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ON TRUNCATED WIENER HOPF OPERATORS AND $BMO(\mathbb{Z})$

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Abstract. We give a tractable estimate for the norm of a truncated Wiener-Hopf operator in terms of the discrete BMO-space. We also improve earlier norm estimates as well as obtain new, more tractable, criteria for compactness.

Key words. Wiener-Hopf operators, Hankel operators, Toeplitz operators.

1. Introduction. The Wiener Hopf operators are defined by the expression

$$L^2(\mathbb{R}^+) \ni F \mapsto \int_0^\infty \Phi(y)F(x+y)dy \in L^2(\mathbb{R}^+)$$

where $\Phi \in L^1(\mathbb{R})$ or, more generally, certain distributios. Given an open interval $I \subset \mathbb{R}$ let $C_0^\infty(I) \subset C^\infty(\mathbb{R})$ denote the set of functions with support within I, and let $\mathcal{D}'(I)$ denote the set of distributions on I. The truncated Wiener Hopf operators, $W_{\Phi,a} : L^2([0,a]) \to L^2([0,a])$, are defined for any $a \in \mathbb{R}^+$ and any distribution $\Phi \in \mathcal{D}'((-a,a))$ by the expression

$$C_0^\infty((0,a)) \ni F \mapsto \Phi(F(\cdot + x)), \tag{1.1}$$

whenever Φ is such that this extends to a bounded operator on $L^2([0,a])$. Whenever a is of no importance we will omit it from the notation. We abbreviate by saying that W_Φ is a TWH-operator. These operators, (or rather, unitarily equivalent ones), also go under the name truncated Hankel operators or Toepliz operators on the Paley-Wiener space. We will in this paper see that the properties of TWH-operators are more similar to those of Hankel operators rather than Toeplitz operators. Let $\mathcal{F}$ denote the Fourier transform on $L^2(\mathbb{R})$, defined as follows

$$\mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(y)\mathrm{e}^{-\mathrm{i}xy}dy. \tag{1.2}$$

We will also use the notation $\hat{f} = \mathcal{F}(f)$ and $\check{f} = \mathcal{F}^{-1}(f)$. Whenever we apply the Fourier transform to a function that is only defined on an interval, it is understood that the function is taken to be zero outside of the interval. We let $P_{[0,a]} : L^2(\mathbb{R}) \to L^2([0,a])$ be the orthogonal projections onto $L^2([0,a])$. Finally, let $\widecheck{L^\infty} \subset \mathcal{D}'(\mathbb{R})$ be the image of $L^\infty(\mathbb{R})$ under $\mathcal{F}^{-1}$, where the transform is interpreted in the distributional sense.

If $\Phi \in L^1(\mathbb{R})$ has support in $(-a,a)$ then $\hat{\Phi} \in L^\infty(\mathbb{R})$ and it follows by standard results that

$$W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Phi}\check{F}). \tag{1.3}$$

Similarly, a calculation yields $W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Psi}\check{F})$ for all $\Psi \in \widecheck{L^\infty}$ such that $\Psi|_{(-a,a)} = \Phi$, where $\Psi|_{(-a,a)}$ denotes the restriction of (the distribution) Ψ to $C_0^\infty((-a,a))$. In particular, setting

$$C_\Phi = \inf\{\|\hat{\Psi}\|_{L^\infty} : \Psi \in \widecheck{L^\infty} \text{ and } \Psi|_{(-a,a)} = \Phi\} \tag{1.4}$$

we immediately obtain

$$\|W_\Phi\| \leqslant C_\Phi.$$

It was shown by R. Rochberg that the two quantities above are comparable. This fact has also recently appeared in [1], where compressions of Toeplitz operators are studied in a more general setting. Based on results by Farforovskaya and Nikolskaya we will improve the constant as follows.

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THEOREM 1.1. *Given any* $\Phi \in \mathcal{D}'((-a,a))$, *the operator* W_Φ *is bounded if and only if* $\Phi = \Psi|_{(-a,a)}$ *for some* $\Psi \in \widetilde{L^\infty}$. *In this case we have that*

$$W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Phi}\check{F}),$$

the infimum in (1.4) is attained and

$$\frac{C_\Phi}{3} \leqslant \|W_\Phi\| \leqslant C_\Phi.$$

However, C_Φ is not easy to compute. Another norm estimate is given in [7], which involves splitting Φ in 3 parts; left, center and right. Loosely speaking, the result says that $\|W_A\|$ is comparable with the BMO-norm of the Fourier transform of certain translations of the left and right part, plus the L^∞-norm of the Fourier transform of the center part. In [7] there is also given a norm estimate involving breaking up W_Φ in two pieces and the discrete $BMO(\mathbb{Z})$-space, defined below. The issue of finding a more tractable norm estimate was raised in [7]. Define $BMO(\mathbb{Z})$ as the space of all sequences σ such that the following semi-norm is finite;

$$\|\sigma\|_{BMO} = \sup_I |I|^{-1}\sum_{k\in I}|\sigma(k)-\sigma_I|.$$

Here $I \subset \mathbb{Z}$ ranges over all sets of the form $\{K_1 < k \leqslant K_2\}$, $(K_1, K_2 \in \mathbb{Z})$, $|I| = K_2 - K_1$ and $\sigma_I = |I|^{-1}\sum_{k\in I}\sigma(k)$.

THEOREM 1.2. *Given any* $\Phi \in L^1(-a,a)$ *set* $\Phi(x) = \sum_{k=-\infty}^{\infty}\phi_k e^{i\pi kx/a}$. *There exists constants* $C_1, C_2 > 0$ *such that*

$$C_1\|((-1)^k\phi_k)_k\|_{BMO(\mathbb{Z})} \leqslant \|W_\Phi\| \leqslant C_2\|((-1)^k\phi_k)_k\|_{BMO(\mathbb{Z})}.$$

The restriction to L^1 is for the introduction only, as there are some complications involved in defining the Fourier series for general elements of $\mathcal{D}'(-a,a)$. We also show that W_Φ is compact if and only if $(\phi_k)_k$ is in $CMO(\mathbb{Z})$ - the closure of the sequences with finite support in $BMO(\mathbb{Z})$.

The proof goes via a new class of "discrete" Hankel operators, which we now introduce. Let $\mathbb{T}$ denote the unit circle and set $\mathbb{T}^+ = \{\zeta \in \mathbb{T} : \operatorname{Im}\zeta > 0\}$, $\mathbb{T}^- = \{\zeta \in \mathbb{T} : \operatorname{Im}\zeta < 0\}$. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform $\mathcal{F}^{-1} : L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ defined via $\mathcal{F}^{-1}(f)(k) = \int_0^{2\pi} f(e^{it})e^{ikt}dt/2\pi$. (Note that we use the same symbol as in (1.2), the type of f will determine which one is intended.) The common denominator of the various Hardy spaces is that the Fourier transform of the elements are in some sense one-sided. It therefore makes some sense to define the discrete Hardy space $H^2(\mathbb{Z})$ as $\mathcal{F}^{-1}(L^2(\mathbb{T}^+))$ and similarly $H^2_-(\mathbb{Z}) = \mathcal{F}^{-1}(L^2(\mathbb{T}^-))$. (We will without comment interpret $L^2(\mathbb{T}^+)$ as functions on $\mathbb{T}$ that vanish on $\mathbb{T}^-$.) In analogy with the classical definition, given $\sigma \in l^\infty(\mathbb{N})$ we define the Hankel operator $H_\sigma : H^2(\mathbb{Z}) \to H^2_-(\mathbb{Z})$ via

$$H_\sigma(f) = P_{H^2_-(\mathbb{Z})}(\sigma \cdot f), \tag{1.5}$$

where $(\sigma \cdot f)(k) = \sigma(k)f(k)$, $\forall k \in \mathbb{Z}$. (We extend this definition to include certain unbounded symbols, but we omit this in the introduction.) Using the notation of Theorem 1.2, we show that W_Φ is equivalent with $H_{((-1)^k\phi_k)}$ under unitary transformations. Moreover, we show

THEOREM 1.3. *H_σ is bounded if and only if $\sigma \in BMO(\mathbb{Z})$ and the norms are comparable.* The proof relies on a characterization in [7] of the $BMO(\mathbb{Z})$-norm of a given σ in terms the operator-norm of an "infinite matrix" R_σ whose "(i,j)"'th entry is given by $\frac{\sigma(i)-\sigma(j)}{i-j}$.

2. A Nehari-type theorem for truncated Wiener-Hopf operators. Given $N \in \mathbb{N}$ and $\phi \in \mathbb{C}^{\{-N,\dots,N\}}$, we define the Toeplitz matrix by

$$T_\phi = \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_N \\ \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-1} \\ \phi_{-2} & \phi_{-1} & \phi_0 & \cdots & \phi_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{-N} & \phi_{-N+1} & \phi_{-N+2} & \cdots & \phi_0 \end{pmatrix}$$

We introduce yet a third meaning of $\mathcal{F}$; when acting on $\phi \in l^2$ we set $\mathcal{F}(\phi) = \sum_k \phi_k z^{-k}$. We will without comment let the type of a function/sequence/distribution s determine the meaning of $\mathcal{F}(s) = \hat{s}$ and $\check{s} = \mathcal{F}^{-1}(s)$. For convenience, we provide a table of the various Fourier transforms used in this paper. Let m denote the normalized arc-length measure on the unit circle $\mathbb{T}$.

$$\begin{cases} \mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R}), & \mathcal{F}(F)(x) = \int_{\mathbb{R}} F(y) e^{-ixy} dy; \quad \mathcal{F}^{-1}(F)(x) = \int_{\mathbb{R}} F(y) e^{ixy} \frac{dy}{2\pi} \\ \mathcal{F}: L^2(\mathbb{T}) \to l^2(\mathbb{Z}), & \mathcal{F}(F)(k) = \int_{\mathbb{T}} F(z) z^{-k} dm(z); \quad \mathcal{F}^{-1}(\sigma)(z) = \sum_k \sigma(k) z^k \\ \mathcal{F}: l^2(\mathbb{Z}) \to L^2(\mathbb{T}), & \mathcal{F}(\sigma)(z) = \sum_k \sigma(k) z^{-k}; \quad \mathcal{F}^{-1}(F)(k) = \int_{\mathbb{T}} F(z) z^k dm(z) \end{cases}$$

For any ϕ as above we set

$$C_\phi = \inf\{\|g\|_{L^\infty(\mathbb{T})} : g \in L^\infty(\mathbb{T}) \text{ and } \check{g}(k) = \phi_k\}. \tag{2.1}$$

We recall the following theorem by Yu. B. Farforovskaya and L. N. Nikolskaya [4].

