Proceedings of the Project Review, Geo-Mathematical Imaging Group (Purdue University, West Lafayette IN), Vol. 1 (2009) pp. 91-108.

INVERSE SCATTERING OF SEISMIC DATA IN THE REVERSE TIME MIGRATION (RTM) APPROACH

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Abstract. In this paper, we revisit the reverse-time imaging procedure. We discuss an inverse scattering transform derived from reverse-time migration (RTM), and establish its relation with generalized Radon transform inversion. In the process, the explicit evaluation of the so-called normal operator is avoided, at the cost of introducing other pseudodifferential operator factors. Using techniques from microlocal analysis, we explain the recently discussed RTM 'artifacts' and provide a method to remove them. We present a seamless integration of reverse-time imaging with downward-continuation based imaging, and establish an explicit relation between RTM and interferometry, and what is sometimes referred to as the 'virtual-source' method.

Key words. inverse scattering, microlocal analysis, reverse time migration

AMS subject classifications. 86A15, 35R30

DOI.

1. Introduction.

Over the past few years, there has been a revived interest in reverse time migration (RTM). Carrying out RTM has become computationallly feasible. RTM is attractive as an imaging procedure because it avoids approximations derived from asymptotic expansions or one-way wave propagation.

In this paper we revisit the original reverse-time imaging procedure [22, 12, 2, 21]; we do this, also, in the context of the integral formulation of Schneider [17] and the inverse scattering integral equation of Bojarski [4].

From reverse-time migration (RTM) to inverse scattering. We discuss an inverse scattering transform derived from RTM, and establish its relation with generalized Radon transform inversion. The explicit evaluation of the normal operator is avoided, at the cost of introducing other pseudodifferential operator factors in the procedure, which is, thus, different from Least-Squares migration-based approaches [15]. We address the following topics:

(i) We explain the recently discussed RTM 'artifacts' [25, 13, 10, 24, 11] and provide a technique to remove them.

(ii) We develop a seamless integration of reverse-time imaging with downward-continuation based imaging.

(iii) We establish an explicit relation between RTM and interferometry [7, 23] and what is sometimes referred to as the 'virtual-source' method [1].

(iv) We point out how the inverse scattering transform simplifies when dual sensor streamer data are available.

The outline of the paper is as follows. In Section 2 we review the principle of inverse scattering in the RTM approach, using a *constant background* and an incident plane wave. In Section 3 we discuss modelling and the Born approximation using the first-order system of partial differential equations describing the scattering of waves. In Section 4 we introduce backpropagation, and discuss retrofocusing and develop the techniques for a seamless integration of the RTM approach with the downward continuation approach to imaging. In Section 5 we introduce inverse scattering in the RTM approach from a microlocal analysis point of view, applicable in the *high-frequency* regime. We obtain an inverse scattering transform and show that it is a Fourier integral operator the canonical relation of which is a graph. In Section 6, we derive a *wave-equation analogue* of

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this inverse scattering transform, which is naturally implemented in terms of solving Helmholtz equations. We conclude with the introduction of annihilators of the data that can be exploited for developing RTM based reflection tomography.

2. Incident plane-wave, constant coefficients, inversion formula. We begin with solving the inhomogeneous wave equation in $\mathbb{R}^2$ given coordinates (x, z),

$$(c^{-2}\partial_t^2 - \partial_x^2 - \partial_z^2)u = A\delta\Big(t - \frac{z}{c}\Big)\, r(x, z), \tag{2.1}$$

for the scattered field, u. In this equation, the right-hand side contains an incoming plane wave with constant amplitude A which travels in the positive z direction, multiplied by a reflectivity $r = r(x, z)$. The (background) medium velocity is a constant denoted by c. (This equation is not entirely the same as the linearization used in the remainder of this paper, but it captures the essential components.)

2.1. Solution of the wave equation in the (t, ξ, ζ) domain. We first review the elementary solution of an initial value problem with nonzero initial value for the first derivative, and then construct the solution of the inhomogeneous wave equation. We do this in the (t, ξ, ζ) domain. The resulting solution for the inhomogeneous equation is given at the end of this subsection in (2.8).

The relevant initial value problem is given by

$$(c^{-2}\partial_t^2 - \partial_x^2 - \partial_z^2)u = 0, \tag{2.2}$$

$$u(t = t_0) = 0, \qquad \partial_t u(t = t_0) = u_1. \tag{2.3}$$

Fourier transformation in the (x, z) variables yields the equation

$$(c^{-2}\partial_t^2 + \xi^2 + \zeta^2)\widetilde{u} = 0,$$

from which it follows that

$$\widetilde{u} = A \exp[\mathrm{i}c\sqrt{\xi^2 + \zeta^2}(t - t_0)] + B \exp[-\mathrm{i}c\sqrt{\xi^2 + \zeta^2}(t - t_0)]. \tag{2.4}$$

The parameters A, B depend on (ξ, ζ) and need to be determined from the initial conditions. These give

$$A + B = 0,$$
$$\mathrm{i}c\sqrt{\xi^2 + \zeta^2}A - \mathrm{i}c\sqrt{\xi^2 + \zeta^2}B = \widetilde{u}_1(\xi, \zeta),$$

and, hence, we find that

$$A = -B = \frac{1}{2\mathrm{i}c\sqrt{\xi^2 + \zeta^2}}\widetilde{u}_1(\xi, \zeta),$$

so that the solution of (2.2) with initial conditions (2.3) follows to be

$$\widetilde{u}(t, \xi, \zeta) = (\exp[\mathrm{i}c\sqrt{\xi^2 + \zeta^2}(t - t_0)] - \exp[-\mathrm{i}c\sqrt{\xi^2 + \zeta^2}(t - t_0)])\frac{1}{2\mathrm{i}c\sqrt{\xi^2 + \zeta^2}}\,\widetilde{u}_1(\xi, \zeta). \tag{2.5}$$

We denote by $U(t, t_0)$ the map from u_1 to $u(t)$ given in (2.5). We note that, from the initial conditions,

$$U(t, t_0)|_{t=t_0} = 0, \qquad \partial_t U(t, t_0)|_{t=t_0} = \mathrm{Id}\,.$$

We next consider the inhomogeneous wave equation

$$(c^{-2}\partial_t^2 - \partial_x^2 - \partial_z^2)u = f, \qquad u(t < 0) = 0. \tag{2.6}$$

We claim that the solution is

$$u(t,.,.) = \int_0^t U(t,s)c^2 f(s,.,.)\,\mathrm{d}s. \tag{2.7}$$

Proof. We have

$$\partial_t u(t,.,.) = U(t,t)c^2 f(t,.,.) + \int_0^t \partial_t U(t,s)c^2 f(s,.,.)\,\mathrm{d}s = \int_0^t \partial_t U(t,s)c^2 f(s,.,.)\,\mathrm{d}s,$$

since $U(t,t) = 0$. Furthermore,

$$\begin{aligned}\partial_t^2 u(t,.,.) &= \partial_t U(t,t)c^2 f(t,.,.) + \int_0^t \partial_t^2 U(t,s)c^2 f(s,.,.)\,\mathrm{d}s\\ &= c^2 f(t,.,.) + \int_0^t \partial_t^2 U(t,s)c^2 f(s,.,.)\,\mathrm{d}s.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(c^{-2}\partial_t^2 - \partial_x^2 - \partial_z^2\right) \int_0^t U(t,s)c^2 f(s,.,.)\,\mathrm{d}s\\ &\qquad = f(t,.,.) + \int_0^t (c^{-2}\partial_t^2 - \partial_x^2 - \partial_z^2)U(t,s)c^2 f(s,.,.)\,\mathrm{d}s\\ &\qquad = f(t,.,.)\end{aligned}$$

which is what we had to show. □

Combining (2.5) and (2.7) we find the following solution for the inhomogeneous wave equation,

$$\widetilde{u}(t,\xi,\zeta) = \int_0^t (\exp[\mathrm{i}c\sqrt{\xi^2+\zeta^2}(t-s)] - \exp[-\mathrm{i}c\sqrt{\xi^2+\zeta^2}(t-s)])\frac{c^2}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}\widetilde{f}(s,\xi,\zeta)\,\mathrm{d}s. \tag{2.8}$$

