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AAK-THEORY ON WEIGHTED SPACES

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Abstract. We extend some of the results of classical AAK-theory to Hankel operators on certain weighted spaces, thereby providing a constructive proof of a theorem by S. Treil and A. Volberg.

1. Introduction. "AAK-theory" is short for the results on Hankel operators on the Hardy space H^2 proved in a sequence of papers by V. M. Adamyan, D. Z. Arov and M. G. Krein around 1970, (see [1], [2], [3], [4]). We will in this paper be concerned with an extension of the theorem on best rational approximation of a function in L^∞/H^∞.

The introduction is split into four parts. The first is a very brief introduction to Hankel operators, the second goes through the classical AAK-theory, the third introduces the weighted spaces we will be working with and finally in the fourth we state the new results obtained in this paper.

1.1. Classical Hankel operators. Let H^2 denote the Hardy space on the unit disc $\mathbb{D}$, which we consider as a subspace of L^2 on the unit circle $\mathbb{T}$ with normalized arc-lenght measure. That is $H^2 = \{f \in L^2 : \hat{f}_n = 0 \ \forall n < 0\}$ where $\hat{f}_n$ denotes the Fourier coefficients of f. Set $H^2_- = L^2 \ominus H^2$. Let z denote the identity function on $\mathbb{T}$ and M_z the operator of multiplication by z on L^2, i.e. $M_z f = zf$.

Classical Hankel operators were defined as bounded operators on $l^2(\mathbb{Z}_+)$ having the following Hankel matrix representation

$$(\gamma_{i+j})_{i,j\in Z^+} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \gamma_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \gamma_n \in \mathbb{C}, \tag{1.1}$$

in the standard basis $e_0 = (1,0,0,\ldots), e_1 = (0,1,0,\ldots)$, etc. It is easy to see that a bounded operator Γ is Hankel if and only if it satisfies the commutation relation

$$\Gamma S = S^*\Gamma, \tag{1.2}$$

where S is the unilateral shift, (i.e $Se_n = e_{n+1}$ for all $n \in \mathbb{Z}_+$).

Another common definition is that $\Gamma : H^2 \to H^2_-$ is a Hankel operator if it satisfies the commutation relation

$$\Gamma M_z = P_- M_z \Gamma, \tag{1.3}$$

where $P_- : L^2 \to H^2_-$ denotes the orthogonal projection onto H^2_-. It is not hard to see that this is satisfied if and only if Γ has a Hankel matrix in the bases $\{z^n\}_{n=0}^\infty$ and $\{z^{-n}\}_{n=1}^\infty$. The second more cumbersome definition has become the popular one mainly due to the deep connections between Hankel operators and the structure of the space H^2, (considered as a space of analytic functions in the unit disc $\mathbb{D}$). For example, setting $\phi(z) = \sum_{n=0}^\infty \gamma_n z^{-n-1}$ we have that

$$\Gamma f = P_- \phi f.$$

Any function ψ such that $P_-\psi = \sum_{n=0}^\infty \gamma_n z^{-n-1}$ is called a symbol for Γ. Let $M_\psi : L^2 \to L^2$ be the operator of multiplication by ψ, i.e. $M_\psi f = f$. Nehari's theorem states that a Hankel operator Γ is bounded if and only if there is a symbol $\psi \in L^\infty$ such that

$$\Gamma = P_- M_\psi$$

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and moreover

$$\|\Gamma\| = \|\psi\|_{L^\infty/H^\infty}. \tag{1.4}$$

This settles the issue of when a Hankel matrix is bounded. Using the celebrated characterization of BMO by Fefferman, this also leads to the theorem that the Hankel matrix (γ_{i+j}) is bounded if and only if

$$\sum_{n=0}^{\infty} \gamma_n z^n \in BMO.$$

1.2. Singular values and the AAK-theorem. Given any operator $\Gamma \in \mathcal{L}(X, \tilde{X})$ we define its singular values as

$$\sigma_n(\Gamma) = \inf\{\|\Gamma - K\| : \mathsf{Rank}\ K \leq n\}, \tag{1.5}$$

or equivalently

$$\sigma_n(\Gamma) = \inf\{\|\Gamma|_{\mathcal{M}}\| : \mathcal{M} \leq X \text{ and } \mathsf{codim}\ \mathcal{M} = n\}, \tag{1.6}$$

where $\mathcal{M} \leq X$ means that $\mathcal{M}$ is a subspace and $\Gamma_{\mathcal{M}}$ denotes the restriction of Γ to $\mathcal{M}$. Also set $\sigma_\infty(\Gamma) = \lim_{n\to\infty} \sigma_n(\Gamma)$ and let $\|\Gamma\|_e$ denote the essential norm of Γ. When there is no risk of confusion we will let the dependence of the singular values on Γ be implicit. Clearly $\|\Gamma\| = \sigma_0$ and $\|\Gamma\|_e = \sigma_\infty$. A vector $u \in X$ will be called a σ-singular vector if $\|u\| = 1$ and

$$\sigma^2 u = \Gamma^* \Gamma u.$$

Let N be the amount of σ_n's such that $\sigma_n > \sigma_\infty$. An application of the spectral theorem shows that there is an orthonormal set $\{u_n\}_{n=0}^N \subset X$ such that each u_n is an σ_n-singular vector and $\|\Gamma|_{\{u_n\}^\perp}\| = \|\Gamma\|_e$. If a σ_n is distinct, (i.e. $\sigma_{n-1} > \sigma_n > \sigma_{n+1}$), then u_n is unique up to multiplication with a unimodular constant.

This paper is concerned with generalizations of the following celebrated result by Adamyan, Arov and Krein.

THEOREM 1.1. *(AAK) Let $\Gamma : H^2 \to H^2_-$ be a Hankel operator and let σ_n be its n'th singular value. Then there is a rank n Hankel operator K such that*

$$\sigma_n = \|\Gamma - K\|.$$

If σ_n is distinct, then r is unique.