THEOREM 2.1.

$$1/3 C_\phi \leqslant \|T_\phi\| \leqslant C_\phi.$$

Note that the right inequality is immediate because T_ϕ is the compression of the Toeplitz operator with symbol g, for any g that appears in (2.1). It is an open problem wether 1/3 is the best possible constant, but it is known that it is not 1. We will now prove Theorem 1.1. It is easy to see that it suffices to show Theorem 1.1 for a fixed value of a, so we set $a = 1$ for the remainder of this section. For $p \geqslant 1$ and any $N \in \mathbb{N}$ let the sampling operator $\mathcal{S}_{p,N} : C([-1,1]) \to \mathbb{C}^{\{-N+1,\dots,N-1\}}$ be defined by

$$\mathcal{S}_{p,N} F = \left(\frac{1}{N^{1/p}} F\left(\frac{k}{N}\right)\right)_{k=-N+1}^{N-1}. \tag{2.2}$$

Note that

$$\|F\|_{L^p} = \lim_{N\to\infty} \|S_{p,N}\|_{l^p}. \tag{2.3}$$

Given $F \in C([0,1])$ we instead let k range from 0 to $N-1$ in (2.2), and denote the corresponding operator by $\mathcal{S}_{p,N}$ as well. Whenever p, N are clear from the context, we will simply write $\mathcal{S}$. Note that if $\Phi \in C([-1,1])$ and $F \in C([0,1])$ we have

$$\big(T_{\mathcal{S}_{1,N}\Phi} \mathcal{S}_{\infty,N} F\big)(k) \approx \big(W_\Phi F\big)(k/N).$$

We will below show a stronger link between the two operators. Let $\chi(S,\cdot)$ denote the characteristic function of a set S. For each $N \in \mathbb{N}$, set

$$b_k^N(x) = \sqrt{N}\chi\left(\left[\frac{k}{N}, \frac{k+1}{N}\right), x\right)$$

and let $P_N : L^2([0,1]) \to L^2([0,1])$ be the orthogonal projection on the subspace spanned by $\{b_k^N\}_{k=0}^{N-1}$. Note that $\{b_k^N\}_{k=0}^{N-1}$ is an orthonormal set in $L^2([0,1])$. We also define $\mathcal{S}_{2,N}^{ri} : \mathbb{C}^{\{0,\dots,N-1\}} \to L^2([0,1])$ by

$$\mathcal{S}_{2,N}^{ri}(\phi) = \sum \phi_k b_k^N, \tag{2.4}$$

where ri stands for right-inverse. Indeed, it is easily verified that $\mathcal{S}_{2,N}\mathcal{S}_{2,N}^{ri}$ is the identity operator. Moreover we have $\mathcal{S}_{2,N}^{ri}\mathcal{S}_{2,N}P_N = P_N$. Note that the compression of the operator $\mathcal{S}_{2,N}^{ri}T_{\mathcal{S}_{1,N}\Phi}\mathcal{S}_{2,N}$ to the subspace $\mathsf{Ran}\ P_N$ is represented by the matrix $T_{\mathcal{S}_{1,N}\Phi}$ in the basis $\{b_k^N\}_{k=0}^{N-1}$. We shall show that for $\Phi \in C^1([-1,1])$, the operators $\mathcal{S}_{2,N}^{ri}T_{\mathcal{S}_{1,N}\Phi}\mathcal{S}_{2,N}$ converge to W_Φ as $N \to \infty$. In order to simplify the notation we set

$$W_\Phi^N = \mathcal{S}_{2,N}^{ri}T_{\mathcal{S}_{1,N}\Phi}\mathcal{S}_{2,N}.$$

For any $\epsilon > 0$ let $\rho_\epsilon : C^1([-1,1]) \to C([-1,1])$ be defined by

$$\rho_\epsilon(\Phi)(x) = \sup\{|\Phi'(y)| : |x-y| \leqslant \epsilon\}.$$

A version of the below proposition also appears in [2], but we include it for the sake of completeness.

Proposition 2.2. *Let $\Phi \in C^1([-1,1])$ be given. Then*

$$\|W_\Phi - W_\Phi^N\| \leqslant \left(1+\sqrt{\frac{1}{3}}\right)\frac{2}{N}\sqrt{\int_{-1}^1 \left|\rho_{2/N}(\Phi)(x)\right|^2 dx}.$$

Proof. Assume that Φ is real valued. We shall first give an estimate of $W_\Phi - W_\Phi^N$ restricted to $\mathsf{Ran}\ P_N$. For each fixed $0 \leqslant x \leqslant 1$ we have

$$W_\Phi b_k^N(x) = \sqrt{N}\int_{\frac{k}{N}-x}^{\frac{k+1}{N}-x}\Phi(y)dy = N^{-1/2}\Phi(y_x - x), \tag{2.5}$$

for some $\frac{k}{N} \leqslant y_x \leqslant \frac{k+1}{N}$ by the mean value theorem. On the other hand, note that $\mathcal{S}_{2,N}b_k^N = e_k$, (where $(e_k)_{k=0}^{N-1}$ denotes the standard basis in $\mathbb{C}^{\{0,\dots,N-1\}}$), so

$$W_\Phi^N b_k^N(x) = (\mathcal{S}_{2,N}^{ri}W_{\mathcal{S}_{1,N}\Phi}e_k)(x) = \sum_{l=0}^{N} N^{-1}\Phi\left(\frac{k-l}{N}\right)b_l^N(x) = N^{-1/2}\Phi\left(\frac{k-l_x}{N}\right)$$

where $l_x \in \mathbb{N}$ is such that $l_x/N \leqslant x < (l_x+1)/N$. Another application of the mean value theorem yields

$$\left|W_\Phi b_k^N(x) - W_\Phi^N b_k^N(x)\right| = \frac{1}{\sqrt{N}}\left|\Phi(y_x - x) - \Phi\left(\frac{k-l_x}{N}\right)\right| \leqslant \frac{1}{N^{3/2}}\rho_{1/N}(\Phi)(\frac{k}{N}-x). \tag{2.6}$$

Now, let $a \in \mathbb{C}^{\{0,\dots,N-1\}}$ be arbitrary but satisfy $\|a\| = 1$. Then

$$\begin{aligned}
&\left|\left(\left(W_\Phi - W_\Phi^N\right)\left(\sum a_k b_k^N\right)\right)(x)\right| = \left|\sum a_k\left(\left(W_\Phi - W_\Phi^N\right)b_k^N\right)(x)\right| \leqslant \\
&\leqslant N^{-3/2}\left|\sum a_k\rho_{1/N}(\Phi)(\frac{k}{N}-x)\right| = \frac{1}{N}\sqrt{\frac{1}{N}\sum\left(\rho_{1/N}(\Phi)(\frac{k}{N}-x)\right)^2} \leqslant \\
&\leqslant \frac{1}{N}\sqrt{\sum\int_{\frac{k}{N}}^{\frac{k+1}{N}}\left(\rho_{2/N}(\Phi)(y-x)\right)^2 dy} = \frac{1}{N}\sqrt{\int_0^1\left(\rho_{2/N}(\Phi)(y-x)\right)^2 dy}
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\|(W_\Phi - W_\Phi^N)\left(\sum a_k b_k^N\right)\|_{L^2} \leqslant \frac{1}{N}\sqrt{\int_0^1\int_0^1\left(\rho_{2/N}(\Phi)(y-x)\right)^2 dydx} = \\
&= \frac{1}{N}\sqrt{\int\limits_{\substack{0<u+v<2\\0<u-v<2}}\left(\rho_{2/N}(\Phi)(v)\right)^2\frac{1}{2}dudv} \leqslant \frac{1}{N}\sqrt{\int_{-1}^1\left(\rho_{2/N}(\Phi)(v)\right)^2 dv}
\end{aligned}$$

which, upon noting that $\|\sum a_k b_k\|_{L_2} = 1$, yields

$$\|(W_\Phi - W_\Phi^N)P_N\| \leqslant \frac{1}{N}\sqrt{\int_{-1}^{1} \left(\rho_{2/N}(\Phi)(x)\right)^2 dx. \tag{2.7}$$