2.2. Modelling: Solving for the scattered field. The application of (2.8) requires Fourier transforming $A\delta(t-\frac{z}{c})r(x,z)$ to the (t,ξ,ζ) domain. We let $\check{r} = \check{r}(\xi,z)$ be the Fourier transform of r with respect to x but not z. Then

$$\int \exp[-\mathrm{i}z\zeta]A\delta\Big(t-\frac{z}{c}\Big)\,\check{r}(\xi,z)\,\mathrm{d}z = cA\exp[-\mathrm{i}\zeta ct]\,\check{r}(\xi,ct). \tag{2.9}$$

We use (2.8) and (2.9) to solve (2.1):

$$\begin{aligned}\widetilde{u}(t,\xi,\zeta) = \int_0^t &(\exp[\mathrm{i}c\sqrt{\xi^2+\zeta^2}(t-s)] - \exp[-\mathrm{i}c\sqrt{\xi^2+\zeta^2}(t-s)])\\ &\frac{c^2}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}cA\exp[-\mathrm{i}\zeta cs]\check{r}(\xi,cs)\,\mathrm{d}s.\end{aligned} \tag{2.10}$$

A second form of this formula is obtained by a change of variable, $cs = \tilde{z}$,

$$\begin{aligned}\widetilde{u}(t,\xi,\zeta) = \int_0^{tc} &(\exp[\mathrm{i}\sqrt{\xi^2+\zeta^2}(ct-\tilde{z})] - \exp[-\mathrm{i}\sqrt{\xi^2+\zeta^2}(ct-\tilde{z})])\\ &\frac{c^2}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}A\exp[-\mathrm{i}\zeta\tilde{z}]\,\check{r}(\xi,\tilde{z})\,\mathrm{d}\tilde{z}.\end{aligned} \tag{2.11}$$

We recognize in this formula a Fourier transformation with respect to $\tilde{z}$. However, the Fourier transform of r is not evaluated at ζ, but at $\zeta \pm \sqrt{\xi^2+\zeta^2}$, because $\tilde{z}$ occurs at several places in the complex exponents. Under the assumption that the support of r is contained in $0 < z < ct$ (in other words, that we consider the field at time t such that the incoming wave front has completely passed the support of the reflectivity), the formula equals

$$\tilde{u}(t,\xi,\zeta) = \exp[\mathrm{i}\sqrt{\xi^2+\zeta^2}ct]\frac{c^2A}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}\tilde{r}(\xi,\zeta+\sqrt{\xi^2+\zeta^2}) - \exp[-\mathrm{i}\sqrt{\xi^2+\zeta^2}ct]\frac{c^2A}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}\tilde{r}(\xi,\zeta-\sqrt{\xi^2+\zeta^2}). \tag{2.12}$$

The field in position coordinates is given by the inverse Fourier transform of this expression, that is

$$u(t,x,z) = \frac{1}{(2\pi)^2}\iint\Big[\exp[\mathrm{i}\sqrt{\xi^2+\zeta^2}ct]\frac{c^2A}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}\tilde{r}(\xi,\zeta+\sqrt{\xi^2+\zeta^2}) - \exp[-\mathrm{i}\sqrt{\xi^2+\zeta^2}ct]\frac{c^2A}{2\mathrm{i}c\sqrt{\xi^2+\zeta^2}}\tilde{r}(\xi,\zeta-\sqrt{\xi^2+\zeta^2})\Big]\exp[\mathrm{i}(x\xi+z\zeta)]\,\mathrm{d}\zeta\,\mathrm{d}\xi. \tag{2.13}$$

The two terms yield complex conjugate contributions after integration. To see this, we change the integration variables in the second term to $(-\xi,-\zeta)$, and use that the property that $r(x,z)$ is real for all (x,z) is equivalent to $\tilde{r}(\xi,\zeta) = \overline{\tilde{r}(-\xi,-\zeta)}$ for all (ξ,ζ). Therefore,

$$u(t,x,z) = \frac{1}{(2\pi)^2}\operatorname{Re}\iint \exp[\mathrm{i}\sqrt{\xi^2+\zeta^2}ct + \mathrm{i}(x\xi+z\zeta)] \frac{c^2A}{\mathrm{i}c\sqrt{\xi^2+\zeta^2}}\tilde{r}(\xi,\zeta+\sqrt{\xi^2+\zeta^2})\,\mathrm{d}\xi\,\mathrm{d}\zeta. \tag{2.14}$$

Geometrical acoustics. We observe that $\pm c\sqrt{\xi^2+\zeta^2}$ is a frequency, τ.

For the positive frequencies (the first term with $\exp[\mathrm{i}\sqrt{\xi^2+\zeta^2}ct]$), $\tilde{u}(t,\xi,\zeta)$ hence depends on $\tilde{r}(\xi,\zeta+\sqrt{\xi^2+\zeta^2})$. For the positive frequencies the ζ vector of the incoming wave field is negative because it propagates in the positive z direction while τ is positive, whence it must equal $-\sqrt{\xi^2+\zeta^2}$.

For the negative frequencies, $\tilde{u}(t,\xi,\zeta)$ depends on $\tilde{r}(\xi,\zeta-\sqrt{\xi^2+\zeta^2})$. For the negative frequencies, the ζ component of the wave vector of the incoming wave field is positive.

If we denote by $(\xi_{\text{in}},\zeta_{\text{in}})$ the wave vector of the incoming field, by $(\xi_{\text{out}},\zeta_{\text{out}})$ that of the scattered field, and by $(\xi_{\text{scat}},\zeta_{\text{scat}})$ that of the component of the reflectivity r involved, then we find that in both cases

$$(\xi_{\text{out}},\zeta_{\text{out}}) = (\xi_{\text{in}},\zeta_{\text{in}}) + (\xi_{\text{scat}},\zeta_{\text{scat}}),$$

in accordance with diffraction tomography [9]. We summarize these observations in the table below.

wave vector	positive frequency	negative frequency
$(\xi_{\text{in}},\zeta_{\text{in}})$	$(0,-\sqrt{\xi^2+\zeta^2})$	$(0,\sqrt{\xi^2+\zeta^2})$
$(\xi_{\text{out}},\zeta_{\text{out}})$	(ξ,ζ)	(ξ,ζ)
$(\xi_{\text{scat}},\zeta_{\text{scat}})$	$(\xi,\zeta+\sqrt{\xi^2+\zeta^2})$	$(\xi,\zeta-\sqrt{\xi^2+\zeta^2})$

2.3. Imaging and reconstruction of r. The basic idea of imaging is to correlate the source (incoming) field with the receiver (scattered) field. Approximating the source field by $A\delta(t-\frac{z}{c})$ this becomes (ignoring the amplitude) evaluating the receiver field at the arrival time of the incoming wave. Thus a first attempt to generate an image of r would be,

$$I_0 = u(z/c,x,z).$$

By some advance knowledge (see comments around (2.18)) we will define instead as our first try for the imaging operator,

$$I_1 = (\partial_t + c\partial_z)u(z/c, x, z). \tag{2.15}$$

Using (2.14) we find that

$$(\partial_t + c\partial_z)u(t, x, z) = \frac{1}{(2\pi)^2}\,\mathrm{Re}\iint \Big(\,1 + \frac{\zeta}{\sqrt{\xi^2+\zeta^2}}\,\Big) \\ \exp[\mathrm{i}\sqrt{\xi^2+\zeta^2}ct + \mathrm{i}(x\xi + z\zeta)]c^2 A\,\widetilde{r}(\xi, \zeta + \sqrt{\xi^2+\zeta^2})\,\mathrm{d}\xi\,\mathrm{d}\zeta. \tag{2.16}$$

Setting $t = z/c$, yields

$$(\partial_t + c\partial_z)u(z/c, x, z) = \frac{1}{(2\pi)^2}\,\mathrm{Re}\iint \Big(\,1 + \frac{\zeta}{\sqrt{\xi^2+\zeta^2}}\,\Big) \\ \exp[\mathrm{i}(x\xi + z(\zeta + \sqrt{\xi^2+\zeta^2})]c^2 A\,\widetilde{r}(\xi, \zeta + \sqrt{\xi^2+\zeta^2})\,\mathrm{d}\xi\,\mathrm{d}\zeta. \tag{2.17}$$