It is easy to see that a rank 1 Hankel operator necessarily has the following form

$$K = \begin{pmatrix} 1 & z_0 & z_0^2 & \cdots \\ z_0 & z_0^2 & z_0^3 & \cdots \\ z_0^2 & z_0^3 & z_0^4 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad |z_0| < 1, \tag{1.7}$$

but it is not true that any finite rank Hankel operator is a sum of such. In fact, a symbol for the above rank one Hankel operator is easily seen to be $(1 - z_0\overline{z})^{-1}$ and in general, any rank n Hankel operator has a symbol of the form $r(\overline{z})$ where r is a rational function with degree[1] n and all poles lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$. In terms of applications, the power of the AAK-theorem comes from the fact that the location of these poles can be easily calculated using the singular vectors. For simplicity, let us

[1]If $r = p/q$ where p and q are polynomials with no common divisors, then the degree of r is defined as $\max(\deg p, \deg q)$.

assume that σ_n is distinct and denote the corresponding singular vector by u_n. The proof of 1.1 then shows that u_n has precisely n roots $(z_j)_{j=1}^n$, counted with multiplicity, and that the poles of the rational function r in Theorem 1.1 are located at $(1/z_j)_{j=1}^n$, again counted with multiplicity. In particular, if u_n has distinct zeroes, then the best rank n Hankel approximant of Γ is a sum of n matrices of the form (1.7) with z_0 replaced by z_j, $j = 1, \dots, n$.

Using Beurling's and Nehari's theorem, a short argument shows that the AAK-theorem is equivalent with the following result

THEOREM 1.2. *Let $\Gamma : H^2 \to H^2_-$ be a Hankel operator and let σ_n be its n'th singular value. Then there is a M_z-invariant subspace $\mathcal{M} \subset H^2$ such that $\mathsf{codim}\ \mathcal{M} = n$ and $\|\Gamma|_{\mathcal{M}}\| = \sigma_n$. If σ_n is distinct, then $\mathcal{M}$ is unique.*

In the case that σ_n is distinct, it follows from the proof that

$$\mathcal{M} = cl(\mathsf{Span}\ \{z^k u_n : k \in \mathbb{Z}_+\}), \tag{1.8}$$

so by Beurling's theorem it follows that $\mathcal{M}$ is completely determined by the zeroes of u_n. We will denote the right hand side of (1.8) by $[u_n]_{M_z}$.

S. Treil and A. Volberg has recently shown that Theorem 1.2 can be extended to certain generalized Hankel operators, that we will introduce in the next section. Their proof relies on a fixed point lemma by Ky Fan, and henceforth it is not clear wether (1.8) holds, which in terms of applications is important as it provides a way to actually calculate $\mathcal{M}$. The purpose of the present paper is to show that (1.8) indeed holds for a large class of "generalized Hankel operators".

1.3. Generalized Hankel operators. Recall the two definitions (1.1) and (1.3) of a Hankel operator. Most extensions of the concept of a Hankel operator are therefore based on the latter definition (1.3). To make the paper more accessible to non-specialists, we have chosen to adopt a definition closer to (1.1). We provide a brief account on various general definitions of Hankel operators and their interconnections in Section 2.

Given a Hilbert space X, let $\mathcal{L}(X)$ denote the set of bounded operators in X. In this paper we shall primarily be concerned with infinite dimensional Hilbert spaces X such that:

(i) $S \in \mathcal{L}(X)$ is an injective operator with closed range and $\mathsf{codim\ Ran}\ S = 1$,

(ii) There exists a be a fixed and non-zero element $x_0 \in X$ that is S-cyclic, i.e.

$$[x_0]_S = X$$

where $[x_0]_S = cl(\mathsf{Span}\ \{S^n x_0 : n \geq 0\})$ and $cl(\cdot)$ denotes the closure.

(iii) B is a left inverse of S such that $B(x_0) = 0$.

Remarks:

1. Note that the unilateral shift on $l^2(\mathbb{Z}_+)$ is such an operator, that $x_0 = e_0$ is cyclic, $S^* e_0 = 0$ that $S^* S = I$. Thus S^* can either be considered as the (unilateral) backward shift or the left inverse of S that annihilates x_0.
2. When working with two Hilbert spaces X and $\tilde{X}$ that both satisfy $(i)-(iii)$, we shall denote the objects S, B etc. that belong to $\tilde{X}$ by $\tilde{S}, \tilde{B}$ etc.

Given X satisfying $(i) - (iii)$, set $x_n = S^n x_0$, $n \geq 0$. It is easy to see that the sequence $(x_n)_{n=0}^\infty$ is minimal, i.e. $x_n \notin \mathsf{Span}\ \{x_m\}_{m \neq n}$, and therefore the action of any operator in $\mathcal{L}(X)$ is uniquely determined by how it acts on $(x_n)_{n=0}^\infty$.

DEFINITION 1.3. *Given Hilbert spaces X and $\tilde{X}$ satisfying $(i) - (iii)$, an operator $\Gamma \in \mathcal{L}(X, \tilde{X})$ is called Hankel if it is represented by a Hankel matrix with respect to $(x_n)_{n=0}^\infty$ and $(\tilde{x_n})_{n=0}^\infty$.*

In analogy with (1.2), it is easy to see that $\Gamma \in \mathcal{L}(X, \tilde{X})$ is Hankel if and only if

$$\Gamma S = \tilde{B} \Gamma. \tag{1.9}$$

We provide some examples of spaces that satisfy $(i) - (iii)$.

EXAMPLE 1.4. *Let X be a Hilbert space of analytic functions on some domain, (typically $R\mathbb{D}$ where $R > 0$), in which Pol is dense and $(z^n)_{n=0}^\infty$ is a minimal sequence. Let M_z denote* *multiplication by the independent variable z and set $S = M_z$. If M_z is bounded below, then X satisfies $(i)-(iii)$ with $x_0 = 1$. Examples of such spaces includes*

- $H^2(R\mathbb{D})$: *Let $R > 0$ and set*

$$H^2(R\mathbb{D}) = \{f \in Hol(R\mathbb{D}) : \|f\|^2 = \sum_{n\geq 0} |\hat{f}_n|^2 R^{2n} < \infty\}.$$

(Here $Hol(\mathbb{D})$ denotes the holomorphic functions on $\mathbb{D}$ and $\hat{f}_n$ denotes the Taylor coefficients of such a function at 0.)