We turn to the estimate for $(W_\Phi - W_\Phi^N)(I-P_N) = W_\Phi(I-P_N)$. Define subsets $S_{k,i,j}^N \subset [\frac{k}{N}, \frac{k+1}{N}]$ for $0 \leqslant k < N$, $j \geqslant 1$ and $0 \leqslant i \leqslant 2^j - 1$ by

$$S_{k,i,j}^N = \left[\frac{k}{N} + \frac{i}{2^j N}, \frac{k}{N} + \frac{i+1}{2^j N}\right],$$

and let $d_{k,i,j}^N$ be functions defined by

$$d_{k,i,j}^N(x) = \sqrt{2^{j-1}N}\left(\chi(S_{k,2i,j}^N, x) - \chi(S_{k,2i+1,j}^N, x)\right)$$

for $0 \leqslant k < N$, $j \geqslant 1$ and $0 \leqslant i \leqslant 2^{j-1} - 1$. It is easy to see that, (for N fixed),

$$\mathsf{Span}\ (\{b_k^N\} \cup \{d_{k,i,j}^N\}) = \mathsf{Span}\ (\{\chi_{S_{k,i,j}^N}\}),$$

and it follows from basic integration theory that the right hand side is dense in $L^2([0,1])$. Moreover, $\{b_k^N\} \cup \{d_{k,i,j}^N\}$ is clearly an orthonormal set, and hence it is a basis for $L^2([0,1])$. Thus $\mathsf{Ran}\ (I - P_N) = \mathsf{Span}\ \{d_{k,i,j}^N\}$. In a similar fashion as in (2.5) and (2.6) we obtain that there are $0 \leqslant \delta_1, \delta_2 \leqslant (2^j N)^{-1}$ such that

$$\begin{aligned}|W_\Phi d_{k,i,j}^N(x)| &= \frac{\sqrt{2^{j-1}N}}{2^j N}\left|\Phi\left(\frac{k}{N} + \frac{2i}{2^j N} + \delta_1 - x\right) - \Phi\left(\frac{k}{N} + \frac{2i+1}{2^j N} + \delta_2 - x\right)\right| \leqslant \\ &\leqslant \frac{1}{\sqrt{2^{j+1}N}}\left|\frac{1}{2^{j-1}N}\rho_{1/N}(\Phi)\left(\frac{k}{N} - x\right)\right| = \frac{\sqrt{2}}{(2^j N)^{3/2}}\left|\rho_{1/N}(\Phi)\left(\frac{k}{N} - x\right)\right|.\end{aligned}$$

Thus

$$\|W_\Phi d_{k,i,j}^N\| \leqslant \frac{\sqrt{2}}{(2^j N)^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N} - \cdot\right)\right\|.$$

Now let $a_{k,i,j} \in \mathbb{C}$ be any numbers (indexed by the same index set as $\{d_{k,i,j}^N\}$) such that $\sum |a_{k,i,j}|^2 = 1$. By repeated use of Cauchy-Schwartz and Minkowski's inequalities we get

$$\begin{aligned}\|W_\Phi\left(\sum a_{k,i,j} d_{k,i,j}^N\right)\| &\leqslant \sum_{k=0}^{N-1}\sum_{j=1}^{\infty}\sum_{i=0}^{2^{j-1}-1} |a_{k,i,j}| \frac{\sqrt{2}}{(2^j N)^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N} - \cdot\right)\right\| \leqslant \\ &\leqslant \sum_{k=0}^{N-1}\sum_{j=1}^{\infty}\sqrt{\sum_i |a_{k,i,j}|^2}\, 2^{(j-1)/2}\frac{\sqrt{2}}{(2^j N)^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N} - \cdot\right)\right\| \leqslant \\ &\leqslant \sum_{k=0}^{N-1}\sqrt{\sum_{j,i}|a_{k,i,j}|^2}\sqrt{\sum_{j\geqslant 1} 2^{-2j}}\,\frac{1}{N^{3/2}}\left\|\rho_{1/N}(\Phi)\left(\frac{k}{N} - \cdot\right)\right\| \leqslant \\ &\leqslant \sqrt{\frac{1}{1-1/4} - 1}\,\frac{1}{N^{3/2}}\sqrt{\sum_{k,j,i}|a_{k,i,j}|^2}\sqrt{\sum_k \int_0^1 \left|\rho_{1/N}(\Phi)\left(\frac{k}{N} - \cdot\right)\right|^2} \leqslant \\ &\leqslant \frac{1}{\sqrt{3}N}\sqrt{\int_0^1\int_0^1 \left|\rho_{2/N}(\Phi)(y-x)\right|^2 dydx} \leqslant \frac{1}{\sqrt{3}N}\sqrt{\int_{-1}^{1}|\rho_{2/N}(\Phi)(x)|^2 dx}.\end{aligned}$$

It follows that

$$\|(W_\Phi - W_\Phi^N)(I-P_N)\| \leqslant \frac{1}{\sqrt{3}N}\sqrt{\int_{-1}^{1}\left|\rho_{2/N}(\Phi)(x)\right|^2 dx} \tag{2.8}$$

which combined with (2.7) yields the first part of the proposition in the case when Φ is real valued, but with constant $(1+1/\sqrt{3})$. For the general case, write $\Phi = \Phi_1 + i\Phi_2$ where Φ_1 and Φ_2 are real valued. Then

$$\begin{aligned}
&\|W_\Phi - W_\Phi^N\| \leqslant \|W_{\Phi_1} - W_{\Phi_1}^N\| + \|W_{\Phi_2} - W_{\Phi_2}^N\| \leqslant \\
&\leqslant \left(1+\sqrt{\frac{1}{3}}\right)\frac{1}{N}\left(\sqrt{\int_{-1}^{1}\left|\rho_{2/N}(\Phi_1)(x)\right|^2 dx} + \sqrt{\int_{-1}^{1}\left|\rho_{2/N}(\Phi_2)(x)\right|^2 dx}\right) \leqslant \\
&\leqslant \left(1+\sqrt{\frac{1}{3}}\right)\frac{2}{N}\sqrt{\int_{-1}^{1}\left|\rho_{2/N}(\Phi)(x)\right|^2 dx}
\end{aligned}$$

as desired.
□

Corollary 2.3. *For every $\Phi \in C^1([-1,1])$ there exists a $C > 0$ such that*

$$\|T_{\mathcal{S}_{1,N}\Phi}\| \leqslant \|W_\Phi\| + \frac{C}{N}.$$

We are now in a position to prove the lower estimate in Theorem 1.1 for $\Phi \in C^1([-1,1])$. For standard results and definitions concerning distributions, we refer to [5]. We omit proofs of simple results such as that $L^\infty(\mathbb{R})$ can be considered as temperate distributions, so that $\widecheck{L^\infty}$ is well defined.

Proposition 2.4. *Given $a > 0$ and $\Phi \in C^1([-a,a])$ there exists a $\Psi \in \widecheck{L^\infty}$ with $\Psi|_{(-a,a)} = \Phi$ and*

$$\frac{1}{3}\|\hat{\Psi}\| \leqslant \|W_{\Phi,a}\|.$$

Proof. It is easily seen that it is sufficient to consider $a = 1$. By Corollary 2.3 we have $\|T_{\mathcal{S}_{1,N}\Phi}\| \leqslant \|W_\Phi\| + \frac{C}{N}$ for some C and every $N \in \mathbb{N}$, and by Theorem 2.1 we get that there exists $\hat{\psi}_N \in L^\infty(\mathbb{T})$ such that

$$\psi_N(k) = \frac{1}{N}\Phi(\frac{k}{N}), \quad -N+1 \leqslant k \leqslant N-1 \tag{2.9}$$

and

$$\|\hat{\psi}_N\|_{L^\infty} \leqslant 3\|T_{\mathcal{S}_{1,N}\Phi}\| \leqslant 3\|W_\Phi\| + \frac{3C}{N}.$$

Let $\Psi_N \in \mathcal{D}'(\mathbb{R})$ be defined by $\Psi_N = \sum_{k=-\infty}^{\infty}\psi_N(k)\delta_{k/N}$, where δ_x is the Dirac distribution at x, and note that

$$\widehat{\Psi_N}(t) = \sum_{k=-\infty}^{\infty}\psi_N(k)e^{-itk/N} = \hat{\psi}_N(e^{-it/N}),$$

so in fact $\Psi_N \in \widecheck{L^\infty}$ and $\|\widehat{\Psi_N}\|_{L^\infty} \leqslant \|W_\Phi\| + \frac{C}{N}$. By standard theorems of functional analysis there exists a subsequence of $(\widehat{\Psi_{N_j}})_{j=1}^\infty$ convergent in the weak-star topology to some $\widehat{\Psi} \in L^\infty(\mathbb{R})$ with $\|\widehat{\Psi}\|_{L^\infty} \leqslant 3\|W_\Phi\|$. Given $F \in C_0^\infty((-1,1))$ we have, using (2.9)

$$\int \Phi F = \lim_{N\to\infty}\int \Psi_N F = \lim_{j\to\infty}\int \Psi_{N_j}F = \lim_{j\to\infty}\int \widehat{\Psi_{N_j}}\check{F} = \int \hat{\Psi}\check{F} = \int \Psi F \tag{2.10}$$

which shows that $\Psi|_{(-1,1)} = \Phi$, and the proof is complete. Note that we use the notation $\int \Psi_N F$ even if Ψ_N is not a function, as opposed to the formally correct $\Psi_N(F)$ or $\langle \Psi_N, F\rangle$. We will in the remainder do this without comment. □