We carry out a coordinate transformation (for the positive frequencies) according to

$$\tilde{\zeta} = \zeta + \sqrt{\xi^2+\zeta^2}.$$

The image of this transformation is the halfplane $\tilde{\zeta} > 0$, while the Jacobian is given by

$$\frac{\partial\tilde{\zeta}}{\partial\zeta} = 1 + \frac{\zeta}{\sqrt{\xi^2+\zeta^2}}, \tag{2.18}$$

and exactly equals the factor $1+\frac{\zeta}{\sqrt{\xi^2+\zeta^2}}$ from the derivative operator $\partial_t + c\partial_z$ we incorporated with advance knowledge into (2.15)[1]. Therefore, by a change of variables, (2.17) equals

$$(\partial_t + c\partial_z)u(z/c, x, z) = \frac{1}{(2\pi)^2}\,\mathrm{Re}\iint_{\tilde{\zeta}>0} \exp[\mathrm{i}(x\xi + z\tilde{\zeta})]c^2 A\,\widetilde{r}(\xi, \tilde{\zeta})\,\mathrm{d}\xi\,\mathrm{d}\tilde{\zeta}, \tag{2.19}$$

or

$$\begin{aligned}(\partial_t + c\partial_z)&u(z/c, x, z) \\ &= \frac{c^2 A}{2(2\pi)^2}\iint_{\tilde{\zeta}>0} [\exp[\mathrm{i}(x\xi + z\tilde{\zeta})]\,\widetilde{r}(\xi, \tilde{\zeta}) + \exp[-\mathrm{i}(x\xi + z\tilde{\zeta})]\,\widetilde{r}(-\xi, -\tilde{\zeta})]\,\mathrm{d}\xi\,\mathrm{d}\tilde{\zeta} \\ &= \frac{c^2 A}{2} r(x, z). \qquad (2.20)\end{aligned}$$

Hence, up to a factor $\frac{c^2 A}{2}$, we have reconstructed r. Therefore, we recover the reconstruction formula,

$$I(x, z) = \frac{2}{c^2 A}(\partial_t + c\partial_z)u(z/c, x, z)$$

yielding $r(x, z)$. Here, ∂_z should be interpreted as the gradient component in the direction of the incoming field. The result extends straightforwardly to dimension 3. We will extend the reconstruction developed for the constant background velocity case to the case of a smoothly varying background in general dimension n in the remainder of the paper.

[1] Without this derivative operator, we would reconstruct a pseudodifferential operator acting on r. The Jacobian $\frac{\partial\tilde{\zeta}}{\partial\zeta}$ can also be described in other ways, in terms of the angle θ between in an outgoing rays it is e.g. given by $\frac{\partial\tilde{\zeta}}{\partial\zeta} = 1 + \cos(\theta)$.

3. Modelling: The Born approximation. While considering the general coefficients case, our point of departure becomes the first-order system for acoustic pressure, p, and particle velocity, v, in $\mathbb{R}^n$,

$$\kappa\,\partial_t p - \sum_{j=1}^{n} \partial_{x_j} v_j = q, \tag{3.1}$$

$$\rho\,\partial_t v_i - \partial_{x_i} p = f_i, \quad i = 1, \ldots, n, \tag{3.2}$$

subject to initial conditions, $p|_{t=0} = 0$ and $v|_{t=0} = 0$. Here, ρ stands for density of mass and κ stands for compressibility. We introduce the causal Green's functions:

$$G^{pq}(.,.,x'),\ G_i^{vq}(.,.,x') \quad \text{if } q(x,t) = \delta(x-x')\delta(t) \text{ and } f(x,t) \equiv 0,$$

and

$$G_j^{pf}(.,.,x'),\ G_{ij}^{vf}(.,.,x') \quad \text{if } q(x,t) \equiv 0 \text{ and } f(x,t) = e_j\delta(x-x')\delta(t).$$

We restrict our configuration to the domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega \sim S^{n-1}$. We denote the time convolution evaluated at t by $\overset{(t)}{*}$.

The perturbed or scattered field, $\{p^{\rm sc}, v^{\rm sc}\}$, under perturbations $\delta\rho$ of ρ and $\delta\kappa$ of κ, satisfies the system of partial differential equations,

$$\kappa\,\partial_t p^{\rm sc} - \sum_{j=1}^{n} \partial_{x_j} v_j^{\rm sc} = q^{\rm sc}, \tag{3.3}$$

$$\rho\,\partial_t v_i^{\rm sc} - \partial_{x_i} p^{\rm sc} = f_i^{\rm sc}, \tag{3.4}$$

and is hence given by

$$p^{\rm sc}(x,t,s) = \int_\Omega G^{pq}(x,.,x') \overset{(t)}{*} q^{\rm sc}(x',.,s)\,\mathrm{d}V(x') + \int_\Omega \sum_{j=1}^{n} G_j^{pf}(x,.,x') \overset{(t)}{*} f_j^{\rm sc}(x',.,s)\,\mathrm{d}V(x'), \tag{3.5}$$

and

$$v_i^{\rm sc}(x,t,s) = \int_\Omega G_i^{vq}(x,.,x') \overset{(t)}{*} q^{\rm sc}(x',.,s)\,\mathrm{d}V(x') + \int_\Omega \sum_{j=1}^{n} G_{ij}^{vf}(x,.,x') \overset{(t)}{*} f_j^{\rm sc}(x',.,s)\,\mathrm{d}V(x'), \tag{3.6}$$

using Duhamel's principle. In the Born approximation, we have

$$q^{\rm sc}(x,t,s) = -\delta\kappa(x)\,\partial_t p^0(x,t,s), \quad f_j^{\rm sc}(x,t,s) = -\delta\rho(x)\,\partial_t v_j^0(x,t,s), \tag{3.7}$$

where p^0, v_j^0 describe the incident field and satisfy (3.1)-(3.2). The coefficients, ρ, κ, describing the background medium are assumed to be smooth. Furthermore, we assume that $\operatorname{supp} q^{\rm sc}$ and $\operatorname{supp} f^{\rm sc}$ are contained in the interior of Ω. Then

$$v_j^{\rm sc}(x,t,s) = \rho(x)^{-1}\partial_t^{-1}\partial_{x_j} p^{\rm sc}(x,t,s) \quad \text{for } x \in \partial\Omega. \tag{3.8}$$

Depending on whether $\delta\rho$ vanishes, we may assume that the supports of the sources, q and f, are also contained in the interior of Ω; however, in applications the sources could be supported close to the boundary $\partial\Omega$.

The scalar-wave Green's function, $G(x,t,x')$, follows to be the pressure, $p(x,t)$, subject to the identifications $f(x,t) \equiv 0$ and $\rho(x)\partial_t q(x,t) = \delta(x-x')\delta(t)$ cf. (3.1)-(3.2). In applications, one often assumes that ρ is constant, and that $\delta\rho \equiv 0$. Then we obtain

$$p^{\rm sc}(x,t,s) = -\partial_t^2 \int_\Omega G(x,.,x') \overset{(t)}{*} p^0(x',.,s)\,\delta c^{-2}(x')\,\mathrm{d}V(x'), \tag{3.9}$$

with $c^{-2} = \rho\kappa$. In this expression, we identify the contrast source, $-\partial_t^2 p^0(x', t, s)\, \delta c^{-2}(x')$. Let n_j be the outer normal to $\partial\Omega$. Via restriction to $\partial\Omega$, this equation defines a map from $\delta c^{-2}(x')$ (with $x' \in \Omega$) to $\{p^{\mathrm{sc}}(x,t,s), n_j(x)(\partial_{x_j} p^{\mathrm{sc}})(x,t,s)\}$ (with $(x,t) \in \partial\Omega \times (0,T)$) from which, also, $n_j(x) v_j^{\mathrm{sc}}(x,t,s)$ can be obtained, cf. (3.8). We extract the single scattering operator, F, with

$$F : \ \delta c^{-2}(x') \to p^{\mathrm{sc}}(x,t,s) \text{ (with } (x,t) \in \partial\Omega \times (0,T)). \tag{3.10}$$

In the further analysis, we consider the case where $p^0(x,t,s) = G(x,t,s)$. Thus, $\hat{p}^0(x,\tau,s) = \int p^0(x,t,s) \exp(-\mathrm{i}\tau t)\,\mathrm{d}t$ can be identified with the solution, $\hat{u}(x,\tau,s)$ say, of the Helmholtz equation,

$$[c(x)^{-2}\tau^2 + \nabla_x^2]\, \hat{u}(x,\tau,s) = \delta(x-s). \tag{3.11}$$

In the inverse scattering problem under consideration, the data are modelled by $p^{\mathrm{sc}}(x,t,s)$. The source at s will be fixed, while the receivers at x are restricted to a set $\Sigma_s \subset \partial\Omega$, signifying limited acquisition aperture and the data are observed over a time interval $(0,T)$; we write $Y_s = \Sigma_s \times (0,T)$. We hasten to mention that present day data acquisition systems can measure not only $p^{\mathrm{sc}}(x,.,s)$ but also $\sum_{j=1}^n n_j(x) v_j^{\mathrm{sc}}(x,.,s)$ (dual sensor streamers) or $\sum_{j=1}^n n_j \partial_{x_j} p^{\mathrm{sc}}(x,t,s)$ ('over-under' towed-streamer acquisition). In the further analysis it will be made explicit how to exploit these extended measurements in inverse scattering in the RTM appproach.