- $D^2(w)$: *Let $w = (w_n)_{n=0}^{\infty}$ be a strictly positive sequence of weights such that $\lim w_{n+1}/w_n = R^{-2} < \infty$ for some $R > 0$, and define*

$$D^2(w) = \{f \in Hol(R\mathbb{D}) : \|f\|^2 = \sum_{n\geq 0} |\hat{f}_n|^2 w_n < \infty\}.$$

Clearly $H^2(R\mathbb{D}) = D^2((R^{2n})_{n\in\mathbb{Z}_+})$. If $w = (1+n)_{n\in\mathbb{Z}_+}$, then $D^2(w)$ is the Dirichlet space.

- $H^2(w)$: *Let $w \in L^1(\mathbb{T})$ be a Helson-Szegö weight, (see [16]) and define $H^2(w)$ as the closure of polynomials in*

$$L^2(w) = \{f : \|f\|^2 = \int |f|^2 w \; dm < \infty\}.$$

Here, M_z is isometric and the sequence (z^n) is minimal by the Helson-Szegö theorem. Note that if we attempt to define $H^2(w)$ as above for a weight w with $\int |\log w| \; dm = \infty$, then, by Szegö's theorem, (z^n) does not become minimal. In fact, we get $H^2(w) = L^2(w)$.

All the above examples are Hilbert spaces of analytic functions on some disc $R\mathbb{D}$, and in addition they satisfy

(iv) $\sigma(S) = cl(R\mathbb{D})$, $S - \lambda$ is bounded below and $\mathsf{codim}\ \mathsf{Ran}\ S - \lambda = 1$ for all $\lambda \in R\mathbb{D}$.

Definition 1.5. *A Hilbert space $\mathcal{H}$ will be called a Hilbert space of analytic functions on $R\mathbb{D}$ if its elements are analytic functions on $R\mathbb{D}$ and $(i)-(iv)$ holds with $x_0 = 1$ and $S = M_z$.*

Some of our results are more natural to present for Hilbert spaces of analytic function than spaces satisfying $(i)-(iv)$. By a modification of the proof in [8] it follows that any Hilbert space X that satisfies $(i)-(iv)$ can be represented as a Hilbert space of analytic functions, i.e. there exists a unitary map $\mathcal{U} : X \to \mathcal{H}$ such that $\mathcal{U}(x_0) = 1$ and $\mathcal{U}S = M_z\mathcal{U}$. We omit a proof of this result. Note that the $\mathcal{U}x_n = z^n$.

1.4. An AAK-theorem for generalized Hankel operators. An operator T on some Hilbert space X is called expansive if $\|Tx\| \geq \|x\|$ for all $x \in X$ and contractive if $\|T\| \leq 1$. We will in this paper show the following results.

Theorem 1.6. *Let $X, \tilde{X}$ be spaces satisfying $(i)-(iii)$ such that S is expansive and $\tilde{B}$ is contractive. Let $\Gamma : X \to \tilde{X}$ be a Hankel operator with singular values $\sigma_0 \geq \sigma_1 \geq \ldots$. Let $u_N \in X$ be a singular vector with singular value σ_N. Then*

$$\|\Gamma|_{[u_N]_S}\| = \sigma_N.$$

In particular, we have

Corollary 1.7. *Let $X, \tilde{X}$ and Γ be as in Theorem 1.6. Let $u_N \in X$ be a singular vector with singular value σ_N and assume that $\sigma_{N-1} > \sigma_N$. Then $\mathsf{codim}\ [u_N]_S \geq N$.*

We will now discuss circumstances under which circumstances we have equality in the above Corollary. First, some definitions.

Definition 1.8. *Given a Hilbert space $\mathcal{H}$ of analytic functions on $R\mathbb{D}$ and an element $u \in \mathcal{H}$, let $(z_j)_{j=1}^N$ denote its zeroes and let $n_j \in \mathbb{N}$ denote the corresponding multiplicities. We define $\mathcal{Z}_R(u)$ to be the set of pairs $\{(z_j, n_j)_{j=1}^N\}$ and $\#(\mathcal{Z}_R(u)) = \sum n_j$. Finally we define*

$$\mathcal{M}(\mathcal{Z}_R(u)) = \{v \in \mathcal{H} : z_j \text{ is a zero of multiplicity at least } n_j,\ \forall j = 1, \ldots, N\}.$$

Note that

$$\text{codim } \mathcal{M}(\mathcal{Z}_R(u)) = \#(\mathcal{Z}_R(u)). \tag{1.10}$$

and that, at least whenever u is a polynomial with no zeroes on $R\mathbb{T}$, we have

$$\mathcal{M}(\mathcal{Z}_R(u)) = [u]_{M_z}. \tag{1.11}$$

We omit a proof of this fact, but note that (1.11) holds for all u in $H^2(R\mathbb{D})$, by Beurling's theorem.

Let $R \geq 1$ and let the space X in the preceding Corollary be a Hilbert space of analytic functions on $R\mathbb{D}$ such that (1.11) holds. If, for some reason, u_N always satisfies

$$\#\mathcal{Z}_R(u_N) = N \tag{1.12}$$

whenever σ_N is distinct, then it follows by Theorem 1.6, (1.6) and (1.10) that

$$\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u_N))}\| = \sigma_N,$$

which would generalize Theorem 1.2 and provide a constructive proof of the result by S. Treil and A. Volberg for the particular space X under consideration. By numerical experiments with finite Hankel operators on $D^2(w)$-spaces, using a wide range of increasing weights w, we have not found one example of a singular vector u_N that does not satisfy (1.12). Unfortunately, we can at present only prove this for the case $X = H^2(R\mathbb{D})$.

THEOREM 1.9. *Let $R \geq 1$, let $\tilde{X}$ satisfy $(i)-(iii)$ and assume that $\tilde{B}$ is a contraction. Let $\Gamma : H^2(R\mathbb{D}) \to \tilde{X}$ be a Hankel operator such that $\Gamma(z^{n+1}) = 0$ for some $n \in \mathbb{N}$. Let $\sigma_N > 0$ be a fixed non-zero singular value of Γ with multiplicity 1. Then there exists a σ_N-singular vector u_N such that*

$$\#(\mathcal{Z}_R(u_N)) = N,$$

and $\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u_N))}\| = \sigma_N$.