THEOREM 2.5. *Given any $a > 0$ and $\Phi \in \mathcal{D}'((-a,a))$, the operator W_Φ is bounded if and only if $\Phi = \Psi|_{(-a,a)}$ for some $\Psi \in \widecheck{L^\infty}$. In this case,*

$$W_\Phi F = P_{[0,a]}\mathcal{F}(\hat{\Psi}\check{F}),$$

the infimum in (1.4) is attained and

$$\frac{C_\Phi}{3} \leqslant \|W_\Phi\| \leqslant C_\Phi.$$

Proof. All claims in the statement follow by standard arguments once we show that given a $\Phi \in \mathcal{D}'((-1,1))$ such that W_Φ is bounded, there exists a $\Psi \in \widecheck{L^\infty}$ such that $\|\hat{\Psi}\|_{L^\infty} \leqslant 3\|W_\Phi\|$ and $\Psi|_{(-1,1)} = \Phi$. We will only prove this. Take a positive function $\eta \in C_0^\infty((-1,1))$ that is symmetric around 0 and satisfies $\|\eta\|_{L^1} = 1$, take a sequence of positive functions $\gamma_k \in C_0^\infty((-1,1))$ such that $\gamma_k(x) = 1$ for $-1 + 4^{-k} < x < 1 - 4^{-k}$, set $\eta_k(x) = 4^k\eta(4^k x)$ and define $\Phi_k \in C_0^\infty(\mathbb{R})$ via

$$\Phi_k(x) = (\gamma_k\Phi) * \eta_k,$$

which is well-defined as $\gamma_k\Phi$ has a natural extension to $\mathbb{R}$ that is "zero on $\mathbb{R}\backslash(-1,1) = 0$", (see [5], Sec 2.3 for details). We have, for any $F \in L^2([2^{-k}, 1-2^{-k}])$, that $\eta_k * F \in C_0^\infty((4^{-k}, 1-4^{-k}))$ and $\hat{\eta_k}\check{F} = \widecheck{\eta_k * F}$, which follows by the symmetry of η. Moreover, by standard properties of distributions with compact support (see [5], Ch. 2 and 7) we get $\widehat{\Phi_k} = \widehat{(\gamma_k\Phi)}\hat{\eta_k}$, where $\widehat{(\gamma_k\Phi)}$ is a function that grows polynomially since distributions with compact support have finite order. Thus, for $F \in C_0^\infty((-1,1))$ we get

$$W_{\Phi_k}(F) = P_{L^2([0,1])}\mathcal{F}\big(\widehat{(\gamma_k\Phi)}\hat{\eta_k}\check{F}\big) = P_{L^2([0,1])}\mathcal{F}\big(\widehat{(\gamma_k\Phi)}\widecheck{(\eta_k * F)}\big) = W_{\gamma_k\Phi}(\eta_k * F) = W_\Phi(\eta_k * F).$$

Now let $\tilde{\Phi_k}(t) = \Phi_k(2^{-k} + t)$ for $|t| < 1 - 2^{1-k}$. The above identity yields, for any $F \in L^2([0, 1-2^{1-k}])$, the following estimate

$$\|W_{\tilde{\Phi_k},(1-2^{1-k})}(F)\| \leqslant \|W_{\Phi_k,1}(F(\cdot - 2^{-k}))\| = \|W_\Phi(\eta_k * F(\cdot - 2^{-k}))\| \leqslant \|W_\Phi\|\|\eta_k\|_{L^1}\|F\|_{L^2}.$$

In particular, $\|W_{\tilde{\Phi_k},(1-2^{1-k})}\| \leqslant \|W_\Phi\|$. By Proposition 2.4 we get the existence of $\Psi_k \in \widecheck{L^\infty}$ with $\Psi_k(t) = \Phi_k(2^{-k} + t)$ for $|t| \leqslant 1 - 2^{1-k}$ and $\|\hat{\Psi}_k\|_{L^\infty} \leqslant 3\|W_\Phi\|$. For any $F \in C_0^\infty((-1,1))$ and k large enough that $\operatorname{supp} F \subset [-1 + 2^{1-k}, 1 - 2^{1-k}]$ we get

$$\int \Psi_k F = \int \Phi_k F(\cdot - 2^{-k}) = \Phi\Big(\gamma_k\big(\eta_k * (F(\cdot - 2^{-k}))\big)\Big) = \Phi\big(\eta_k * (F(\cdot - 2^{-k}))\big).$$

Let $\mathcal{D}(-1,1)$ denote the set of test functions on $(-1,1)$ with the usual topology. It is a standard matter to check that $(\eta_k * (F(\cdot - 2^{-k}))$ goes to F in $\mathcal{D}((-1,1))$ as $k \to \infty$, and hence Ψ_k converge to Φ in $\mathcal{D}'((-1,1))$. The proof is now easily completed with a similar calculation as (2.10), we omit the details. □

3. Technicalities. We will in this section collect a number of definitions and results, mainly from distribution theory, that would otherwise disrupt the flow of the text. The first 7 chapters of [5] contain all the necessary background material.

LEMMA 3.1. *Given any $\Phi \in \mathcal{D}'(-a,a)$ such that W_Φ is bounded, there exists a sequence $\Psi_1, \Psi_2, \ldots$ such that $\Psi_k \in C^\infty(\mathbb{R})$, $\|\Psi_k\| \leqslant 3\|W_\Phi\|$ and $\lim_{k\to\infty} \Psi_k = \Phi$ in $\mathcal{D}'(-a,a)$.*

Proof. The proof of Theorem 2.5 almost provides such a sequence, the only issue is that we do not know that the Ψ_k's produced there are functions near the edges. However, it is easily seen that this issue can be resolved by considering the sequence $(\Psi_k * \eta_k)_{k=1}^{\infty}$ instead. □

Given an open interval I, let $\bar{I}$ denote the closure of I and let $|I|$ denote its length. We define $C_I^1(\mathbb{R})$ as the set of functions F on $\mathbb{R}$ with support in $\bar{I}$ such that $F|_I$ is continuously differentiable and $F'|_I$ has a continuous extension to $\bar{I}$. Note that both F and F' (as functions on $\mathbb{R}$) are allowed to have discontinuities at the endpoints of I. We make $C_I^1(\mathbb{R})$ into a Banach space by giving it the norm $\|F\|_{C_I^1} = \|F\|_{L^\infty} + \|F'\|_{L^\infty}$. Moreover, we let $C_{I,0}^1(\mathbb{R}) \subset C_I^1(\mathbb{R})$ consist of those elements that are continuous at the endpoints of I. This notation is not to be confused with e.g. $C_0^\infty(I)$, which means C^∞-functions with support inside of I.

Lemma 3.2. *Given any interval I and $F \in C_{I,0}^1(\mathbb{R})$ we have $\hat{F} \in L^1(\mathbb{R})$ and*

$$\|\hat{F}\|_{L^1} \leqslant (2 + |I|^{3/2})\|F'\|_{L^2}.$$

Proof. For $F \in C_{I,0}^1(\mathbb{R})$ we have $F(x) = \int_{-\infty}^x F'(y)dy$ so $\|F\|_{L^\infty} \leqslant \|F'\|_{L^2}\sqrt{|I|}$ by Hölder's inequality and therefore $\|\hat{F}\|_{L^\infty} \leqslant |I|^{3/2}\|F'\|_{L^2}$. Moreover $\widehat{F'}(\xi) = i\xi\hat{F}(\xi)$ so by Parselval's formula and Cauchy-Schwartz inequality we get

$$\int_{|\xi|>1/2} |\hat{F}(\xi)|d\xi = \int_{|\xi|>1/2} |\hat{F'}(\xi)/\xi|d\xi \leqslant \sqrt{2\int_{1/2}^{\infty} \xi^{-2}d\xi}\|\hat{F'}\|_{L^2} \leqslant 2\|F'\|_{L^2}.$$

These combined easily gives the desired inequality. □

Lemma 3.3. *Let I be an open interval and let $\Phi \in \mathcal{D}'(I)$ be such that $\Phi = \Psi|_I$ for some $\Psi \in \widecheck{L^\infty}$. Then Ψ determines a continuous extension of Φ to $C_{I,0}^1(\mathbb{R})$ via*

$$\Phi(F) = \int_{\mathbb{R}} \hat{\Psi}\check{F}.$$

This extension is independent of Ψ.

Proof. By Lemma 3.2 we clearly have that $F \mapsto \int \hat{\Psi}\check{F}$ defines a continuous linear functional on $C_{J,0}^1(\mathbb{R})$ for any interval J, so by the Riesz representation theorem there exists a $\sigma-$finite Borel-measure ν on $\mathbb{R}$ such that

$$\int \hat{\Psi}\check{F} = \int F'd\nu \tag{3.1}$$

for all $F \in C_{J,0}^1(\mathbb{R})$ and all intervals J. Moreover, standard computations show that (3.1) coincides with Φ on $C_0^\infty(I)$. Now, suppose that $\Psi_1|_I = \Psi_2|_I = \Phi$ and let ν_1 and ν_2 be corresponding measures as in (3.1). It is easy to see that $\nu_1|_I - \nu_2|_I = c\lambda|_I$ for some $c \in \mathbb{C}$, where λ denotes the Lebesgue measure on $\mathbb{R}$. By (3.1), the uniqueness follows if we show that ν_1 and ν_2 are absolutely continuous.

Let ν be as in (3.1) and assume that it is not absolutely continuous with respect to Lebesgue measure λ. Let E be a set such that $|E| = 0$ but $\nu(E) > 0$. Since each finite Borel-measure is regular, (see e.g [3]), we may assume that E is compact. Let $(F_k)_{k=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R})$ such that $F_k(x) = 1$ for all $x \in E$, $\lim_{k\to\infty} |\operatorname{supp} F_k| = 0$ and each F_k is monotonously increasing or decreasing between values of 0 and 1. Also suppose that $F_{k+1} \leqslant F_k$. By the regularity of ν, it follows that

$$\nu(E) = \lim_{k\to\infty} \int F_k d\nu.$$

Now take a point $x \in \mathbb{R}\backslash E$ such that $\nu(\{x\}) = 0$. Let $G_k \in C_0^\infty(\mathbb{R})$ be functions, bounded by 2 and supported on intervals around x of length $\int F_k(x)dx$ such that $\int G_k(x)dx = \int F_k(x)dx$. Define

$H_k(x) = \int_{-\infty}^{x} F_k(y) - G_k(y)$. Then $H_k \in C_0^1(\mathbb{R})$ and, if I is an interval that contains the support of all H_k's, we have

$$|\nu(E)| = \left|\lim_{k\to\infty} \int H_k' d\nu\right| = \left|\int \hat{\Psi}\check{H}_k\right| \leqslant 2\pi(2 + |I|^{3/2})\|\hat{\Psi}\|_{L^\infty}\|H_k'\|_{L^2}$$

by Lemma 3.2. This is a contradiction, because it is easy to see that $\lim_{k\to\infty}\|H_k'\|_{L^2} = 0$.
□

Given $\Phi \in \mathcal{D}'(I)$ such that $\Phi = \Psi|_I$ for some $\Psi \in \widecheck{L^\infty}$, we will due to the above lemma in the future identify Φ with its extension to $C_{I,0}^1(\mathbb{R})$.

