with a larger constant in the uncertainty relation (Feng and Winful, 2001).

3. Paraxial Gaussian Beams in 2D. Approximate paraxial Gaussian beams in inhomogeneous media can be described using dynamic ray tracing with complex initial conditions along a real ray, and this provides a major computational advantage for the calculation of high-frequency Gaussian beams in smoothly varying media. Overviews of paraxial Gaussian beams using dynamic ray tracing are given by Kravtsov and Berczynski (2007), Popov (2002), Cerveny (2001), where the complex part of the eikonal gives rise to a Gaussian amplitude shape.

In 2D, the ray-centered coordinates are given by (s, n) where s is the coordinate along the ray and n is transverse to the ray. A paraxial Gaussian beam connected with such a central ray is given by (Cerveny et al., 1982)

$$u(s,n,t) = \sqrt{\frac{v(s)}{q(s)}}\exp\left\{-\mathrm{i}\omega\left(t - \int_{s_0}^{s}\frac{ds}{v(s)}\right) + \frac{\mathrm{i}\omega}{2}\frac{p(s)}{q(s)}n^2\right\} \tag{3.1}$$

where $v(s)$ is the velocity along the central ray, and $p(s)$ and $q(s)$ are the complex solutions of the dynamic ray equations. These can be constructed by a linear combination of real fundamental solutions of the dynamic ray equations with complex initial conditions. Thus, the complex $p(s)$ and

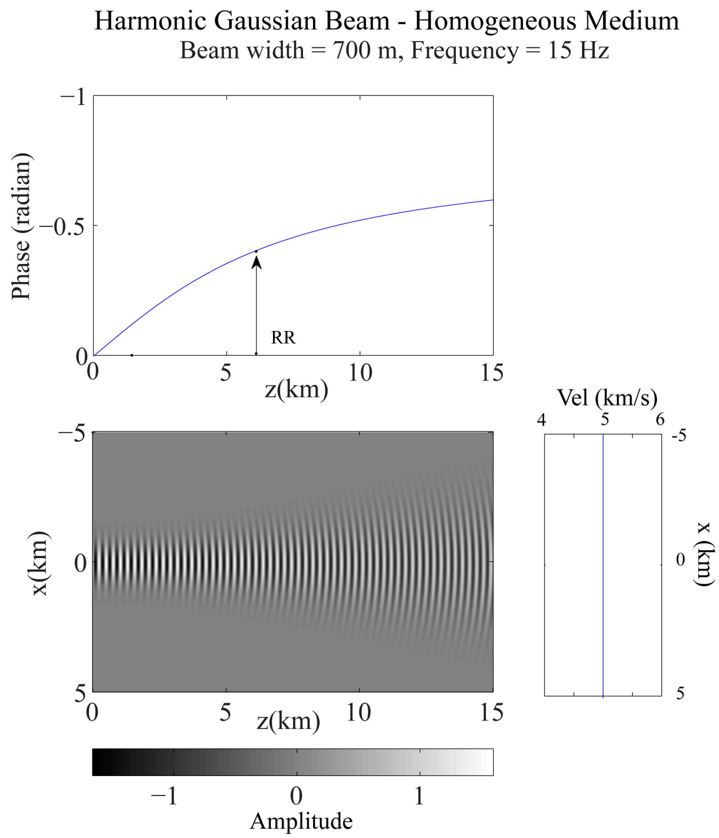

FIG. 2. *Harmonic Gaussian beam for a frequency of 15 Hz in a homogeneous medium. The beam-waist is at $z = 0$ with a beam-width of 700 m. The beam-field is shown at the bottom and the Gouy phase anomaly at the top. The Rayleigh range is indicated by RR.*