Numerical experiments indicates that the above theorem fails to hold whenever $R < 1$. We include an example to support this conclusion;

EXAMPLE 1.10. *Let $\Gamma : H^2(R\mathbb{D}) \to H^2(\mathbb{D})$ be given by*

$$\begin{pmatrix} 4 & 3 & 2 & 1 & 0 & \dots \\ 3 & 2 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $R = 0.9$ we then have $(\sigma_N)_{N=1}^4 \approx (67, 2, 0.4, 0.1)$ and

$$\#(\mathcal{Z}_R(u_N))_{N=1}^4 = (0, 1, 2, 1)$$

and for $R = 0.8$ we get $(\sigma_N)_{N=1}^4 \approx (101, 4, 0.8, 0.3)$ and $\#(\mathcal{Z}_R(u_N))_{N=1}^4 = (0, 0, 0, 0)$.

We then show that we can remove the assumptions that $\Gamma(z^n) = 0$ and σ_N has multiplicity 1 in Theorem 1.9, at the cost of assuming a bit more about $\tilde{X}$. For simplicity we present a special case here, a more general result is given in Theorem 5.8.

THEOREM 1.11. *Let $R \geq 1$ and let $w = (w_j)$ be an increasing sequence of strictly positive weights. Let $\Gamma : H^2(R\mathbb{D}) \to D^2(w)$ be a Hankel operator and let N be such that*

$$\sigma_{N-1} > \sigma_N = \ldots = \sigma_{N+\mu} > \sigma_{N+\mu+1} \geq \sigma_\infty.$$

Then there exists mutually orthogonal σ_N−singular vectors $u_N, \ldots, u_{N+\mu}$ such that

$$N \leq \#(\mathcal{Z}_R(u_N)) \leq N + r,$$

and $\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u_N))}\| = \sigma_N$.

Recall that Theorems 1.1 and 1.2 are equivalent for classical Hankel operators, by Beurling's and Nehari's theorems, and it is therefore natural to ask wether Theorem 1.11 is equivalent to a generalized version of Theorem 1.1. Unfortunately, despite extensive research in this direction, (see e.g. [5], [11] and [17]), for most generalized Hankel operators there are no versions of Nehari's theorem that accurately describes the operator norm as in (1.4). Thus it is not clear wether Theorem 1.1 can be generalized accordingly, but preliminary results here are negative. We also point out that all constructive proofs of the AAK-theorem relies on Nehari's theorem, and as this is not available in our setting, the proof presented here differs significantly from previous proofs.

2. A brief review of generalized Hankel operators. Recall the two definitions (1.1) and (1.3) of a Hankel operator. As mentioned before, most modern definitions of a generalized Hankel operator is an extension of (1.3). The most general one goes as follows; one exchanges the space H^2 and the operator M_z on H^2 with an arbitrary Hilbert space X_1 and some operator $S_1 \in \mathcal{L}(X_1)$[2]; $H^2_- \subset L^2$ gets replaced by arbitrary Hilbert spaces $X_{2-} \subset X_2$ and it is assumed that X_{2-} is complemented in X_2, i.e. there exist some $X_{2+} \subset X_2$ such that $X_{2+} \cap X_{2-} = \{0\}$ and $X_{2+} + X_{2-} = X_2$. Finally, one lets $S_2 \in \mathcal{L}(X_2)$ be such that $S_2(X_{2+}) \subset X_{2+}$, P_- be the projection onto X_{2-} parallel with X_{2+}. Clearly, X_{2+} plays the role of H^2 in L^2, and S_2 that of M_z on L^2. An operator $\Gamma : X_1 \to X_{2-}$ is now called Hankel if it satisfies the relation

$$\Gamma S_1 = P_- S_2 \Gamma. \tag{2.1}$$

Some of these operators have Hankel matrices in certain bases of X_1 and X_{2-}, but most of them do not. We refer to N. K. Nikolskii [16], sec B:1.7, for more details.

Treil and Volberg additionally assumes that $X_{2-} \perp X_{2+}$. Under this assumption they show the following Nehari type theorem.

THEOREM 2.1. *Assume that S_1 is expansive and that S_2 is a contraction and let $\Gamma : X_1 \to X_{2-}$ be a Hankel operator. Then there exists an operator T such that $\Gamma = P_- T$, $TS_1 = S_2 T$ and $\|\Gamma\| = \|T\|$.*

To see why this is a generalization of Nehari's theorem, note that in the classical case, (i.e. when $X_1 = H^2$, $X_2 = L^2$, $X_{2-} = H^2_-$ and $S_1 = S_2 = M_z$), the commutation relation $TS_1 = S_2 T$ implies that $T = M_\phi$ for some $\phi \in L^\infty$ and clearly $\|M_\phi\| = \|\phi\|_{L^\infty}$. However, there is no way of calculating T explicitly, and in most cases one does not have a good estimate for $\inf\{\|T\| : P_- T = \Gamma\}$, (like $\|\Gamma\| = \|\phi\|_{L^\infty / H^\infty}$ in Nehari's theorem), so this theorem is of limited use in proving Theorem 1.11. Despite this, the authors give a (non-constructive) proof of the following similar theorem.

THEOREM 2.2. *Assume that S_1 is expansive and that S_2 is a contraction and let $\Gamma : X_1 \to X_{2-}$ be a Hankel operator. Let $\sigma_N > 0$ be a fixed non-zero singular value of Γ. Then there exists a S_1-invariant subspace $\mathcal{M}$ with $\mathsf{codim}\ \mathcal{M} = N$ such that $\|\Gamma|_{\mathcal{M}}\| = \sigma_N$.*

Other popular (but more restrictive) generalizations of Hankel operators is to consider them as certain bilinear or sesquilinear forms of a certain kind. This viewpoint has been adopted by N. Arcozzi, R. Rochberg, E. Sawyer, and B. D. Wick ([5]) as well as M. Cotlar and C. Sadosky ([10]). ARSW define a Hankel form on the Dirichlet space D^2 as a bounded bilinear form $G : D^2 \times D^2$ which can be evaluated, (at least on polynomials), as

$$G(f,g) = \langle fg, \phi \rangle_{D^2} = \sum_{n \geq 0} (n+1) \widehat{(fg)}_n \overline{\hat{\phi}_n}, \quad f, g \in Pol_+, \tag{2.2}$$