Rakesh [16] showed that, with this acquisition geometry, F is a Fourier integral operator of order $(n-1)/2$ subject to

Assumption 1. *There are no rays connecting s with $x \in \Sigma_s$ with traveltime t such that $(x,t) \in Y_s$. For all ray pairs connecting $x \in \Sigma_s$ via some scattering point in the suburface Ω to s with total time t such that $(x,t) \in Y_s$, the receiver rays intersect Σ_s transversally at x.*

4. Backpropagation and retrofocusing. We construct the inverse scattering in the RTM approach by backpropagation (this section), imaging followed by "amplitude and illumination" correction (next section).

Here, we develop the (common-source) backpropagation procedure. We continue to assume that the density ρ is constant. Motivated by (A.4) in Appendix A, substituting (3.8), while suppressing q^{sc}, it is natural to introduce the field

$$\hat{p}^{\mathrm{f:r}}(x',\tau,s) = \int_{\partial\Omega} \Bigg[-\sum_{j=1}^n n_j(x)(\partial_{x_j}\hat{p}^{\mathrm{sc}})(x,\tau,s)\, \overline{\hat{G}(x,\tau,x')} + \hat{p}^{\mathrm{sc}}(x,\tau,s) \sum_{j=1}^n n_j(x) \overline{(\partial_{x_j}\hat{G})(x,\tau,x')} \Bigg] \mathrm{d}A(x) \quad \text{for } x' \in \Omega\backslash\partial\Omega,$$

or, using reciprocity,

$$\hat{p}^{\mathrm{f:r}}(x',\tau,s) = \int_{\partial\Omega} \Bigg[-\sum_{j=1}^n n_j(x)(\partial_{x_j}\hat{p}^{\mathrm{sc}})(x,\tau,s)\, \overline{\hat{G}(x',\tau,x)} + \hat{p}^{\mathrm{sc}}(x,\tau,s) \sum_{j=1}^n \overline{(\partial_{x_j}\hat{G})(x',\tau,x)} n_j(x) \Bigg] \mathrm{d}A(x) \quad \text{for } x' \in \Omega\backslash\partial\Omega. \tag{4.1}$$

Subjecting the entire equation to time reversal yields

$$\hat{P}^{\mathrm{f:r}}(x',\tau,s) := \hat{p}^{\mathrm{f:r}}(x',-\tau,s) = \int_{\partial\Omega} \hat{G}(x',\tau,x) \Bigg[-\sum_{j=1}^n n_j(x)(\partial_{x_j}\hat{p}^{\mathrm{sc}})(x,-\tau,s) \Bigg] \mathrm{d}A(x) + \int_{\partial\Omega} \sum_{j=1}^n (\partial_{x_j}\hat{G})(x',\tau,x) \Bigg[\hat{p}^{\mathrm{sc}}(x,-\tau,s)\, n_j(x) \Bigg] \mathrm{d}A(x) \quad \text{for } x' \in \Omega\backslash\partial\Omega, \tag{4.2}$$

which upon introduction of an appropriate surface measure attains the form of equation (3.5). Thus, one can interpret the scattered field, that is, 'data' through time reversal in terms of q and f sources. (The field $P^{\mathrm{f:r}}(x',t,s)$ can be viewed as the solution to a system of 'adjoint' equations in the context of optimization.)

The data are only available on a finite time interval, and hence (4.1) should be subjected to (A.2) and understood in terms of time correlations. In the context of the seismic migration literature, equation (4.1) can be called the time-correlation analogue of the 'Rayleigh integral' [3]. To account for the actual acquisition aperture, the domain of integration $\partial\Omega$ is replaced by Σ_s. Equation (4.1) defines a map from (dual sensor data) $\{p^{\mathrm{sc}}(x,t,s), n_j(x)(\partial_{x_j}p^{\mathrm{sc}})(x,t,s)\}$ (with $(x,t)\in\Sigma_s\times(0,T)$) to $p^{\mathrm{f:r}}(x',t,s)$ (with $x'\in\Omega\backslash\partial\Omega$).

We proceed with expressing p^0 and $p^{\mathrm{f:r}}$ near the boundary Σ_s, in terms of one-way wave propagation. We introduce coordinates (z,x) on $\mathbb{R}^n$, and we assume that $n=(n_z,n_x)=(n_z,0)$ at $\Sigma_s\subset\partial\Omega$, where $z=0$. We summarize the procedure of directional wavefield decomposition [19, 20, 18]:

$$(4.3)\qquad \mathsf{Q}\begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix}\mathsf{Q}^{-1} = \begin{pmatrix} \mathrm{i}B_+ & 0 \\ 0 & \mathrm{i}B_- \end{pmatrix}, \quad A = A(z,x,D_x,D_t) = c(z,x)^{-2}D_t^2 - D_x^2.$$

Here,

$$(4.4)\quad \mathsf{Q}_x(z) = \tfrac{1}{2}\begin{pmatrix} (Q_{+,x}(z)^*)^{-1} & -Q_{+,x}(z)\mathcal{H} \\ (Q_{-,x}(z)^*)^{-1} & Q_{-,x}(z)\mathcal{H} \end{pmatrix},$$

$$\mathsf{Q}_x^{-1}(z) = \begin{pmatrix} Q_{+,x}(z)^* & Q_{-,x}(z)^* \\ \mathcal{H}Q_{+,x}(z)^{-1} & -\mathcal{H}Q_{-,x}(z)^{-1} \end{pmatrix},$$

in which $\mathcal{H}$ denotes the Hilbert transform in time, while $\mathcal{H}\,(Q_{\pm,x}(z)^*Q_{\pm,x}(z))^{-1} = \mp\mathrm{i}B_{\pm,x}(z)$ and $B_\pm^2 = A$. We have the property,

$$\int (Q_\pm(z,x,D_x,\tau)\hat{u}(z,x,\tau))\,\overline{\hat{v}(z,x,\tau)}\,\mathrm{d}x = \int \hat{u}(z,x,\tau)\,\overline{(Q_\pm^*(z,x,D_x,\tau)\hat{v}(z,x,\tau))}\,\mathrm{d}x$$

for arbitrary u,v, and similarly for $B_\pm(z,x,D_x,\tau)$.

Assuming upcoming waves in the *surface* receivers at $(0,x)\in\Sigma_s$, we can write

$$(4.5)\qquad p^{\mathrm{sc}}(0,x,.,s) \sim Q_{-,x}(0)^*p_-^{\mathrm{sc}}(0,.,.,s), \partial_z p^{\mathrm{sc}}(0,x,.,s) \sim -\mathcal{H}Q_{-,x}(0)^{-1}p_-^{\mathrm{sc}}(0,.,.,s),$$

so that $\partial_z p^{\mathrm{sc}}(0,x,.,s) \sim \mathrm{i}B_{-,x}(0)p^{\mathrm{sc}}(0,.,.,s)$. Correspondingly,

$$(4.6)\quad G(0,x,.,z',x') \sim -Q_{-,x}^*(0)\left[\,G_-(0,z')\,\tfrac{1}{2}Q_{-,.}(z)\mathcal{H}\,\delta(.-x')\delta(.)\,\right],$$

$$\partial_z G(0,x,.,z',x') \sim -Q_{-,x}(0)^{-1}\left[\,G_-(0,z')\,\tfrac{1}{2}Q_{-,.}(z')\,\delta(.-x')\delta(.)\,\right],$$

so that $\partial_z G(0,x,.,z',x') \sim \mathrm{i}B_{-,x}(0)G(0,x,.,z',x')$. Then we have

$$\int_{\Sigma_s}\left[\,-\sum_{j=1}^n n_j(x)(\partial_{x_j}\hat{p}^{\mathrm{sc}})(x,\tau,s)\,\overline{\hat{G}(x,\tau,x')}\,\right]\mathrm{d}A(x)$$

$$\sim -\int (\mathrm{i}B_-(0,x,D_x,\tau)p^{\mathrm{sc}}(0,x,\tau,s))\,\overline{\hat{G}(0,x,\tau,z',x')}\mathrm{d}x$$

$$= \int p^{\mathrm{sc}}(0,x,\tau,s)\,\overline{(\mathrm{i}B_-(0,x,D_x,\tau)\hat{G}(0,x,\tau,z',x'))}\mathrm{d}x$$

$$\sim \int_{\Sigma_s}\hat{p}^{\mathrm{sc}}(x,\tau,s)\,\sum_{j=1}^n\overline{(\partial_{x_j}\hat{G})(x',\tau,x)}n_j(x)\mathrm{d}A(x).$$