$q(s)$ can be written as

$$p(s) = \epsilon p_1(s) + p_2(s)$$
$$q(s) = \epsilon q_1(s) + q_2(s) \tag{3.2}$$

where $(q_1(s), p_1(s))$ and $(q_2(s), p_2(s))$ are the fundamental real solutions. ϵ is the complex beam parameter that specifies the position of the beam waist and the beam-width of the Gaussian beam (Cerveny et al., 1982) and is given by

$$\epsilon = S_0 - \mathrm{i}L_0^2 \tag{3.3}$$

where S_0 is the position of the beam-waist and $L(s_0) = \sqrt{\frac{2v_0}{\omega}}L_0$ specifies the beam-width. The position of the beam-waist determines where the planar part of the beam is located. ϵ is just one choice for the complex beam parameter and is equal to $1/M(s_0)$ where $M(s_0) = p(s_0)/q(s_0)$.

4. Harmonic Gaussian beams in homogeneous media. For paraxial Gaussian beams in a homogeneous medium, the dynamic ray tracing can be solved analytically. In Figure 1, an example is shown for a harmonic beam with a frequency of 15 Hz, and a beam-waist located at $z = 0$ with a beam-width of 350 m. The background velocity is 4 km/s and the Rayleigh distance is 1.44 km. "z" is now the propagation direction of the beam and "x" is the transverse direction. The lower plot in Figure 1 is the beam-field which is seen to spread with distance z. The top plot in Figure 1 shows the Guoy phase anomaly which for the 2D case reaches a value of $-\pi/8$ at the Rayleigh

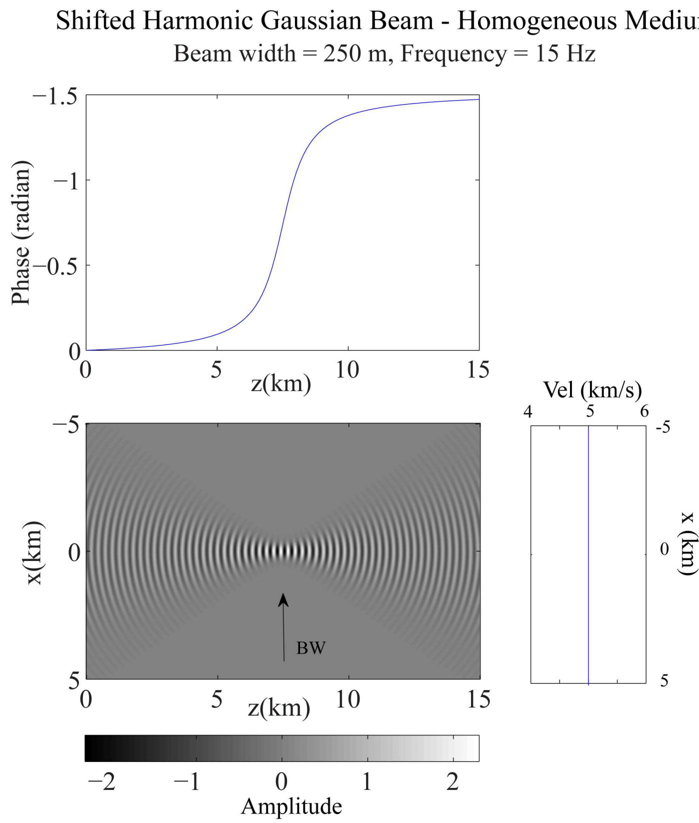

Fig. 3. *Harmonic Gaussian beam for a frequency of 15 Hz in a homogeneous medium. The beam-waist is located at 7.5 km with a beam-width of 250 m. The beam-field is shown at the bottom and the Gouy phase anomaly at the top. The beam-waist is indicated by BW.*

distance noted by RR in the plot. The remaining $-\pi/8$ is accumulated from the Rayleigh distance out to greater distances. For smaller initial beam-widths, the Rayleigh distance would be closer approaching $z = 0$, and more and more of the Gouy phase would be accumulated closer to the beam-waist. This would then give rise to the $-\pi/4$ phase shift for secondary wavelets used in the 2D Huygens-Fresnel integral.