[2] $\mathcal{L}(X_1)$ denotes all bounded operators on X_1

where $\phi \in D^2$ is a "symbol" for G. If $\Gamma : D^2 \to D^2$ denotes the Hankel operator, (in the sense of Definition 1.3), represented by the matrix

$$\begin{pmatrix} \overline{\phi_0} & \overline{\phi_1} & \overline{\phi_2} & \cdots \\ \overline{\phi_1} & \overline{\phi_2} & \overline{\phi_3} & \cdots \\ \overline{\phi_2} & \overline{\phi_3} & \overline{\phi_4} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

then it is easy to see that $G(f,g) = \langle \Gamma f, \overline{g} \rangle_{D^2} = \sum_{n \geq 0} (n+1) \widehat{\Gamma f}_n \hat{g}_n$. The main goal of ARSW is to obtain a boundedness criterion for G in terms of the symbol ϕ, i.e. a generalization of Nehari's theorem. They prove that G is bounded if and only if $|\phi'|^2 dA$ is a certain Carleson-type measure on $\mathbb{D}$. For the purposes of the present paper, this is again of limited use because the Carleson norm of $|\phi'|^2 dA$ is only comparable with the operator norm of G, not equal.

Finally we wish to point out that M. Cotlar and C. Sadosky have written a number of papers about Hankel forms on Hardy spaces with two Helson-Szegö weights, (they consider more general weights, but we restrict attention to these), see e.g. [10], [11], [12]. Given two such weights w_1 and w_2 they define a sesquilinear form

$$G : H^2(w_1) \times \overline{H^2(w_2)} \to \mathbb{C}$$

to be Hankel if it satisfies $G(zf, g) = G(f, \overline{z} g)$. (They also consider more general versions where z and $\overline{z}$ are replaced by isometric operators S_1 and S_2). In our setting, these forms correspond to Hankel operators $\Gamma : H^2(w_1) \to L^2 \ominus \overline{H^2(w_2)}$. They do provide exact versions of the Nehari theorem and AAK-results for these operators. Corresponding results for Hankel operators between $H^2(w_1)$ and $H^2(w_2)$ are not known to us. This paper also fails to provide such theorems because the backward shift on $H^2(w_2)$ is not a contraction unless w_2 is constant.

3. Preliminaries. We will in section 3.1 provide a class of spaces that will serve as models for any space X that satisfy $(i)-(iii)$. In the following three sections we present lemmas that will be needed in the proof of Theorems 1.6, 1.9 and 1.11.

3.1. Representation. Let $c_{00} \subset \times_{n=0}^{\infty} \mathbb{C}$ denote the subset of sequences with finitely many nonzero elements and let S denote the shift "operator" on c_{00}. We let $\mathbb{M}_\infty$ denote the set of infinite matrices. Given $W = (w_{m,n})_{m,n \in \mathbb{Z}_+} \in \mathbb{M}_\infty$ we will let W' denote the "adjoint" $W' = (\overline{w_{n,m}})$. W will be called Hermitian symmetric if $W = W'$ and positive if

$$a'Wa = \sum \overline{a_m} w_{m,n} a_n \geq 0$$

for all $a \in c_{00}$, (where a' denotes the transpose and the conjugate of a). In this case we will write $\|a\|_W^2$ for the above expression. We let $\{e_n\}_{n \in \mathbb{Z}_+}$ denote the usual basis for c_{00}, i.e. $e_0 = (1, 0, 0, \ldots)$ and $e_n = S^n e_0$. If W is strictly positive, (i.e. $a'Wa > 0$ whenever $a \neq 0$), then l_W will denote the completion of c_{00} with respect to $\| \cdot \|_W$. If $\{e_n\}_{n \in \mathbb{Z}_+}$ is minimal, then each element in l_W can be represented as an infinite sequence in the obvious way.

Given a space X that satisfies $(i)-(iii)$, set $W = (w_{m,n}) = (\langle x_n, x_m \rangle_X)$. It is clear that the map such that $x_n \mapsto e_n$ is unitary and intertwines the operator S on X with the shift S on l_W. Thus we will often work with spaces l_W instead of X. Note that $\{x_n\}$ is a minimal set so the same will be true for $\{e_n\}$. We will write $X \cong l_W$ to denote that these spaces are unitarily equivalent in the above sense.

If $W \in \mathbb{M}_\infty$ is Hermitian symmetric, strictly positive and such that $\{e_n\}_{n \in \mathbb{Z}_+}$ is minimal, then each operator on l_W is represented by a matrix in $\mathbb{M}_\infty$ with respect to the natural basis $\{e_n\}_{n \in \mathbb{Z}_+}$. We will not separate between an operator on l_W and its matrix, so e.g. for the shift S on l_W we have

$$S=\begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that whenever $A,B\in\mathcal{L}(l_W)$ are such that the sums involved in the matrix computation of AB contain finitely many non-zero numbers, then the matrix AB coincides with the operator AB. Given two strictly positive Hermitian symmetric matrices $W,\tilde{W}$ and an operator $A\in\mathcal{L}(l_W,l_{\tilde{W}})$, note that the adjoint operator A^* is usually not given by the matrix A'. In fact, under suitable conditions on $A,W,\tilde{W}$, we have

$$\langle a,A^*b\rangle_{\tilde{W}}=\langle Aa,b\rangle_{\tilde{W}}=(Aa)'\tilde{W}b=a'A'\tilde{W}b=a'WW^{-1}A'\tilde{W}b$$

so

$$A^*=W^{-1}A'\tilde{W}. \tag{3.1}$$

3.2. Some lemmas on Hankel matrices. This section contain a number of results which are generalizations of lemmas used by D. Clark [9] and P. Hartman [15]. Let $\mathbb{M}_n$ denote the set of $(n+1)\times(n+1)$ matrices and let $hank(n)\subset\mathbb{M}_n$ denote the set of Hankel matrices of the form

$$h((\gamma_j)_{j=0}^n)=\begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_n & 0 & \dots & 0 \end{pmatrix}, \quad \gamma_j\in\mathbb{C}. \tag{3.2}$$