Exploiting this equivalence in (4.1) leads to

$$\hat{p}^{\text{f:r}}(x',\tau,s) \sim 2\int_{\Sigma_s} \hat{p}^{\text{sc}}(x,\tau,s) \sum_{j=1}^{n} \overline{(\partial_{x_j}\hat{G})(x',\tau,x)}n_j(x)\mathrm{d}A(x) \quad \text{for } x' \in \Omega\backslash\partial\Omega \tag{4.7}$$

for the backpropagated field.

Remark 4.1. *Using (4.7) we observe that* $p^{\text{f:r}}(x',t,s)$ *can be identified with the solution,* $w^*(x',t,s)$ *say, of an adjoint equation:*

$$[c(x')^{-2}\tau^2 + \nabla_{x'}^2]\,\widehat{w^*}(x',\tau,s) = -2\int_{\Sigma_s}\sum_{j=1}^{n}\overline{\hat{p}^{\text{sc}}(x,\tau,s)}\,n_j(x)\partial_j\delta(x'-x)\mathrm{d}A(x'). \tag{4.8}$$

That is, $\hat{p}^{\text{f:r}}(x',\tau,s) = \overline{\widehat{w^*}(x',\tau,s)}$. *We note that the incident field,* $\hat{p}^0$*, was already identified with the solution of the (forward) Helmholtz equation (3.11).*

Remark 4.2. *Both* $p^{\text{f:r}}$ *and* p^0 *can be subjected to a projection, in the subsurface, extracting locally the upcoming or downgoing wave constituents. The relevant operators are of the form* $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathcal{P}_\pm\begin{pmatrix} 1 \\ \partial_z \end{pmatrix}$ *with*

$$\mathcal{P}_+ = \mathsf{Q}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathsf{Q} = \tfrac{1}{2}\begin{pmatrix} I & (-\mathrm{i}B_+)^{-1} \\ -\mathrm{i}B_+ & I \end{pmatrix},\ \mathcal{P}_- = \tfrac{1}{2}\begin{pmatrix} I & -(\mathrm{i}B_-)^{-1} \\ -\mathrm{i}B_- & I \end{pmatrix}.$$

Remark 4.3. *We can express* p^0 *and* $p^{\text{f:r}}$ *in terms of one-way wave propagation, subject to the so-called double-square-root (DSR) assumption. For the incident field, considering the downward radiating component of a density normalized point injection source, we obtain*

$$p^0(z',x',.,s) \sim Q^*_{+,x'}(z')G_+(z',0)\tfrac{1}{2}Q_{+,.}(0)\mathcal{H}\,\delta(.-s)\delta(.). \tag{4.9}$$

On the other hand, upon substituting (4.6) and (4.5) into (4.1), the backpropagated field takes the form

$$p^{\text{f:r}}(z',x',.,s) \sim Q^*_{-,x'}(z')G_-(0,z')^* p^{\text{sc}}_-(0,.,.,s). \tag{4.10}$$

Using these representations, one can design a seamless integration of reverse-time (full-wave) based with downward-continuation (one-way-wave) based inverse scattering.

By backsubstituting $\hat{p}^{\text{sc}}$ from (3.9) into (4.1), we obtain a map $\delta c^{-2}(x')$ to $p^{\text{f:r}}(x',t,s)$. We analyze this map in the Section 6.

5. High-frequency inverse scattering: RTM formulation.

5.1. Modelling operator. Assumption 2. *The incident field* $p^0(x',t,s)$ *is free of caustics.*

The source field, p^0, in the absence of caustics, can be asymptotically represented as

$$G(x',t,s) = \int a(x',s,\tau)\exp[\mathrm{i}\tau(t-T(x',s))]\,\mathrm{d}\tau, \tag{5.1}$$

where $T(x',s)$ denotes the travel time along a ray connecting x' with s; T satisfies an eikonal equation. To leading order, a is given by $A(x',s)\tau^m$, where $m = \frac{n-3}{2}$; A satisfies a transport equation. The contrast source in (3.9) can then, to leading order, be written in the form

$$-\partial_t^2 D_t^m\, E_L A\delta c^{-2}(x',t,s), \quad E_L g(x',t,s) = \delta(t-T(x',s))g(x'). \tag{5.2}$$

Here, s is viewed as a parameter.

The adjoint of E_L is the restriction $R_L = E_L^*$ which can be written as a time correlation evaluated at zero lag,

$$R_L f(x') = \int \delta(t' - T(x', s))\, f(x', t')\mathrm{d}t'. \tag{5.3}$$

The Green's function associated with the wave equation admits an oscillatory integral representation of the type,

$$G(y, t, x') = \int a(y_I, x', \eta_J, \tau) \exp[\mathrm{i}\phi(y, x', t, \eta_J, \tau)]\, \mathrm{d}\eta_J \mathrm{d}\tau. \tag{5.4}$$

The adjoint, $U^* E_a$, of $R_a U$ has the kernel

$$\int \overline{a(y_I, x', \eta_J, \tau)} \exp[-\mathrm{i}\phi(y, x', t, \eta_J, \tau)]\, \mathrm{d}\eta_J \mathrm{d}\tau. \tag{5.5}$$

There exists a generating function $S = S(y_I, x', \eta_J, \tau)$ such that

$$y_J = -\frac{\partial S}{\partial \eta_J},\ \eta_I = \frac{\partial S}{\partial y_I}, \qquad t = -\frac{\partial S}{\partial \tau},\ \xi' = \frac{\partial S}{\partial x'} \tag{5.6}$$

and

$$\phi(x', y, t, \eta_J, \tau) = S(y_I, x', \eta_J, \tau) + \langle \eta_J, y_J \rangle + \tau t. \tag{5.7}$$

The Green's function defines the solution operator, U, to the wave equation. We let R_a denote the restriction to the receiver manifold $\Sigma_s \subset \partial\Omega$, which is locally described by $z = 0$ if $x = (z, y)$. Assumption 1 excludes the presence of grazing rays.

The modelling of scattered waves, p^{sc}, in the 'high-frequency' Born approximation can then be written as the composition of operators, $F_0 = -\partial_t^2 D_t^m\, R_a U E_L A$, with

$$\begin{aligned}(F_0 \delta c^{-2})(y, t, s) = -\partial_t^2 D_t^m\, R_a U E_L A \delta c^{-2}(y, t, s) = \int\!\!\int \tau^{m+2} a(y_I, x', \eta_J, \tau)\, A(x', s) \\ \exp[\mathrm{i}(S(y_I, x', \eta_J, \tau) + \langle \eta_J, y_J \rangle + \tau(t - T(x', s)))]\, \mathrm{d}\eta_J \mathrm{d}\tau\ \delta c^{-2}(x')\, \mathrm{d}x'.\end{aligned} \tag{5.8}$$

5.2. Inversion formula. The imaging operator, $F_0^* = A R_L U^* E_a\, (-D_t^m \partial_t^2)$, acts on data as

$$\begin{aligned}(F_0^* d)(x') &= \int\!\!\int \tau^{m+2} A(x', s)\, \overline{a(y_I, x', \eta_J, \tau)} \\ &\quad \exp[-\mathrm{i}(S(y_I, x', \eta_J, \tau) + \langle \eta_J, y_J \rangle + \tau(t - T(x', s)))]\, \mathrm{d}\eta_J \mathrm{d}\tau\ d(y, t, s)\, \mathrm{d}y\mathrm{d}t \\ &= \int \overline{\tau^m A(x', s)\ \exp[-\mathrm{i}\tau T(x', s)]} \\ &\quad \left\{ \int\!\!\int \tau^2 \overline{a(y_I, x', \eta_J, \tau) \exp[\mathrm{i}(S(y_I, x', \eta_J, \tau) + \langle \eta_J, y_J \rangle + \tau t)]}\, d(y, t, s)\, \mathrm{d}y\mathrm{d}t\ \mathrm{d}\eta_J \right\} \mathrm{d}\tau \\ &= A(x', s) \int \delta(t' - T(x', s))\ \int\!\!\int \tau^{m+2} \overline{a(y_I, x', \eta_J, \tau)} \\ &\quad \exp[-\mathrm{i}(S(x', y_I, \eta_J) + \langle \eta_J, y_J \rangle + \tau(t - t'))]\, d(y, t, s)\, \mathrm{d}y\mathrm{d}t\, \mathrm{d}\eta_J \mathrm{d}\tau\ \mathrm{d}t'.\end{aligned} \tag{5.9}$$

Upon carrying out the integration over t', we recover the classical RTM imaging condition (see, for example, [5]).