Figure 2 shows a similar case with the beam-waist at $z = 0$, but now with a larger beam-width of 700 m. The beam is shown in the lower plot and spreads more slowly compared to the earlier example, now with a Rayleigh distance of 5.77 km. For this case the phase is not even up to $-\pi/4$ by 15 km at the right side of the model. Since this beam-width would be typical for Gaussian beams using for imaging in oil reservoirs, the Gouy phase anomaly would influence the beam calculations for all distances of interest for this application.

In Figure 3, the beam-waist is now located in the center of the model at 7.5 km noted by BW in the lower beam field plot, and the beam-width at the beam-waist is 250 m. This shows that the Guoy phase anomaly is continuous across the beam-waist, with half resulting from the converging beam to the beam-waist and half from the beam diverging from the beam-waist. As the beam-width shrinks to zero, the Gouy phase anomaly in the limit would approach the discontinuous $-\pi/2$ phase jump for ray theory at a line caustic. The Gouy phase anomaly thus provides a continuous wave solution to explain the phase jump at caustics for ray theory.

In Figure 4, a larger beam-width of 700 km is used with the beam-waist located at 7.5 km. The Gouy phase is now much more gradual over the entire propagation length of the beam. For this case, the complete $-\pi/2$ has still not accumulated over the 15 km of the model. Interestingly,

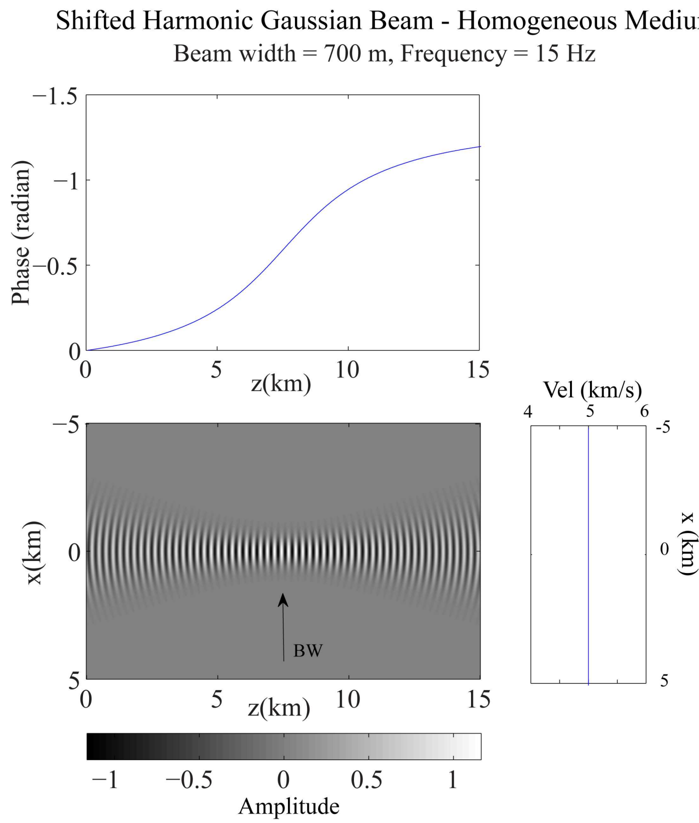

FIG. 4. *Harmonic Gaussian beam for a frequency of 15 Hz in a homogeneous medium. The beam-waist is located at 7.5 km with a beam-width of 700 m. The beam-field is shown at the bottom and the Gouy phase anomaly at the top. The beam-waist is indicated by BW.*

as the beam-width gets larger, the Gouy phase flattens with propagation distance but influences a greater and greater propagation distance of the beam. Thus for very slightly confined or "squeezed" plane waves, the Gouy phase will still be an important effect when considering large propagation distances.