Similarly, let $Hank(n)\subset\mathbb{M}_\infty$ be the set of Hankel matrices Γ such that $\Gamma(e_{n+1})=0$. Let $F_n\in Hank(n)$ denote the Hankel matrix which is 1 on its anti-diagonal and zero elsewhere, i.e.

$$F_n=\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Note that

$$F^{-1}=F'=F. \tag{3.3}$$

Lemma 3.1. *Let $\Gamma=h((\gamma_j)_{j=0}^n)\in hank(n)$ be such that $\gamma_n\neq 0$. Then Γ is invertible and $\Gamma^{-1}=F_n\Pi F_n$, where $\Pi\in hank(n)$.*

Proof. Let $\pi_0,\dots,\pi_n$ solve the equation system

$$\Gamma\begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_n \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

It is then easy to see that Γ^{-1} is given by the Hankel matrix

$$\Gamma^{-1}=\begin{pmatrix} 0 & \dots & \pi_0 \\ \vdots & \ddots & \vdots \\ \pi_0 & \dots & \pi_n \end{pmatrix}.$$

□

Let W and $\tilde{W}$ in $\mathbb{M}_n$ be any invertible Hermitian symmetric matrices. Let $\mathcal{D}(W,\tilde{W}) \subset hank(n)$ be the set of matrices Γ such that all roots of the polynomial

$$p_\Gamma(\sigma) = \det(\sigma I - W\overline{\Gamma}\tilde{W}\Gamma)$$

are distinct. Note that by (3.1), the roots of p_Γ are the singular values of Γ as a Hankel operator on certain finite dimensional Hilbert spaces. Also recall that all normed topologies on $hank(n)$ are equivalent, as $hank(n)$ is finite dimensional.

LEMMA 3.2. *$\mathcal{D}(W,\tilde{W})$ is dense in $hank(n)$ for all invertible matrices W and $\tilde{W}$.*

Proof. Let $R(p_\Gamma, p'_\Gamma)$ denote the resolvent of p_Γ and p'_Γ. It is a standard fact (see e.g. [13]) that p_Γ has a root of multiplicity higher than one if and only if

$$R(p_\Gamma, p'_\Gamma) = 0.$$

Let $\Gamma = h((\alpha_j + i\beta_j)_{j=0}^n)$ be arbitrary with $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$ and $\beta_0, \ldots, \beta_n \in \mathbb{R}$. Clearly $R(p_\Gamma, p'_\Gamma)$ can be considered as a real polynomial in the variables $\alpha_0, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$, and it is a standard fact that the lemma holds if we show that this polynomial is not identically zero. This in turn follows if we show that there exists one Γ such that p_Γ has distinct and non-zero roots. We will prove this by induction. The existence of such a Γ is trivial for $n = 0$. Assume that $h((\gamma_j)_{j=0}^{n-1}) \in hank(n-1)$ has distinct and non-zero roots and put $\Gamma(\gamma_n) = h((\gamma_j)_{j=0}^n) \in hank(n)$, where γ_n is a free parameter. By the inductive hypothesis, the roots of $p_{\Gamma(0)}$ are distinct. Moreover, the roots are all non-zero whenever $\gamma_n \neq 0$, because then $W\overline{\Gamma}\tilde{W}\Gamma$ is invertible by Lemma 3.1. Finally, a standard argument shows that the roots of $p_{\Gamma(\gamma_n)}$ depend continuously on γ_n, and thus $\Gamma(\gamma_n)$ has distinct non-zero roots for small non-zero values of γ_n. □

Define $I_r \in \mathbb{M}_\infty$ via

$$I_r = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & r & 0 & \cdots \\ 0 & 0 & r^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.4}$$

LEMMA 3.3. *Let W and $\tilde{W}$ be Hermitian symmetric strictly positive infinite matrices and assume that each I_r is bounded on l_W and converges SOT to I as $r \to 1^-$. Let $\Gamma : l_W \to l_{\tilde{W}}$ be a Hankel operator. Then $I_r\Gamma I_r$ is a Hankel operator for each $r < 1$ and $I_r\Gamma I_r$ converges SOT to Γ as $r \to 1^-$. If in addition the I_r's are contractions on l_W, then we also have*

$$\lim_{r \to 1^-} \sigma_n(I_r\Gamma I_r) = \sigma(\Gamma).$$

Proof. That $I_r\Gamma I_r$ is Hankel is immediate by writing out its matrix. Given $u \in l_W$ we have that

$$\|\Gamma u - I_r\Gamma I_r u\| \leq \|(I - I_r)\Gamma u\| + \|I_r\Gamma(I - I_r)u\| \leq \|(I - I_r)\Gamma u\| + \|I_r\Gamma\|\|(I - I_r)u\| \to 0$$

as $r \to 1^-$, because $\|I_r\Gamma\|$ is uniformly bounded by the uniform boundedness principle. The final statement is proved exactly as Lemma 4.2 in [9]. □

Given any Hermitian symmetric strictly positive infinite matrix W, let P_n denote the orthogonal projection onto $\mathsf{Span}\ \{e_j\}_{j=n}^\infty$ in l_W. We will write $\tilde{P}_n$ for the corresponding operator in $l_{\tilde{W}}$. The following Lemma generalizes Lemma 1 in [15] of P. Hartman.

LEMMA 3.4. *Let $W \in \mathbb{M}_\infty$ be a diagonal strictly positive matrix and let $\tilde{W}$ be a Hermitian symmetric strictly positive infinite matrix. Let $\Gamma : l_W \to l_{\tilde{W}}$ be a Hankel operator, let $r < 1$ be fixed and assume that $\|\tilde{P}_n I_r\|_{\mathcal{L}(l_{\tilde{W}})} \to 0$ as $n \to \infty$. Then there exists a sequence of Hankel operators $\Gamma_n \in Hank(n)$, $n \in \mathbb{N}$, with distinct non-zero singular values such that*

$$\Gamma_n \to I_r\Gamma I_r$$

as $n \to \infty$.