LEMMA 3.4. *Let $\Psi \in \widecheck{L^\infty}$ be given and set $\Phi = \Psi|_{(-a,a)}$. For any $F, G \in C_{(0,a)}^1$ we have $\int F(\cdot + y)G(y)dy \in C_{(-a,a),0}^1$ and*

$$\langle W_\Phi F, \bar{G}\rangle_{L^2([0,a])} = \Psi\left(\int F(\cdot + y)G(y)dy\right).$$

Proof. For $F \in C_0^\infty((0,a))$ the formula can be shown e.g. by approximating the integrals with Riemann sums. We omit the details and assume that this has been established. Let $\gamma_k \in C_0^\infty((0,a))$ be functions such that $\gamma_k(x) = 1$ for $1/k \leqslant x \leqslant a - 1/k$ and moreover, $0 \leqslant \gamma_k(x) \leqslant 1$ and $|\gamma_k'(x)| \leqslant 2k$ for all x. Also let $\gamma_k'(x) \geqslant 0$ for $0 \leqslant x \leqslant 1/k$ and $\gamma_k'(x) \leqslant 0$ for $a - 1/k \leqslant x \leqslant a$. Clearly $\lim_{k\to\infty}\langle W_\Phi(\gamma_k F), \bar{G}\rangle = \langle W_\Phi F, \bar{G}\rangle$. Lets write $F \circ G$ for $\int F(\cdot + y)G(y)dy$. Then

$$\frac{d}{dx}\left(F\circ G - (\gamma_k F)\circ G\right) = \big(F'\circ G - (\gamma_k F')\circ G\big) + \big((F(0)\delta_0 - F(a)\delta_a)\circ G - (\gamma_k' F)\circ G\big).$$

The first parenthesis converges to zero in L^∞. The second term is easily seen to be bounded by $\|F'\|_{L^\infty}\|G\|_{L^\infty}/k$ in the intervals $(-a + 1/k, -1/k)$ and $(1/k, a - 1/k)$, whereas it is bounded by $4\|F\|_{L^\infty}\|G\|_{L^\infty}$ everywhere. Thus $\lim_{k\to\infty}\|\big((\gamma_k F)\circ G\big)' - (F\circ G)'\|_{L^2} = 0$ so by Lemma 3.2 and 3.3 we obtain

$$\begin{aligned}\langle W_\Phi F, \bar{G}\rangle = \lim_{k\to\infty}\langle W_\Phi(\gamma_k F), \bar{G}\rangle = \lim_{k\to\infty}\Phi\Big((\gamma_k F)\circ G\Big) = \\ = \lim_{k\to\infty}\int \hat{\Psi}\mathcal{F}^{-1}\big((\gamma_k F)\circ G\big) = \int \hat{\Psi}\widecheck{F\circ G} = \Psi(F\circ G),\end{aligned}$$

as desired. □

It will be convenient to move the discussion to the circle, so we will introduce a new class of operators, unitarily equivalent with the TWH-operators, which resemble Hankel operators. Recall that m denotes the normalized arc-length measure on the unit circle $\mathbb{T}$, set $\mathbb{T}^+ = \{\zeta \in \mathbb{T} : \mathrm{Re}\,\zeta > 0\}$ and define $\mathbb{T}^-$ analogously. Write $L^2(\mathbb{T}^+)$ for the L^2-space on $\mathbb{T}^+$ with measure m. As before we will let the meaning of expressions like $\mathcal{F}^{-1}(f)$ be determined by the type of f; if $f \in l^2(\mathbb{Z})$ then $\mathcal{F}^{-1}(f)(z) = \sum f(k)z^k$, $z \in \mathbb{T}$, and if $f \in L^2(\mathbb{T})$ then $\mathcal{F}^{-1}(f)(k) = \int_{\mathbb{T}} f z^k dm$, $k \in \mathbb{Z}$.

Given $\Theta \in \mathcal{D}'(\mathbb{T}\backslash\{1\})$ we define the operator $\Gamma_\Theta : L^2(\mathbb{T}^+) \to L^2(\mathbb{T}^-)$ by

$$F \ni C_0^\infty(\mathbb{T}^+) \to \int \Theta(\zeta)F(z\zeta)dm(\zeta), \quad z \in \mathbb{T}^- \tag{3.2}$$

whenever this extends to a bounded operator on $L^2(\mathbb{T}^+)$. Formally, we should write $\Theta(F(z\ \cdot))$ instead of $\int \Theta(\zeta)F(z\zeta)dm(\zeta)$, but we believe that the latter notation is more readable and therefore we will continue to abuse notation in the above way. The reader should keep in mind that Θ not necessarily is a function.

We now show that this new class is unitarily equivalent with the set of TWH-operators (for any fixed a). It will be convenient to set the value of a to $1/2$. Thus let $\Phi \in \mathcal{D}'(-1/2, 1/2)$ be given such that $W_\Phi : L^2([0,1/2]) \to L^2([0,1/2])$ is bounded. Set

$$\Theta(e^{2\pi iy}) = \Phi(1/2 - y), \quad 0 < y < 1. \tag{3.3}$$

Given any $F \in L^2([0,1/2])$ we also define $\tilde{F} \in L^2(\mathbb{T}^+)$ via $\tilde{F}(e^{2\pi it}) = F(1/2-t)$ and note that this transformation is unitary. Moreover, for any $0 < x < 1/2$ and $F \in C_0^\infty((0,1/2))$ we have

$$
\begin{aligned}
(W_\Phi F)(x) &= \int_0^{1/2} \Phi(y-x)F(y)dy = \int_0^{1/2} \Phi(1/2-y-x)F(1/2-y)dy = \\
&= \int_0^{1/2} \Theta(e^{2\pi ix}e^{2\pi iy})\tilde{F}(e^{2\pi iy})dy = \int_{\mathbb{T}^+} \Theta(\bar{z}\zeta)\tilde{F}(\zeta)dm(\zeta) = \Gamma_\Theta(\tilde{F}), \quad z = e^{-2\pi ix}.
\end{aligned} \tag{3.4}
$$

Thus Γ_Θ and W_Φ are equivalent under simple unitary transformations. Endow $C^1(\mathbb{T})$ with the norm

$$\|F\|_{C^1(\mathbb{T})} = \|F\|_{L^\infty} + \|F'\|_{L^\infty}.$$

Here, and in the remainder of the paper, $F'(z)$ denotes $\lim_{t\to 0}(F(ze^{it}) - F(z)/t)$ whenever F is a function on $\mathbb{T}$. By Theorem 2.5, Lemma 3.3 and the Hahn-Banach theorem we conclude that it is no restriction to assume that $\Theta \in \left(C^1(\mathbb{T})\right)^*$ in the definition of Γ_Θ, (3.2). For any $\Theta \in \left(C^1(\mathbb{T})\right)^*$, the Riesz representation theorem and the Hahn-Banach theorem show that there exists a finite measure μ on $\mathbb{T}$ and a constant $c \in \mathbb{C}$ such that

$$\Theta(F) = \int F' d\mu + cF(1), \tag{3.5}$$

and $\|\mu\| \leqslant \|\Theta\|$, where $\|\mu\|$ denotes the variational norm for measures. Given two measures μ_1, μ_2 representing Θ as above, it is also not hard to see that $\mu_1 - \mu_2 = c'm$, where $c' \in \mathbb{C}$ is a constant. Let $C_0^1(\mathbb{T}\backslash\{1\})$ be the set of functions in $C^1(\mathbb{T})$ with support in $\mathbb{T}\backslash\{1\}$. The formula (3.5) also holds for any $\Theta \in \left(C_0^1(\mathbb{T}\backslash\{1\})\right)^*$, and it is easy to see that μ is uniquely defined by Θ except for the value of $\mu(\{1\})$ and multiples of m.

Definition 3.5. *Given $\Theta \in \left(C_0^1(\mathbb{T}\backslash\{1\})\right)^*$ let μ be the unique measure such that $\mu(\{1\}) = 0$, $\mu(\mathbb{T}) = 0$ and (3.5) holds with $c = 0$. The extension of Θ to $(C^1(\mathbb{T}))^*$ given by $\Theta(F) = \int F' d\mu$ will be called the canonical extension of Θ.* Given $\Phi \in \mathcal{D}'((-1/2,1/2))$ such that W_Φ is bounded, it follows by Theorem 2.5 and Lemma 3.3 that the distribution Θ defined via (3.3) is in $\left(C_0^1(\mathbb{T}\backslash\{1\})\right)^*$. We denote its canonical extension by Θ as well and we define $\mathcal{C}$ to be the map such that $\Theta = \mathcal{C}(\Phi)$. We record for future reference the following immediate consequence of the proof of Lemma 3.3.

Lemma 3.6. *If $\Phi = \Psi|_{(-1/2,1/2)}$ where $\Psi \in \breve{L^\infty}$, then the μ that gives $\mathcal{C}(\Phi)$ via (3.5) is absolutely continuous.* The next lemma is also almost immediate.

Lemma 3.7. *Let $\Psi \in \breve{L^\infty}$ be given and set $\Theta = \mathcal{C}(\Psi|_{(-1/2,1/2)})$. Given $\{F \in C^1(\mathbb{T}) : F(1) = 0\}$ we define $\tilde{F}$ via $\tilde{F}(1/2-t) = F(e^{2\pi it})$. Then*

$$\Psi(\tilde{F}) = \Theta(F),$$

where $\Psi(\tilde{F})$ is defined as in Lemma 3.3.