5. Harmonic Gaussian beams in a quadratic waveguide. In order to investigate heterogeneous media, paraxial Gaussian beam solutions in a quadratic waveguide are now investigated. For given initial conditions, the dynamic ray equations can be solved either analytically or numerically by using the Runga-Kutta method. For the case of a quadratic velocity profile, the central ray along the waveguide will be a straight line, and a waveguide will be formed for a positive second derivative of the velocity in the x direction. The individual paraxial rays making up the beam solution will then fold back on themselves and form multiple focus points along the waveguide depending on the initial conditions.

Figure 5 shows a Gaussian beam guided along the waveguide for a frequency of 15 Hz and an initial beam-width of 200 m. The beam-field is shown in the lower plot with two focus points along the z axis, each noted by BW. The upper plot displays the Gouy phase anomaly which shows two flat areas separated by two areas of rapid phase advance associated with the beam focus locations. For the case of multiple focus points along the beam care needs to be taken to ensure that the phase is properly unwrapped to allow for a continuous phase advance with distance. For an isotropic media the Gouy phase anomaly will always be an advance, however, for anisotropic media there can be situations where the phase anomaly can decrease (Cerveny, 2000; Chapman, 2004).

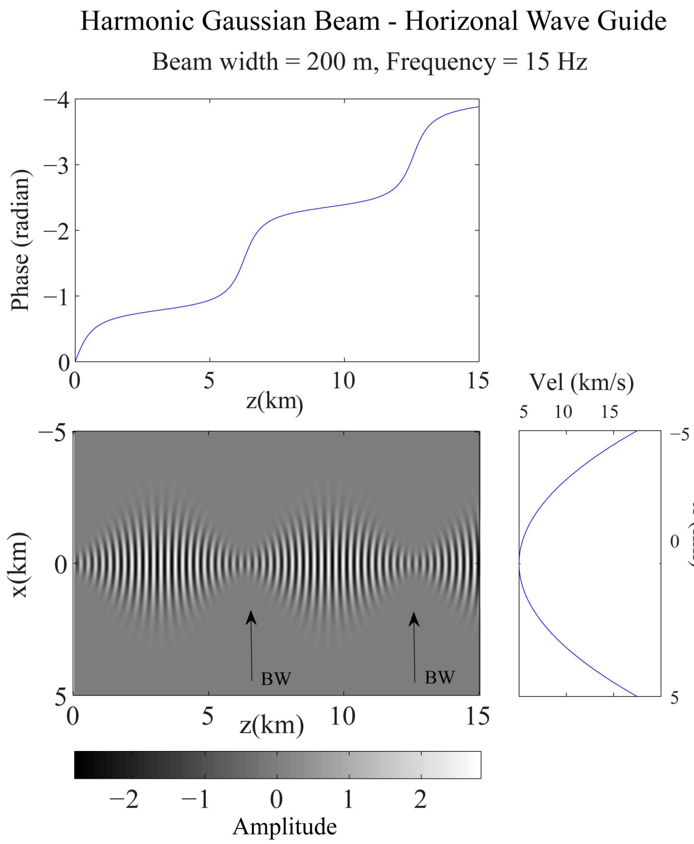

FIG. 5. *Harmonic Gaussian beam for a frequency of 15 Hz in a quadratic waveguide. The beam-waist $z = 0$ with a beam-width of 200 m. The beam-field is shown at the bottom where BW shows the beam focus locations. The Gouy phase anomaly is shown at the top.*

6. Time-domain Gaussian beams in homogeneous media. In order to investigate time-domain beam solutions, the harmonic results for different frequencies are convolved with a Gabor wavelet which can be written as

$$f(t) = \mathrm{e}^{-(2\pi f_M (t-t_i)/\gamma)^2} \cos(2\pi f_M (t - t_i) + \nu) \tag{6.1}$$

where f_M is the center frequency and γ determines the number of wave cycles under the Gaussian envelope (Cerveny, 2001). For the examples shown, $f_M = 15Hz$, $\gamma = 3.5$ and $\nu = 0$.