Proof. Let $\Gamma = (\gamma_{i+j})$ and let $\Gamma_n \in Hank(n)$ be defined by the sequence $(r^j\gamma_j)_{j=0}^n$. Set $g_k^n = \sum_{j\geq n} r^j\gamma_{j+k}\tilde{e_j}$. Note that $g_k^n = \widetilde{P_n}I_r\Gamma e_k$ so

$$\|g_k^n\|_{l_{\tilde{W}}} \leq \|\widetilde{P_n}I_r\|_{\mathcal{L}(l_{\tilde{W}})}\|\Gamma\|_{\mathcal{L}(l_W,l_{\tilde{W}})}\|e_k\|_{l_W}. \tag{3.5}$$

Given $f \in l_W$ and $g \in l_{\tilde{W}}$, let $g \otimes f : l_W \to l_{\tilde{W}}$ denote the operator $g \otimes f(h) = \langle h, f\rangle g$. As W is diagonal we have $\|I_rP_n\| = r^n$ so

$$I_r\Gamma I_r(I - P_n) \to I_r\Gamma I_r$$

as $n \to \infty$. Moreover

$$\Gamma_n = I_r\Gamma I_r(I - P_n) - \sum_{j=0}^{n} r^j g_j^{n+1-j} \otimes (e_j/\|e_j\|^2)$$

so $\Gamma_n \to I_r\Gamma I_r$ holds if we show that $\lim_n \|\sum_{j=0}^n r^j g_j^{n+1-j} \otimes (e_j/\|e_j\|^2)\| = 0$. Note that

$$\|g_j^{n+1-j} \otimes (e_j/\|e_j\|^2)\| \leq \|\widetilde{P_n}I_r\|\|\Gamma\|$$

by (3.5). Let C be an upper bound for the right hand side, let $\epsilon > 0$ be arbitrary and pick N such that $r^NC/(1-r) < \epsilon/2$. Then

$$\|\sum_{j=0}^{n} r^j g_j^{n+1-j} \otimes e_j\| \leq \sum_{j=0}^{N}\|g_j^{n+1-j}\| + \frac{\epsilon}{2} \leq \sum_{j=0}^{N}\|\widetilde{P_n}I_r\|\|\Gamma\| + \frac{\epsilon}{2},$$

so by choosing n large enough it is clear that this will be less than ϵ.

It follows that $\Gamma_n \to I_r\Gamma I_r$ as $n \to \infty$. It remains to show that Γ_n has distinct non-zero singular values. This might not be true, but a short argument (which we omit) shows that Lemma 3.2 can be applied to show that it is always possible to find an arbitrarily small perturbation of Γ_n within $Hank(n)$ such that the perturbation has distinct non-zero singular values, and hence the desired conclusion follows. □

3.3. On singular vectors for *SOT*-convergent sequences of compact operators. A statement similar to the below proposition appears in Clark [9] and is claimed to be well known. As we have not found a proof of either our or Clark's result, we include a proof.

PROPOSITION 3.5. *Let X be a Hilbert space and let $(T_k)_{k=1}^\infty$ be a sequence in $\mathcal{L}(X)$ of compact positive operators that converge strongly to T. Let $\mathcal{E}_T$ respectively $\mathcal{E}_{T_k}$ denote the associated spectral projection measures. Assume that*

$$\lim_{k\to\infty}\sigma_n(T_k) = \sigma_n(T)$$

for all $n \in \mathbb{Z}_+$ and let $\Sigma \subset (\sigma_\infty(T), \infty)$ be an open interval such that

$$cl(\Sigma) \cap \{\sigma_n(T) : n \in \mathbb{Z}_+\} = \Sigma \cap \{\sigma_n(T) : n \in \mathbb{Z}_+\} = \{\sigma_N(T)\}$$

for some $N \in \mathbb{Z}_+$. Then $\mathcal{E}_{T_k}(\Sigma)$ converges strongly to $\mathcal{E}_T(\Sigma)$.

Proof. As T is positive, it follows by the spectral theorem that each $\sigma_n(T) > \sigma_\infty(T)$ is an eigenvalue to T. Set $M = \#\{n : \sigma_n(T) > \sigma_\infty(T)\}$. Then there exists an orthonormal set $\{f_n\}_{n=0}^M$ such that $\sigma_n(T)f_n = Tf_n$. Similarly let $\{f_n^k\}_{n=1}^\infty$ satisfy $\sigma_n(T_k)f_n^k = T_kf_n^k$. It is no restriction to assume that N is such that

$$\sigma_{N-1}(T) < \sigma_N(T) = \ldots = \sigma_{N+\mu}(T) < \sigma_{N+\mu+1}(T)$$

for some $\mu \in \mathbb{Z}_+$. Let $\alpha_n^k \in \mathbb{C}$ be coefficients such that $\sum_{n=0}^{\infty} \alpha_n^k f_n^k = f_N$, and note that we are done if we show that

$$\lim_k \sum_{n=N}^{N+r} \alpha_n^k f_n^k = f_N.$$

We have

$$(T - T_k) f_N = \sigma_N(T) f_N - \sum_n \sigma_n(T_k) \alpha_n^k f_n^k = \sum_n (\sigma_N(T) - \sigma_n(T_k)) \alpha_n^k f_n^k.$$

Let $\epsilon > 0$ be such that $\sigma_{N-1}(T) < \sigma_N(T) - \epsilon$ and $\sigma_N(T) + \epsilon < \sigma_{N+\mu+1}(T)$. For sufficiently large k we then have

$$\|(T - T_k) f_N\| = \sum_n (\sigma_N(T) - \sigma_n(T_k))^2 |\alpha_n^k|^2 \geq \epsilon^2 \sum_{n \notin \{N,\dots,N+\mu\}} |\alpha_n^k|^2$$

so that $\lim_k \sum_{n \notin \{N,\dots,N+\mu\}} |\alpha_n^k|^2 = 0$ as desired. □

3.4. Some results on Cauchy duals. Let $W \in \mathbb{M}_\infty$ be Hermitian symmetric and strictly positive. By the Cauchy dual of l_W we shall simply mean the dual with respect to the standard pairing l^2-pairing;

$$\langle x, y \rangle = \sum x_k \overline{y_k},$$

which is well defined at least for $y \in c_{00}$. We denote this dual by l_W^{C*}. Without going into details, note that in the case when W^{-1} "exists", the Cauchy dual is simply given by

$$l_W^{C*} = l_{W^{-1}}. \tag{3.6}$$

This follows by the calculation

$$\|y\|^2_{l_W^{C*}} = \sup \frac{|\langle x, y \rangle|}{\|x\|_{l_W}} = \sup \frac{|\langle W^{-1/2} z, y \rangle|}{\|z\|_{l^2}} = \sup \frac{|\langle z, W^{-1/2} y \rangle|}{\|z\|_{l^2}} = \frac{\|W^{-1/2} y\|^2_{l^2}}{\|W^{-1/2} y\|_{l^2}} = \|y\|_{l_{W^{-1}}},$$

where $y \in c_{00}$.