It is possible to modify the imaging operator and obtain an inverse scattering transform, like in the constant background case. We introduce

$$C_0 = A^{-1} \sum_{j=0}^{n} (\Xi_j R_L)\, \Xi_j U^* E_a\, (-2)\, \mathrm{i} \mathcal{N} \mathcal{S}\, (-D_t^{-m} \partial_t^{-1})$$

in which $\mathcal{S} = \mathcal{S}(D_t)$ with

$$\mathcal{S}(\tau) = \frac{2}{\tau^2}, \tag{5.10}$$

and $\mathcal{N} = \mathcal{N}(y, D_y, D_t)$ with

$$\mathcal{N}(y, \eta, \tau) = \tau\, [c(0, y)^{-2} - \tau^{-2} \|\eta\|^2]^{1/2} \tag{5.11}$$

corresponds with the boundary differential operator $n_j(x)\partial_j$ (outgoing wave constituents), that is, $\mathcal{N}$ corresponds with $B_{-,\cdot}(0)$; $\Xi = \Xi(x', D_{x'}, D_t)$ is given by

$$\Xi_0(x', \xi', \tau) = c(x')^{-1} \tau, \quad \Xi_j(x', \xi', \tau) = \xi'_j,\ j = 1, \ldots, n. \tag{5.12}$$

We have

$$\begin{aligned}(C_0 d)(x') \approx \int A(x', s)^{-1} \sum_{j=0}^{n} \Xi_j(x', \tau \partial_{x'} T(x', s), -\tau)\, \overline{\exp[-\mathrm{i}\tau T(x', s)]} \\ \Bigg\{ \iint \tau^{-m} \mathcal{S}(\tau) (\mathrm{i}\tau)^{-1} \overline{a(y_I, x', \eta_J, \tau)}\, \Xi_j(x', -\partial_{x'} S(y_I, x', \eta_J, \tau), -\tau) \\ \overline{\exp[\mathrm{i}(S(y_I, x', \eta_J, \tau) + \langle \eta_J, y_J \rangle + \tau t)]}\, (-2)\, \mathrm{i} \mathcal{N}(y, D_y, D_t) d(y, t, s)\, \mathrm{d}y \mathrm{d}t\ \mathrm{d}\eta_J \Bigg\}\, \mathrm{d}\tau.\end{aligned} \tag{5.13}$$

We note that we can incorporate a pseudodifferential cutoff in combination with the application of $\mathcal{N}$ to ensure that Assumption 1 is satisfied.

Theorem 5.1. *("Kirchhoff"-RTM) Let ψ denote a pseudodifferential cutoff with a conically compact support (Σ_s). Under Assumptions 1, 2, it holds true that*

$$C_0\, \psi F_0 = \Psi_0 + R_0, \tag{5.14}$$

where $\Psi_0 = \Psi_0(x', D_{x'})$ is a pseudodifferential operator of order 0 *with principal symbol* 1 *microlocally where there is illumination, and $R_0 = R_0(x', D_{x'})$ is a pseudodifferential operator of order* -1.

6. Inverse scattering: Wave-equation analogue. We initially follow the imaging principle proposed by Claerbout [6], including the division by the amplitude squared of the incident field (Baysal *et al.* [2]), while deriving an inverse scattering formulation for the common-source acquisition geometry

Theorem 6.1. *("Wave-equation" RTM) Let $p^{\mathrm{f:r}}$ be given by (4.1). Let Ξ_j, $j = 0, 1, \ldots, n$ be given by (5.12) and $\mathcal{S}$ by (5.10). Let C be given by*

$$\begin{aligned}(Cd)(x') = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{|\hat{p}^0(x', \tau, s)|^2}\, \mathcal{S}(\tau)\, (\mathrm{i}\tau)^{-1} \\ \Bigg[-\sum_{j=0}^{n} (\Xi_j(x', D_{x'}, \tau) \overline{\hat{p}^0(x', \tau, s)})\ (\Xi_j(x', D_{x'}, \tau) \hat{p}^{\mathrm{f:r}}(x', \tau, s)) \Bigg]\, \mathrm{d}\tau.\end{aligned} \tag{6.1}$$

Under Assumptions 1, 2, it holds true that

$$C\,\psi F = \Psi + R, \tag{6.2}$$

where $\Psi = \Psi(x', D_{x'})$ is a pseudodifferential operator of order 0 *with principal symbol* 1 *microlocally where there is illumination, and $R = R(x', D_{x'})$ is a pseudodifferential operator of order* -1.

Proof. We will exploit the occurrence of a cross-correlation Green's function. The reciprocity relation of the time-correlation type generates a Green's function through cross correlations: for $x', x'' \in \Omega\backslash\partial\Omega$

$$\hat{H}(x', \tau, x'') = \int_{\partial\Omega} \Bigg[-\sum_{j=1}^{n} n_j(x)(\partial_{x_j}\hat{G})(x, \tau, x'')\,\overline{\hat{G}(x', \tau, x)} + \hat{G}(x, \tau, x'')\,\sum_{j=1}^{n} n_j(x)\overline{(\partial_{x_j}\hat{G})(x', \tau, x)}\Bigg]\,\mathrm{d}A(x); \tag{6.3}$$

with integration over the full boundary,

$$\hat{H}(x', \tau, x'') = \hat{G}(x', \tau, x'') - \overline{\hat{G}(x'', \tau, x')} = 2\mathrm{i}\,\mathrm{Im}\{\hat{G}(x', \tau, x'')\}; \tag{6.4}$$

clearly $\hat{H}(x', -\tau, x'') = \overline{\hat{H}(x', \tau, x'')}$, while

$$H(x', t, x'') = G(x', t, x'') - G(x', -t, x'').$$

In the above, x plays the role of receiver and x'' the role of 'virtual' source. Upon replacing $\partial\Omega$ in (6.3) by Σ_s, $\hat{H}$ is replaced by $\hat{H}^{\Sigma_s}$.