In Figure 6, the beam-waist is located at 7.5 km noted by BW in the lower beam-field plot with a beam-width of 200 m at the center frequency. The time-domain traces at different z distances are shown in the top plot. The vertical line shows the predicted travel-times $t = z/v$ for each centered Gabor wavelet, not accounting for the Gouy phase anomaly. An individual peak or trough on the traces can be seen to move to the left to earlier times as distance increases. In addition, there is a rapid increase in the phase advance between 6 and 9 km around the beam-waist. This slight advance in the travel-time can be intuitively understood as when a beam solution is confined or "squeezed" at a focus, it slightly "squirts" forward.

Figure 7 shows a similar result, but now for a beam-width of 700 m at the center frequency. Now the beam-field in the lower plot stays collimated for longer distances than in the previous example. The traces displayed in the top plot still show the time advance in the signals with distance, however, now the time advance is less abrupt at the focus distance. Nonetheless, all pulsed beam solutions will have this time advance with distance. This could potentially influence the interpretation of

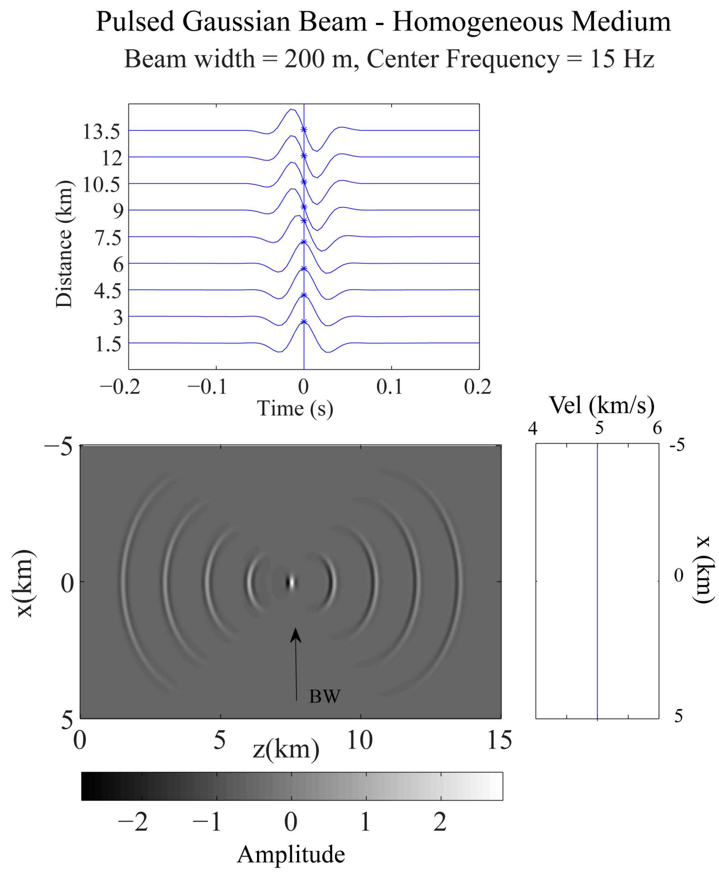

Fig. 6. *A time-domain Gaussian beam is shown using a Gabor wavelet for a center frequency of 15 Hz and a $\gamma = 4$ in a homogeneous medium for the beam-waist located at 7.5 km with a beam-width of 200 m at the center frequency. The beam-field is shown at the bottom and the time-domain traces for different z distances along the beam are shown at the top. The vertical line shows the predicted travel-times $t = z/v$ for each centered Gabor wavelet not accounting for the Gouy phase anomaly.*

travel-times when using sources that generate beamed signals, for example from transducer sources in the laboratory.