Proof. By definition, the formula is certainly true for $F \in C_0^\infty(\mathbb{T}\backslash\{1\})$. This combined with (3.1), (3.5) and the absolute continuity of μ and ν easily implies the general case. □

Let $z : \mathbb{T} \to \mathbb{T}$ denote the identity function $z(\zeta) = \zeta$. Given a distribution $\Theta \in \mathcal{D}'(\mathbb{T})$ we define the Fourier transform as the sequence $\hat{\Theta} \in \mathbb{C}^{\mathbb{Z}}$ given by

$$\hat{\Theta}(j) = \Theta(z^{-j}),$$

and similarly $\check{\Theta}(j) = \Theta(z^j)$. For $\Theta \in \left(C_0^1(\mathbb{T}\backslash\{1\})\right)^*$ we define $\hat{\Theta}$ to be the Fourier transform of the canonical extension. Note that in the case when Θ is a function in $L^1(\mathbb{T})$, this definition can disagree with the traditional definition by a constant sequence. Similarly, if $\Phi \in \mathcal{D}'((-1/2,1/2))$ is such that W_Φ is bounded and $\Theta = \mathcal{C}(\Phi)$, then a short calculation shows that it is natural to define $\check{\Phi}(k) = (-1)^k\hat{\Theta}(k)$. Again, for functions Φ this definition can be off by a constant sequence with respect to the usual definition. We omit the proof of the next result.

LEMMA 3.8. *If $\Theta \in \mathcal{D}'^k(\mathbb{T})$ then there exists $C > 0$ such that $|\hat{\Theta}(j)| < C|j|^k$. Conversely, if $\sigma = (\sigma_j)_{j=-\infty}^{\infty} \in \mathbb{C}^{\mathbb{Z}}$ satisfies $|\sigma_j| < C|j|^k$ for some $C > 0$ and $k \in \mathbb{N}$, then there is a unique $\Theta \in \mathcal{D}'^{k+2}(\mathbb{T})$ such that $\sigma = \hat{\Theta}$.* Sequences σ as in the above lemma will be called polynomially bounded.

4. Discrete Hardy spaces, Hankel operators and $BMO(\mathbb{Z})$**..** Recall the definition of $H^2(\mathbb{Z})$ and the discrete Hankel operators defined in (1.5). We extend this definition to polynomially bounded sequences σ by setting

$$H_\sigma(f) = P_{H^2_-(\mathbb{Z})}(\sigma \cdot f), \quad f \in \mathcal{F}^{-1}\left(C_0^\infty(\mathbb{T}^+)\right), \tag{4.1}$$

whenever this extends to a bounded operator on $H^2(\mathbb{Z})$. Note that $\mathcal{F}^{-1}P_{L^2(\mathbb{T}_-)} = P_{H^2_-(\mathbb{Z})}\mathcal{F}^{-1}$. Given $\Theta \in \mathcal{D}'(\mathbb{T})$ and $F \in C^\infty(\mathbb{T}_+)$, we will use the notation $\Theta(F(z\,\cdot)) = \Theta(F(z\zeta))$, i.e. we think of Θ as acting on functions in the variable ζ. By standard results about distributions we get that

$$\begin{aligned} &\mathcal{F}^{-1}\big(\Gamma_\Theta(F)\big) = P_{H^2_-(\mathbb{Z})}\mathcal{F}^{-1}\left(\Theta(F(z\zeta))\right) = \\ &= P_{H^2_-(\mathbb{Z})}\left(\left(\int \Theta\Big(F(z\zeta)\Big)z^k dm(z)\right)_{k=-\infty}^{\infty}\right) = P_{H^2_-(\mathbb{Z})}\left(\left(\Theta\Big(\int F(z\zeta)z^k dm(z)\Big)\right)_{k=-\infty}^{\infty}\right) = \\ &= P_{H^2_-(\mathbb{Z})}\left(\left(\Theta\big(\zeta^{-k}\check{F}(k)\big)\right)_{k=-\infty}^{\infty}\right) = P_{H^2_-(\mathbb{Z})}\left(\hat{\Theta}\cdot\check{F}\right). \end{aligned} \tag{4.2}$$

Upon noting that $\check{F} \in H^2(\mathbb{Z}^+)$ we get

$$\mathcal{F}^{-1}\Gamma_\Theta\mathcal{F} = H_{\hat{\Theta}} \tag{4.3}$$

in analogy with the classical theory. Conversely, given H_σ for some polynomially bounded sequence σ, we can define a distribution $\Theta \in \mathcal{D}'(\mathbb{T})$ via Lemma 3.8 and by (4.2) we get $\mathcal{F}^{-1}\Gamma_\Theta\mathcal{F} = H_\sigma$. We summarize a number of elementary observations in the following 3 propositions.

PROPOSITION 4.1. *Let $\Phi \in \mathcal{D}'((-1/2, 1/2))$ be such that W_Φ is bounded, set $\Theta = \mathcal{C}(\Phi)$ and set $\sigma = \hat{\Theta}$. Then W_Φ, Γ_Θ and H_σ are unitarily equivalent and $\Phi \in \mathcal{D}'^1((-1/2, 1/2))$. If Γ_Θ is bounded for some $\Theta \in \mathsf{Ran}\ \mathcal{C}$, or if $\sigma \in \mathbb{C}^{\mathbb{Z}}$ satisfies $\lim_{k\to\pm\infty}\sigma(k)/k = 0$, $\sigma(0) = 0$ and H_σ is bounded, then there exists a unique $\Phi \in \mathcal{D}'((-1/2, 1/2))$ such that the first statement is true.*

Proof. The only part of the Proposition that is not immediate from the previous developments is that the first statement is true if $\lim_{k\to\pm\infty}\sigma(k)/k = 0$, $\sigma(0) = 0$ and H_σ is bounded. We prove this. By Lemma 3.8 and (4.3) there exists a $\Theta \in \mathcal{D}'(\mathbb{T})$ such that Γ_Θ and H_σ are unitarily equivalent, with $\sigma = \hat{\Theta}$. Moreover, by (3.4) and Theorem 1.1 there is a $\Psi \in \breve{L^\infty}$, $\Phi = \Psi|_{(-1/2,1/2)}$ such that W_Φ which is unitarily equivalent with Γ_Θ via (3.4). But setting $\tau = \widehat{\mathcal{C}(\Phi)}$, W_Φ is also equivalent with $\Gamma_{\mathcal{C}(\Phi)}$ and H_τ under the usual unitary transformations. Let μ be the measure that gives $\mathcal{C}(\Phi)$ via (3.5), which by Lemma 3.6 is absolutely continuous. By the Riemann-Lebesgue lemma it follows that $\lim_{k\to\pm\infty}\hat{\mu}(k) = 0$. Thus

$$\lim_{k\to\pm\infty}\frac{\tau(k)}{k} = \lim_{k\to\pm\infty}\frac{\widehat{\mathcal{C}(\Phi)}(k)}{k} = \lim_{k\to\pm\infty} -i\hat{\mu}(k) = 0.$$

We are done if we show that $\sigma = \tau$. Since $\Gamma_\Theta = \Gamma_{\mathcal{C}(\Phi)}$, it follows by Proposition 4.2 below that $\sigma = \tau + c_0(1)_{k=-\infty}^{\infty}$ for some $c_0 \in \mathbb{C}$, (where $(1)_{k=-\infty}^{\infty} = (\ldots, 1, 1, 1, 1, \ldots)$), and since $\sigma(0) = \tau(0) = 0$, we get $c_0 = 0$. □

PROPOSITION 4.2. *Let $\Theta \in \mathcal{D}'(\mathbb{T})$ be such that Γ_Θ is bounded. Then there exists a unique $\tilde{\Theta} \in \mathsf{Ran}\ \mathcal{C}$ such that $\Gamma_\Theta = \Gamma_{\tilde{\Theta}}$. Moreover, there is an $N \in \mathbb{N}$ and $c_0, \ldots, c_N$ such that*

$$\Theta - \tilde{\Theta} = c_0\delta_1 + c_1\delta_1' + \ldots + c_N\delta_1^{(N)}.$$

Proof. Any distribution on a compact set if a distribution of finite order, (Theorem 2.3.1 in [5]). If $\Gamma_\Theta = \Gamma_{\tilde{\Theta}}$, then clearly $(\Theta - \tilde{\Theta})|_{\mathbb{T}\setminus\{1\}} = 0$, so $\Theta - \tilde{\Theta}$ is a finite order distribution with support in $\{1\}$. By Theorem 2.3.4 in [5], it is necessarily of the form $c_0\delta_1 + c_1\delta_1' + \ldots + c_N\delta_1^{(N)}$. □

PROPOSITION 4.3. *Let $\sigma \in \mathbb{C}^{\mathbb{Z}}$ be polynomially bounded such that H_σ in bounded. Then there exists a unique $\tilde{\sigma}$ such that $\lim_{k\to\pm\infty} \tilde{\sigma}(k)/k = 0$, $\tilde{\sigma}(0) = 0$ and $H_{\tilde{\sigma}} = H_\sigma$. Moreover, there is an $N \in \mathbb{N}$ and $c_0, \ldots, c_N$ such that*

$$\sigma - \tilde{\sigma} = c_0(1)_{k=-\infty}^{\infty} + c_1(k)_{k=-\infty}^{\infty} + \ldots + c_N\left(\delta_1^{(N)}(z^{-k})\right)_{k=-\infty}^{\infty}.$$

Proof. Let τ, Φ, Θ etc. be as in the proof of Proposition 4.1. Then $H_\tau = H_\sigma$, $\Gamma_\Theta = \Gamma_{\mathcal{C}\Phi}$, and hence it follows from Proposition 4.2 that $\sigma - \tau = c_0(1)_{k=-\infty}^{\infty} + c_1(k)_{k=-\infty}^{\infty} + \ldots + c_N(\delta_1^{(N)}(z^{-k}))_{k=-\infty}^{\infty}$. As in the proof of Proposition 4.1 we conclude that $\tilde{\sigma} = \tau$ whenever $\tilde{\sigma}$ is a sequence with the properties listed above. □