Substituting (3.9) into (4.1) using (6.3) yields

$$\hat{p}^{\mathrm{f:r}}(x', \tau, s) = \tau^2 \int_{\Omega} \hat{H}^{\Sigma_s}(x', \tau, x'')\hat{p}^0(x'', \tau, s)\,\delta c^{-2}(x'')\,\mathrm{d}V(x''), \tag{6.5}$$

which compares directly to the scattering equation (3.9) with $x' \in \Omega\backslash\partial\Omega$. This may be referred to as the outcome of retrofocusing the scattered field in receivers. As before, we will assume that $p^0(x'', t, s) = G(x'', t, s)$. It then follows that the kernel of the composition $C\,\psi F$ is given by $(2\pi)^{-1}\int N(x', x'', \tau; s)\,\mathrm{d}\tau$, with

$$N(x', x'', \tau; s) = \frac{1}{|\hat{p}^0(x', \tau, s)|^2}\,\mathcal{S}(\tau)\tau^2(\mathrm{i}\tau)^{-1} \Bigg[\sum_{j=1}^{n}(\partial_{x'_j}\overline{\hat{p}^0(x', \tau, s)})\;(\partial_{x'_j}\hat{H}^{\Sigma_s}(x', \tau, x''))\,\hat{p}^0(x'', \tau, s) - \overline{\hat{p}^0(x', \tau, s)}\;\tau^2 c(x')^{-2}\,\hat{H}^{\Sigma_s}(x', \tau, x'')\,\hat{p}^0(x'', \tau, s)\Bigg]. \tag{6.6}$$

Because $N(x', x'', -\tau; s) = \overline{N(x', x'', \tau; s)}$, we have

$$(2\pi)^{-1}\int N(x', x'', \tau; s)\,\mathrm{d}\tau = \mathrm{Re}\;\pi^{-1}\int_{\mathbb{R}_{\geq 0}} N(x', x'', \tau; s)\,\mathrm{d}\tau.$$

Up to various (pseudodifferential) factors, $(2\pi)^{-1}\int N(x', x'', \tau; s)\,\mathrm{d}\tau$ generates the kernel of the normal operator F^*F (cf. (3.10)) which is given by

$$\int\int_{\Sigma_s} \tau^4\;\overline{\hat{p}^0(x', \tau, s)}\,\overline{\hat{G}(x, \tau, x')}\;\hat{p}^0(x'', \tau, s)\,\hat{G}(x, \tau, x'')\,\mathrm{d}A(x)\mathrm{d}\tau.$$

This kernel is a conormal distribution. To show this, we consider the propagation of singularities following the composition of F^* with F, thus accounting for the integration over x while using the method of stationary phase. We view the pair consisting of a source ray connecting s with x'' and a receiver ray connecting $x = r$ with x''. We let T denote travel time along a ray, and write $t_{\text{inc}}(x'', s) = T(x'', s)$ for the travel time along the source ray. If t denotes the so-called two-way time associated with a reflection off a scatterer at x'', the travel time along the receiver ray must be given by $t - t_{\text{inc}}(x'', s)$ which equals $T^{(i)}(r, x'')$ if i labels branches in the case of multi-pathing. We have $\rho = \tau\partial_r t$ and $\sigma = \tau\partial_s t$ defining the directions of the receiver ray at the receiver and of the source ray at the source, respectively. Under the assumption that no caustics develop in the incident field, r, ρ and t and determine t_{inc} and σ, and hence the geometry of the ray pair, uniquely (see [14]). But then the wavefront set of the above mentioned kernel must be contained in the diagonal of $T^*\Omega\backslash 0 \times T^*\Omega\backslash 0$.

We revisit the alternative kernel representation, which involves regrouping the factors in the integrand,

$$\int \overline{\hat{p}^0(x', \tau, s)}\; \hat{H}^{\Sigma_s}(x', \tau, x'')\, \hat{p}^0(x'', \tau, s)\, \mathrm{d}\tau,$$

see Fig. 1. From the method of stationary phase, it follows that x' and x'' must lie on a common receiver ray (the particular receiver, which must be contained in Σ_s, giving the stationary contribution to the integration over x in (6.3)). In comparison with the previous analysis, the travel time $T^{(i)}(r, x'')$ has been correlated out, whereas $T^{(i)}(x', x'') = T^{(i)}(r, x') - T^{(i)}(r, x'')$; the same reasoning, based on the mean value theorem, applies.

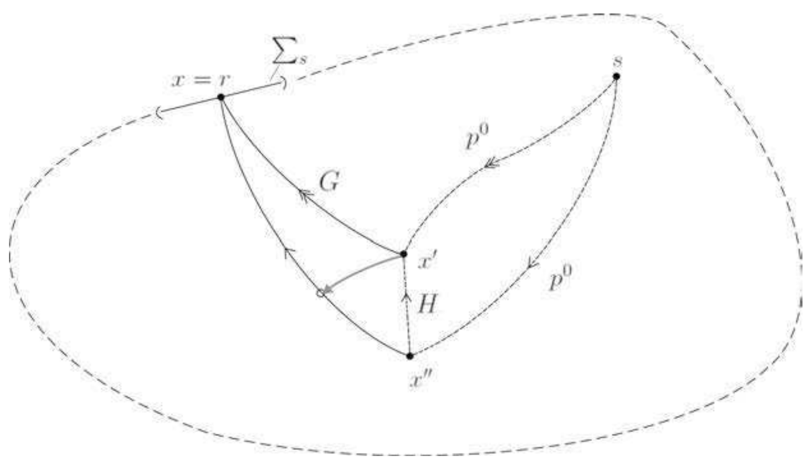

FIG. 1. *Retrofocusing: The phase of* $(2\pi)^{-1} \int N(x', x'', \tau; s)\, \mathrm{d}\tau$ *can be stationary only if* x' *lies on the ray connecting* x'' *with* $x = r$. *Dashed: A description in terms of interferometry. The rays indicate the propagation of singularities.*

We proceed with determining the symbol of $(2\pi)^{-1} \int N(x', x'', \tau; s)\, \mathrm{d}\tau$. We consider the near-field expansion

$$\overline{\hat{p}^0(x', \tau, s)}\, \hat{p}^0(x'', \tau, s) = |\hat{p}^0(x', \tau, s)|^2 \exp[-\mathrm{i}\tau\, \partial_{x'} T(x', s) \cdot (x' - x'')] + \text{l.o.t.} \tag{6.7}$$

To arrive at the near-field expansion of $\hat{H}^{\Sigma_s}(x', \tau, x'')$, we first consider the early time representation,

$$\tfrac{1}{2} H^{\Sigma_s}(x', t, x'') = \int_{\mathbb{R}_{\geq 0}} F(x', \tau, x'')\, \frac{[\exp(\mathrm{i}\tau t) - \exp(-\mathrm{i}\tau t)]}{2\pi}\, \mathrm{d}\tau, \tag{6.8}$$

in which

$$F(x', \tau, x'') = \frac{1}{(2\pi)^{n-1}} \frac{c(x'')^2}{2} \frac{1}{\mathrm{i}\tau} \left[\int_{E_\nu(x')} (\tau c(x')^{-1})^{n-1} \exp[\mathrm{i}\tau c(x')^{-1} \nu \cdot (x' - x'')]\, c(x')^{-1} \mathrm{d}\nu + \text{l.o.t.} \right], \tag{6.9}$$

whence

$$\tfrac{1}{2} \hat{H}(x', \tau, x'') = \begin{cases} F(x', \tau, x'') & \text{if} \quad \tau > 0 \\ -F(x', -\tau, x'') & \text{if} \quad \tau < 0. \end{cases} \tag{6.10}$$

Substituting (6.7) and (6.10) into (6.6) yields

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} N(x', x'', \tau; s)\, \mathrm{d}\tau = \frac{1}{2\pi} \operatorname{Re} \int_{\mathbb{R}_{\geq 0}} \frac{1}{2^{n-1}\pi^{n-1}}\ c(x')^2 (-) \\ \int_{E_\nu(x')} (\tau c(x')^{-1})^{n-1} \Bigg[\overbrace{\sum_{j=1}^{n} (\partial_{x'_j} T)(x', s)\, c(x')^{-1} \nu_j - c(x')^{-2}}^{\sum_{j=1}^{n} \partial_{x'_j} T(x', s)\, (c(x')^{-1} \nu_j - \partial_{x'_j} T(x', s))} \Bigg] \\ \exp[\mathrm{i}\tau\, (c(x')^{-1} \nu - \partial_{x'} T(x', s)) \cdot (x' - x'')]\, c(x')^{-1} \mathrm{d}\tau \mathrm{d}\nu + \text{l.o.t.} \end{aligned} \tag{6.11}$$

With $\partial_{x'} T(x', s)$ fixed, we form $\xi = \tau\, (c(x')^{-1}\nu - \partial_{x'} T(x', s))$, the superposition of source and receiver co-vectors, associated with the source and receiver rays connecting the image point x' to the source (at s) and a receiver (at x), respectively, see Fig. 2. The integration in ξ is over a part $E_\xi(x')$, obtained from $E_\nu(x')$, of a sphere with radius $c_0^{-1}(x')$ centered at $\partial_{x'} T(x', s)$. We then change variables of integration, from (τ, ν) to ξ, with

$$\mathrm{d}\xi = \left[c(x')^{-2} - \sum_{j=1}^{n} (\partial_{x'_j} T)(x', s)\, c(x')^{-1} \nu_j \right] c(x') (|\tau| c(x')^{-1})^{n-1} \mathrm{d}\tau \mathrm{d}\nu,$$

so that

$$\frac{1}{2\pi} \int_{\mathbb{R}} N(x', x'', \tau; s)\, \mathrm{d}\tau = \frac{1}{(2\pi)^n} \int_{E_\xi(x')} \exp[\mathrm{i}\xi \cdot (x' - x'')]\, \mathrm{d}\xi + \text{l.o.t.}$$

that is, the principal symbol of $(2\pi)^{-1} \int_{\mathbb{R}} N(x', x'', \tau; s)\, \mathrm{d}\tau$ equals 1 where there is illumination. □

Traditionally, in RTM-based wave-equation imaging, the terms with operators $\varXi_j(x', D_{x'}, \tau)$, $j = 1, \ldots, n$ in (6.1) are missing. The 'true-amplitude' imaging condition generating (6.1) has the property that it 'annihilates' field constituents propagating from the source at s towards x', and hence acts as a 'one-way filter'.