7. Conclusion. In this paper, we illustrate the Gouy phase anomaly using harmonic and time-domain Gaussian beams, where the Gouy phase affects all spatially confined solutions to the wave equation. For harmonic Gaussian beams, the Gouy phase anomaly varies continuously with distance. This is in contrast to ray solutions which have discontinuous phase anomalies at caustic points which are counted using the KMAH index. However, as the beam-width at a focus point gets smaller, the Gouy phase anomaly for beams will concentrate around the focus point and approach the discontinuous phase jump at a caustic for ray solutions. The propagation of Gaussian beams in heterogeneous media is illustrated using a quadratic velocity waveguide, where multiple focus points for beams can be generated. However, care must be taken to ensure that the phase is properly unwrapped to allow for a continuous Gouy phase anomaly with distance.

Finally, time-domain Gaussian beams are investigated using a Gabor wavelet. The travel-times of the time-domain signals are slightly advanced by the Gouy phase anomaly at focus points of a beam. Intuitively this can be understood as when a wave solution gets “squeezed” at a focus, it slightly “squirts” forward and increases its apparent speed in the propagation direction. This could be important for interpreting travel-times for experiments using beamed signals, such as from transducers in the laboratory. It could also have an effect on travel-times for Gaussian beams and

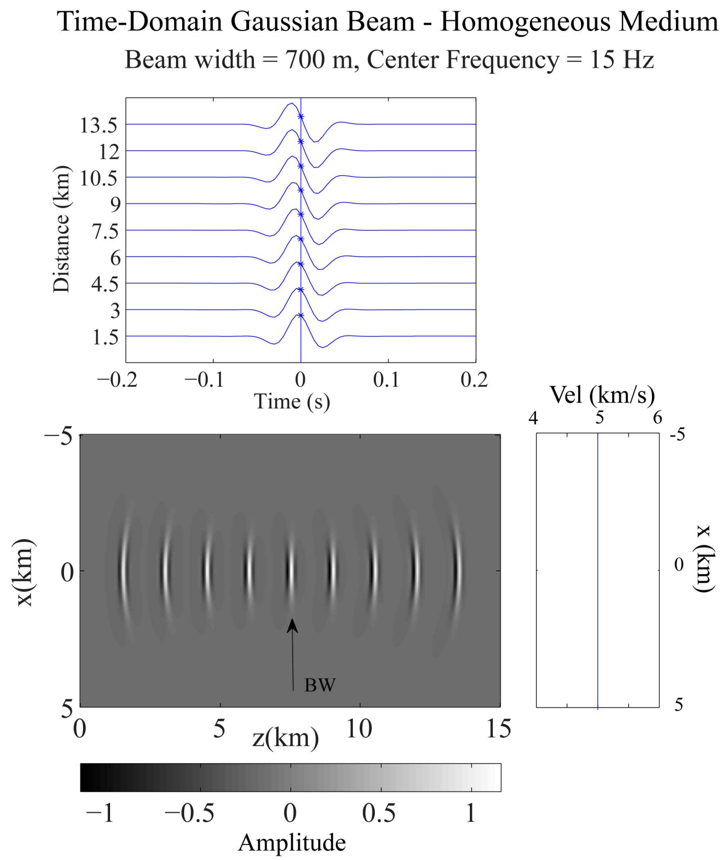

Fig. 7. *A time-domain Gaussian beam is shown using a Gabor wavelet with a center frequency of 15 Hz and a $\gamma = 4$ in a homogeneous medium for the beam-waist located at 7.5 km with a beam-width of 700 m at the center frequency. The beam-field is shown at the bottom and the time-domain traces for different z distances along the beam are shown at the top. The vertical line shows the predicted travel-times $t = z/v$ for each centered Gabor wavelet not accounting for the Gouy phase anomaly.*

other confined wave solutions used for seismic imaging.

Acknowledgments. This work was supported in part by the National Science Foundation and the Air Force Geophysics Laboratory and partly by the members of the Geo-Mathematical Imaging Group (GMIG) at Purdue University.

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