We will now begin the proof that $\|\Gamma_\Theta\|$ and $\|\hat{\Theta}\|_{BMO}$ are comparable for $\Theta \in \mathsf{Ran}\ \mathcal{C}$. Define $R_\Theta : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ via

$$\langle R_\Theta a, b\rangle = \sum \frac{\hat{\Theta}(i) - \hat{\Theta}(j)}{i - j} a(i)\overline{b(j)}.$$

In the above formulas we interpret $0/0$ as 0. Let $\mathbb{Z}_e$ and $\mathbb{Z}_o$ be the even/odd integers respectively, and let $\{e_k\}_{k\in\mathbb{Z}}$ denote the standard basis for $l^2(\mathbb{Z})$, (i.e. $e_k(j) = \delta(k - j)$ where δ is the Kronecker symbol). Define $P_e : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ via $P_e(\sum_{k\in\mathbb{Z}} a(k)e_k) = \sum_{k\in\mathbb{Z}_e} a(k)e_k$ and set $P_o = I - P_e$. Define $f_l^+ \in L^2(\mathbb{T}^+)$ via $f_l^+(z) = \sqrt{2}z^{-l}\chi(\mathbb{T}^+, z)$ for all $l \in \mathbb{Z}$ and define f_l^- analogously. Note that each of the sets $\{f_k^+\}_{k\in\mathbb{Z}_e}$, $\{f_k^+\}_{k\in\mathbb{Z}_o}$ forms an orthonormal basis for $L^2(\mathbb{T}^+)$ and the same is true if $+$ is exchanged with $-$ everywhere.

LEMMA 4.4. *Given $\Theta \in \mathsf{Ran}\ \mathcal{C}$, $a \in \mathsf{Ran}\ P_o$ and $b \in \mathsf{Ran}\ P_e$ we have*

$$\left\langle \Gamma_\Theta \sum a_l f_l^+, \sum b_l f_l^- \right\rangle = \frac{i}{2\pi}\langle R_\Theta P_o a, P_e b\rangle.$$

The corresponding formula with o and e switched also holds. In particular,

$$\|P_o R_\Theta P_e\| = \|P_e R_\Theta P_o\| = 2\pi\|\Gamma_\Theta\|.$$

Proof. Assume first that $\Theta \in L^1(\mathbb{T})$. These formulas can of course be obtained by evaluating some multiple integrals, but this provides little intuition for what is going on. We therefore prefer the following argument. By (4.2) and some simple calculations we have

$$\langle \Gamma_\Theta f_k^+, f_l^- \rangle = \left\langle \hat{\Theta} \cdot \breve{f_k^+}, \breve{f_l^-} \right\rangle = \sum_m \left(\hat{\Theta} \cdot \breve{f_k^+} \cdot \overline{\breve{f_l^-}}\right)(m) = \sum_m \left(\hat{\Theta} \cdot \breve{f_k^+} \cdot \breve{f_l^+}\right)(m). \tag{4.4}$$

We have $\breve{f_0^+} = \frac{1}{\sqrt{2}}e_0 + \sum_{k\in\mathbb{Z}_o} \frac{\sqrt{2}i}{\pi k}e_k$, or, written out as a sequence;

$$\breve{f_0^+} = \sqrt{2}\left(\ldots, \frac{-i}{5\pi}, 0, \frac{-i}{3\pi}, 0, \frac{-i}{\pi}, \frac{1}{2}, \frac{i}{\pi}, 0, \frac{i}{3\pi}, 0, \frac{i}{5\pi}, \ldots\right).$$

By the formula $\breve{f_k^+}(n) = \widetilde{z^{-k}f_0^+}(n) = \breve{f_0^+}(n - k)$ for all $n, k \in \mathbb{Z}$ we see that any $\breve{f_k^+}$ is obtained by a translation of $\breve{f_0^+}$. We thus get

$$\breve{f_k^+} \cdot \breve{f_l^+} = \frac{i}{2\pi}\frac{e_k - e_l}{k - l} \tag{4.5}$$

whenever $k-l$ is an odd number, which combined with (4.4) yields the desired formula.

If Θ is not in L^1, then by Proposition 4.1 there is a $\Phi \in \mathcal{D}'((-1/2,1/2))$ such that $\Theta = \mathcal{C}(\Phi)$ and W_Φ is bounded. Lemma 3.1 shows that there exists a sequence $\Psi_1, \Psi_2, \ldots \in C^\infty(\mathbb{R})$ such that $\|\widehat{\Psi_k}\|_{L^\infty} \leqslant 3\|W_\Phi\|$, and $\Phi(F) = \lim_{k\to\infty} \int \Psi_k F$ for all $F \in C_0^\infty((-1/2,1/2))$. By standard functional analysis, we can choose a subsequence $(\Psi_{k_j})_{j=1}^\infty$ such that $(\widehat{\Psi_{k_j}})_{j=1}^\infty$ is convergent in the weak*-topology of L^∞. Denote the limit by $\hat{\Psi}$ and note that $\Phi = \Psi|_{(-1/2,1/2)}$. Put $\Theta_k = \mathcal{C}(\Psi_k|_{(-1/2,1/2)})$ and note that $\Theta_k \in L^1$. Moreover, for any $l \in \mathbb{Z}$ we have by Lemmas 3.2 and 3.7 that

$$\widehat{\Theta_{k_j}}(l) - \widehat{\Theta_{k_j}}(0) = \Theta_{k_j}(z^{-l}-1) = \int_{-1/2}^{1/2} \Psi_{k_j}(x)\big(e^{-2\pi i l(1/2-x)}-1\big) =$$
$$= \int \hat{\Psi}_{k_j} \mathcal{F}^{-1}\big((e^{-2\pi i l(1/2-x)}-1)\chi_{(-1/2,1/2)}(x)\big) \to \int \hat{\Psi}\mathcal{F}^{-1}\big((e^{-2\pi i l(1/2-x)}-1)\chi_{(-1/2,1/2)}(x)\big) =$$
$$= \Psi\big((e^{-2\pi i l(1/2-x)}-1)\chi_{(-1/2,1/2)}(x)\big), \qquad \text{as } j \to \infty.$$

But $\Theta = \mathcal{C}(\Phi)$, so the right hand side equals $\hat{\Theta}(l) - \hat{\Theta}(0)$, again by Lemma 3.7. Now, it is clearly sufficient to verify the desired identity for $a \in \mathsf{Ran}\ P_o$ and $b \in \mathsf{Ran}\ P_e$ with finite support. Let such a and b's be fixed and put $A = \sum a_l f_l^+$, $B = \sum b_l f_l^-$. By a calculation similar to (3.4) is easy to see that there exists functions $\alpha, \beta \in C^1_{[0,1/2]}$ such that $\langle \Gamma_\Theta A, B\rangle = \langle W_\Phi \alpha, \beta\rangle$. By Lemmas 3.2, 3.3, 3.4, 3.7 and the first part of the proof we thus get

$$\langle \Gamma_\Theta A, B\rangle = \langle W_\Phi \alpha, \beta\rangle = \Psi\left(\int \alpha(\cdot+y)\overline{\beta(y)}dy\right) = \int \hat{\Psi}\mathcal{F}^{-1}\left(\int \alpha(\cdot+y)\overline{\beta(y)}dy\right) =$$
$$\lim_{j\to\infty} \int \widehat{\Psi_{k_j}}\mathcal{F}^{-1}\left(\int \alpha(\cdot+y)\overline{\beta(y)}dy\right) = \lim_{j\to\infty}\langle \Gamma_{\Theta_{k_j}} A, B\rangle = \lim_{j\to\infty} \frac{i}{2\pi}\langle R_{\Theta_{k_j}} P_o a, P_e b\rangle$$
$$\lim_{j\to\infty} \frac{i}{2\pi}\langle R_{\Theta_{k_j} - \widehat{\Theta_{k_j}}(0)\delta_1} P_o a, P_e b\rangle = \frac{i}{2\pi}\langle R_{\Theta - \hat{\Theta}(0)\delta_1} P_o a, P_e b\rangle = \frac{i}{2\pi}\langle R_\Theta P_o a, P_e b\rangle.$$

□ The calculation leading to (4.5) also shows why the "oo" and "ee"-cases are not part of Lemma 4.4; infinitely many terms would appear on the right hand side of (4.5). Nevertheless we have

Lemma 4.5. *Given $\Theta \in \mathsf{Ran}\ \mathcal{C}$ we have*

$$\|P_e R_\Theta P_e\| \leqslant (3\pi + \pi^2)\|\Gamma_\Theta\|$$

and

$$\|P_o R_\Theta P_o\| \leqslant (3\pi + \pi^2)\|\Gamma_\Theta\|.$$

Proof. We only do the ee-case, the other is identical. As earlier, we interpret $x/0$ as 0, regardless of x. Let $a, b \in \mathsf{Ran}\ P_e$ be arbitrary. Then

$$|\langle R_\Theta P_e a, P_e b\rangle| = \left|\sum_{i,j\in\mathbb{Z}_e} \frac{\hat{\Theta}(i) - \hat{\Theta}(j)}{i-j} a(i)\overline{b(j)}\right| =$$
$$= \left|\sum_{i,j\in\mathbb{Z}_e} \frac{i-1-j}{i-j}\frac{\hat{\Theta}(i-1) - \hat{\Theta}(j)}{i-1-j} a(i)\overline{b(j)} + \sum_{i,j\in\mathbb{Z}_e} \frac{\hat{\Theta}(i) - \hat{\Theta}(i-1)}{i-j} a(i)\overline{b(j)}\right| \leqslant \tag{4.6}$$
$$\leqslant \sup_{i,j\in\mathbb{Z}_e}\left\{\frac{i-1-j}{i-j}\right\}\|P_e R_\Theta P_o\|\|a\|\|b\| + \frac{1}{2}\left|\sum_{l,k\in\mathbb{Z}} \frac{1}{l-k}\tilde{a}(l)\overline{\tilde{b}(k)}\right|,$$