The 'true-amplitude' image Cd is artifact free under the assumption of absence of source caustics (receiver caustics are allowed). This coincides with the result in Section 5.1 in Nolan and Symes [14].

Corollary 6.2. *We consider the multi-source application. The composition* $\left(\frac{\partial}{\partial t}\right)^{-1} F \frac{\partial}{\partial s} C$ *defines pseudodifferential operators of order* 1 *which are annihilators of the data.*

The annihilators characterize the range of F. They can be exploited to develop wave-equation reflection tomography.

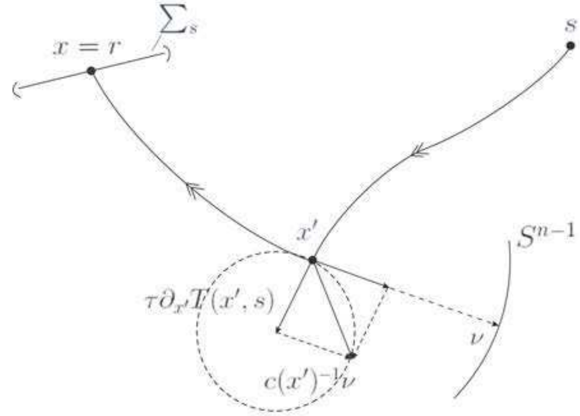

FIG. 2. *Mapping from ν to ξ.*

7. Discussion. We presented a microlocal analysis point of view of inverse scattering in the RTM approach, while making use of the notion of backpropagation. The resulting inverse scattering transform is a Fourier integral operator the canonical relation of which is a graph, and lends itself, for example, directly to a sparse discretization using wave packets or 'curvelets' [8]. We explained the recently discussed RTM 'artifacts' and provided a technique to remove them. We also presented a seamless integration of reverse-time imaging with downward-continuation based imaging, and established an explicit relation between RTM and interferometry.

In practice, inverse scattering in the RTM approach is carried out using data generated by a large number of sources, while the results are integrated or averaged. In this situation, annihilators with the source coordinate(s) being the "redundant" one(s) can be constructed and used to develop reflection tomography. However, under the assumptions presented in the main text, it is also possible to apply the inverse scattering transform to data corresponding with a sparse set of sources. The incorporation of (sparsity promoting) regularization techniques can still yield a proper image. The resulting approach can be applied both in exploration seismology and global seismology. We discussed a way to implement the RTM-based inverse scattering transform in terms of solving a couple of inhomogeneous Helmholtz equations.

Appendix A. Time reversal and wavefield extrapolation.

In preparation of developing the proper backpropagation component of reverse-time-migration derived inverse scattering, we briefly review the reciprocity relation including time reversal and its implications. We denote the time correlation evaluated at t by $\overset{(-t)}{*}$. For two general functions f, g we have

$$f \overset{(-t)}{*} g = \int_{\mathbb{R}} f(t - t') g(-t') \, \mathrm{d}t'.$$

The reciprocity relation of the time-correlation type, using (3.3)-(3.4), implies

$$\begin{aligned}\text{(A.1)}\quad &\int_{\partial\Omega}\left[\sum_{j=1}^{n} n_j(x)v_j^{\mathrm{sc}}(x,.,s) \overset{(-t)}{*} p(x,.) + p^{\mathrm{sc}}(x,.,s) \overset{(-t)}{*} \sum_{j=1}^{n} n_j(x)v_j(x,.)\right]\mathrm{d}A(x)\\ &= \int_{\Omega}\left[p^{\mathrm{sc}}(x,.,s) \overset{(-t)}{*} q(x,.) + \sum_{j=1}^{n} v_j^{\mathrm{sc}}(x,.,s) \overset{(-t)}{*} f_j(x,.)\right]\mathrm{d}V(x).\\ &\quad + \int_{\Omega}\left[q^{\mathrm{sc}}(x,.,s) \overset{(-t)}{*} p(x,.) + \sum_{j=1}^{n} f_j^{\mathrm{sc}}(x,.,s) \overset{(-t)}{*} v_j(x,.)\right]\mathrm{d}V(x).\end{aligned}$$

This relation is directly connected to the energy balance, where the boundary integral can be identified with flux. In the boundary integral, we can make use of relation (3.8). In expression (A.1), $\partial\Omega$ is viewed as the manifold containing the receivers.

We will make use of Fourier representations, for example,

$$\text{(A.2)}\qquad p^{\mathrm{sc}}(x,t,s) = \int_{\mathbb{R}} \hat{p}^{\mathrm{sc}}(x,\tau,s)\,\exp(\mathrm{i}\tau t)\,\mathrm{d}\tau.$$

We have

$$\begin{aligned}\text{(A.3)}\quad &\int_{\Omega}\left[\hat{p}^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{q}(x,\tau)} + \sum_{j=1}^{n}\hat{v}_j^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{f}_j(x,\tau)}\right]\mathrm{d}V(x)\\ &+ \int_{\Omega}\left[\hat{q}^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{p}(x,\tau)} + \sum_{j=1}^{n}\hat{f}_j^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{v}_j(x,\tau)}\right]\mathrm{d}V(x)\\ &= \int_{\partial\Omega}\left[\sum_{j=1}^{n} n_j(x)\hat{v}_j^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{p}(x,\tau)} + \hat{p}^{\mathrm{sc}}(x,\tau,s)\,\sum_{j=1}^{n}\overline{n_j(x)\hat{v}_j(x,\tau)}\right]\mathrm{d}A(x).\end{aligned}$$

As before, we assume $\delta\rho \equiv 0$. Let $x' \in \Omega\backslash\partial\Omega$. With $q(x,t) = \delta(x-x')\delta(t)$ and $f(x,t) \equiv 0$, we obtain

$$\begin{aligned}\text{(A.4)}\quad &\hat{p}^{\mathrm{sc}}(x',\tau,s) + \int_{\Omega}\hat{q}^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{G}^{pq}(x,\tau,x')}\mathrm{d}V(x)\\ &= \int_{\partial\Omega}\left[\sum_{j=1}^{n} n_j(x)\hat{v}_j^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{G}^{pq}(x,\tau,x')} + \hat{p}^{\mathrm{sc}}(x,\tau,s)\sum_{j=1}^{n}\overline{n_j(x)\hat{G}_j^{vq}(x,\tau,x')}\right]\mathrm{d}A(x),\end{aligned}$$

while with $q(x,t) \equiv 0$ and $f(x,t) = e_i\delta(x-x')\delta(t)$, we obtain

$$\begin{aligned}\text{(A.5)}\quad &\hat{v}_i^{\mathrm{sc}}(x',\tau,s) + \int_{\Omega}\hat{q}^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{G}_i^{vq}(x,\tau,x')}\mathrm{d}V(x)\\ &= \int_{\partial\Omega}\left[\sum_{j=1}^{n} n_j(x)\hat{v}_j^{\mathrm{sc}}(x,\tau,s)\,\overline{\hat{G}_i^{pf}(x,\tau,x')} + \hat{p}^{\mathrm{sc}}(x,\tau,s)\sum_{j=1}^{n}\overline{n_j(x)\hat{G}_{ji}^{vf}(x,\tau,x')}\right]\mathrm{d}A(x).\end{aligned}$$

We have the symmetries, $G_i^{vq}(x,t,x') = G_i^{pf}(x',t,x)$ and $G_{ij}^{vf}(x,t,x') = G_{ji}^{vf}(x',t,x)$.

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