where $\tilde{a}(l) = a(2l)(\hat{\Theta}(2l) - \hat{\Theta}(2l-1))$ and $\tilde{b}(k) = b(2k)$. Clearly $\|\tilde{b}\| = \|b\|$ and moreover, by Lemma 4.4 we have $\sup_{l\in\mathbb{Z}}\{|\hat{\Theta}(2l) - \hat{\Theta}(2l-1)|\} \leqslant 2\pi\|\Gamma_\Theta\|$ so

$$\|\tilde{a}\| \leqslant 2\pi\|\Gamma_\Theta\|\|a\|.$$

Let Im log denote the imaginary part of the logarithm defined in the right half plane and note that $\sum_k -z^k/k = 2i\mathsf{Im}\ \log(1-z)$ for $z \in \mathbb{T}$. By these calculations and Lemma 4.4 we may continue the above calculation as follows:

$$|\langle R_\Theta P_e a, P_e b\rangle| \leqslant \frac{3}{2}2\pi\|\Gamma_\Theta\|\|a\|\|b\| + \left|\left\langle i\mathsf{Im}\ \log(1-z)\check{\tilde{a}}(z), \check{\tilde{b}}(z)\right\rangle_{L^2(\mathbb{T})}\right| \leqslant$$
$$\leqslant 3\pi\|\Gamma_\Theta\|\|a\|\|b\| + \|\mathsf{Im}\ \log(1-z)\|_{L^\infty(\mathbb{T})}\|\check{\tilde{a}}\|_{L^2(\mathbb{T})}\|\check{\tilde{b}}\|_{L^2(\mathbb{T})} \leqslant (3\pi+\pi^2)\|\Gamma_\Theta\|\|a\|\|b\|.$$

□

THEOREM 4.6. *There exists $C_1, C_2 > 0$ such that $C_1\|\Gamma_\Theta\| \leqslant \|\hat{\Theta}\|_{BMO} \leqslant C_2\|\Gamma_\Theta\|$ for all $\Theta \in \mathsf{Ran}\ \mathcal{C}$.*

Proof. By Lemma 4.4 we have $2\pi\|\Gamma_\Theta\| \leqslant \|R_\Theta\|$. Conversely,

$$R_\Theta = P_e R_\Theta P_o + P_o R_\Theta P_e + P_e R_\Theta P_e + P_o R_\Theta P_o \tag{4.7}$$

so

$$\|R_\Theta\| \leqslant \sum_{x\in\{o,e\};y\in\{o,e\}} \|P_y R_\Theta P_x\| \leqslant (10\pi + 2\pi^2)\|\Gamma_\Theta\|$$

by Lemma 4.4 and 4.5. The theorem now follows from Theorem 6.2 in [7], which states that $\|R_\Theta\|$ and $\|\hat{\Theta}\|_{BMO(\mathbb{Z})}$ are bounded by each other. □

5. Compactness.. We first recall some standard results on Hadamard-Schur multipliers. Let $\mathcal{L}(l^2(\mathbb{Z}))$ denote all bounded operators on $l^2(\mathbb{Z})$, which we identify with matrices via the canonical basis $\{e_k\}_{k\in\mathbb{Z}}$. Given $A = (a_{ij}) \in \mathcal{L}(l^2(\mathbb{Z}))$ and $B = (b_{ij}) \in \mathcal{L}(l^2(\mathbb{Z}))$ we define the Hadamard-Schur product via

$$A \diamond B = (a_{ij}b_{ij}).$$

If $A \diamond B \in \mathcal{L}(l^2(\mathbb{Z}))$ for all $B \in \mathcal{L}(l^2(\mathbb{Z}))$, then A is called a Hadamard-Schur multiplier. In this case we write

$$\|A\|_{HS} = \sup_{B\in\mathcal{L}(l^2(\mathbb{Z}))} \frac{\|A \diamond B\|_{\mathcal{L}(l^2(\mathbb{Z}))}}{\|B\|_{\mathcal{L}(l^2(\mathbb{Z}))}}.$$

Given $\omega \in L^1(\mathbb{T})$ we define $A_\omega = (a_{ij}) \in \mathcal{L}(l^2(\mathbb{Z}))$ via $a_{ij} = \int_T \omega z^{j-i} dm$.

LEMMA 5.1. *Let $\omega \in L^1(\mathbb{T})$ be given. Then*

$$\|A_\omega\|_{HS} \leqslant \|\omega\|_{L^1(\mathbb{T})},$$

and $A_\omega \diamond B$ is compact whenever B is.

Proof. Let $D_z \in \mathcal{L}(l^2(\mathbb{Z}))$ be given by $D_z(a) = (a_j z^j)_{j=-\infty}^{\infty}$. It is not hard to see that

$$A_\omega \diamond B = \int_T \omega(z) D_{\bar{z}} B D_z dm(z), \tag{5.1}$$

where the integral is interpreted in the WOT-sense, (see [6]). Thus

$$\|A_\omega \diamond B\| \leqslant \int_T |\omega(z)|\|D_{\bar{z}} B D_z\| dm(z) = \|\omega\|_{L^1(\mathbb{T})}\|B\|.$$

Moreover, a short argument shows that when B is compact, (5.1) holds as a Bochner-integral. The compactness of $A_\omega \diamond B$ thus follows as the set of compact operators is closed in $\mathcal{L}(l^2(\mathbb{Z}))$. □

Given an interval $I \subset \mathbb{Z}$ and $\sigma \in \mathbb{C}^{\mathbb{Z}}$ we set $Osc(\sigma, I) = |I|^{-1}\sum_{k\in I}\sigma(k) - \sigma_I$, where σ_I is the average of σ over I. Following [7] we define $CMO(\mathbb{Z}) \subset BMO(\mathbb{Z})$ to by requiring that

$$\lim_{|I| \text{fixed}, I \to \pm\infty} Osc(\sigma, I) = 0, \quad \lim_{|I|\to\infty}\sup_{|I| \text{fixed},} Osc(\sigma, I) = 0.$$

Note that by [7] we know that $CMO(\mathbb{Z})$ is the closure of the sequences with finite support in $BMO(\mathbb{Z})$.

Theorem 5.2. *Let $\Theta \in \mathsf{Ran}\ \mathcal{C}$ be given. Then Γ_Θ is compact if and only if $\hat{\Theta} \in CMO(\mathbb{Z})$.*

Proof. We first note that the above theorem is true if Γ_Θ is replaced with R_Θ, by Theorem 6.2 in [7]. By Lemma 4.4, Γ_Θ is unitarily equivalent with both $P_e R_\Theta P_o$ and $P_o R_\Theta P_e$. The "if"-part of the theorem thus follows easily. Conversely, by (4.7) one sees that the "only if"-part follows once we establish the following claim: If Γ_Θ is compact, then the same is true for $P_e R_\Theta P_e$ and $P_o R_\Theta P_o$. To this end, we define $\iota_e, \iota_o : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ via $\iota_e(a) = \sum_{k=-\infty}^{\infty} a(k)e_{2k}$ and $\iota_o(a) = \sum_{k=-\infty}^{\infty} a(k)e_{2k-1}$. Given a sequence $\sigma \in \mathbb{C}^{\mathbb{Z}}$ we define the "diagonal operator" D_σ via $D_\sigma(a) = (\sigma(k)a(k))_{k=-\infty}^{\infty}$. Let $a, b \in l^2(\mathbb{Z})$ be arbitrary. Returning to the second line of (4.6), a careful calculation shows that it can be rewritten as follows:

$$\langle R_\Theta \iota_e a, \iota_e b\rangle = \sum_{i,j\in\mathbb{Z}} \left(1 - \frac{1}{2(i-j)}\right)\frac{\hat{\Theta}(2i-1) - \hat{\Theta}(2j)}{2i-1-2j} a(i)\overline{b(j)} + \sum_{i,j\in\mathbb{Z}} \frac{\hat{\Theta}(2i) - \hat{\Theta}(2i-1)}{2(i-j)} a(i)\overline{b(j)} =$$

$$= \left\langle \left(\iota_e^* R_{\hat{\Theta}} \iota_o - \frac{1}{2} A_{2i\mathsf{Im}\ \log(1-z)} \diamond (\iota_e^* R_{\hat{\Theta}} \iota_o)\right) a, b \right\rangle + \left\langle \frac{1}{2} A_{2i\mathsf{Im}\ \log(1-z)} D_{(\hat{\Theta}(2i) - \hat{\Theta}(2i-1))_{i=-\infty}^{\infty}} a, b \right\rangle$$

Now, $\iota_e^* R_{\hat{\Theta}} \iota_o$ is essentially the same object as $P_e R_{\hat{\Theta}} P_o$, (unitarily equivalent), so by Lemma 5.1 it follows that the operator in the first bracket is compact. By Lemma 4.4 we have

$$|\hat{\Theta}(2i) - \hat{\Theta}(2i-1)| = 2\pi \left|\langle \Gamma_\Theta f_{2i}^+, f_{2i-1}^-\rangle\right| \leqslant 2\pi \|\Gamma_\Theta f_{2i}^+\|,$$

which by standard facts about compact operators shows that $\lim_{i=\pm\infty} |\hat{\Theta}(2i) - \hat{\Theta}(2i-1)| = 0$, and hence the operator in the second bracket is compact as well. □

A few simple estimates shows that $\sigma \in CMO(\mathbb{Z})$ whenever σ is a sequence such that $\lim_{k=\pm\infty} |\sigma(k)| = 0$. Thus, by the Riemann Lebesgue-lemma, we conclude in particular that Γ_Θ is a compact operator when Θ coincides with an absolutely continuous measure on $\mathbb{T}$. However, this can also be deduced from Hartman's theorem on compact Hankel operators. We refer to Section 3 of [2], where questions concerning theorems of AAK-type for TWH-operators are numerically investigated.

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