Example 3.6. *The Cauchy dual of the Dirichlet space is the Bergman space. More generally, if $w = (w_n)$ is a sequence and w^{-1} denotes the sequence (w_n^{-1}), then*

$$D^2(w) = D^2(w^{-1}).$$

In these examples, S is expansive in l_W if and only if it is a contraction in $l_{W^{-1}}$. This is false in general, which follows by considering

$$W = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & \dots \\ 0 & 0 & 4 & 0 & \dots \\ 0 & 0 & 0 & 4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1/4 & 0 & \dots \\ 0 & 0 & 0 & 1/4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Curiously, this can not happen if the off diagonal entries in the first row and column are zero.

Lemma 3.7. *Assume that $W = (w_{i,j})_{i,j \geq 0}$ is such that $w_{0,i} = w_{i,0} = 0$ for all $i > 0$. Then S is a contraction on l_W^{C*} if and only if S is expansive on l_W.*

Proof. For simplicity we assume that W is bounded above and below as an operator on l^2. Let $d_\infty(W) = S'WS = (w_{i,j})_{i,j\geq 1}$. S is expansive (resp. contractive) if and only if $d_\infty(W) \geq W$ (resp. $d_\infty(W) \leq W$). Due to the special form of the matrix W, it is easy to see that $d_\infty(W^{-1}) = (d_\infty(W))^{-1}$, and hence

$$d_\infty(W) \geq W \iff (d_\infty(W))^{-1} \leq W^{-1} \iff (d_\infty(W^{-1})) \leq W^{-1},$$

from which the lemma follows. The first equivalence above is a general fact about invertible positive self-adjoint operators, which can be seen as follows: Let W_1 and W_2 be such operators. Then $W_1 \geq W_2$ if and only if $W_1 = CW_2$ for some expansive invertible operator C. But C is expansive if and only if C^{-1} is a contraction and

$$W_1^{-1} = W_2^{-1}C^{-1} = (W_2^{-1}C^{-1})^* = (C^{-1})^* W_2^{-1}.$$

□

The main result of this section is the following.

LEMMA 3.8. *S is a contraction on l_W^{C*} if and only if B is a contraction on l_W.*

Proof. S on l_W^{C*} is the adjoint of B on l_W. □

EXAMPLE 3.9. *Let w be any non-constant Helson-Szegö weight w. With the obvious extension of the concept of Cauchy duality we have $(L^2(w))^{C*} = L^2(w^{-1})$, but the Cauchy dual of $H^2(w)$ is not equal to $H^2(v)$ for any $v \in L^1$. This follows by the above lemma because S is always isometric on $H^2(w)$ whereas B is a contraction on $H^2(v)$ if and only if v is constant.*

4. A matrix positivity lemma and proof of Theorem 1.6. Recall that $c_{00} \subset \times_{n=0}^{\infty}\mathbb{C}$ denotes the subset of sequences with finitely many nonzero elements and that S denotes the shift "operator" on c_{00}. For two Hermitian symmetric matrices W_1, W_2 we write $W_1 \leq W_2$ if $W_2 - W_1$ is positive.

Let X be a fixed Hilbert space and let $S \in \mathcal{L}(X)$. Given any $u \in X$, we associate with it the Toeplitz-matrix

$$T_{u,S} = \begin{pmatrix} \langle u,u\rangle & \langle Su,u\rangle & \langle S^2u,u\rangle & \cdots \\ \langle u,Su\rangle & \langle u,u\rangle & \langle Su,u\rangle & \cdots \\ \langle u,S^2u\rangle & \langle u,Su\rangle & \langle u,u\rangle & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{4.1}$$

as well as the following matrix

$$N_{u,S} = \begin{pmatrix} \langle u,u\rangle & \langle Su,u\rangle & \langle S^2u,u\rangle & \cdots \\ \langle u,Su\rangle & \langle Su,Su\rangle & \langle S^2u,Su\rangle & \cdots \\ \langle u,S^2u\rangle & \langle Su,S^2u\rangle & \langle S^2u,S^2u\rangle & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{4.2}$$

PROPOSITION 4.1. *If S is a contraction, then*

$$N_{u,S} \leq T_{u,S}.$$

If S is expansive, then

$$T_{u,S} \leq N_{u,S}.$$

To simplify notation we shall write $\|a\|_{N_{u,S}}$ for $\|a\|_{l_{N_{u,S}}}$. Recall that S denotes the shift operator on $l_{N_{u,S}}$. Given $a \in c_{00}$, it is easy to see that $\|a\|_{N_{u,S}} = \|\sum a_n S^n u\|_X$ and hence $\|Sa\|_{N_{u,S}} =$

$\|S\sum a_n S^n u\|_X$. By these identities, the proposition is easily seen to be a consequence of the following result about matrices.

PROPOSITION 4.2. *Let $W = (w_{m,n})$ be Hermitian symmetric, set $t_m = w_{0,m}$ and $t_{-m} = w_{m,0}$ $\forall m \in \mathbb{Z}_+$, and let $T = (t_{n-m})$ be the corresponding Toeplitz matrix. If the shift S on c_{00} satisfies*

$$(Sa)'W(Sa) \leq a'Wa, \quad \forall a \in c_{00}, \tag{4.3}$$

then $W \leq T$. If

$$(Sa)'W(Sa) \geq a'Wa, \quad \forall a \in c_{00}, \tag{4.4}$$

then $W \geq T$.

Proof. First assume that (4.3) holds and in addition that W is a strictly positive matrix. Then S is a contraction on the space l_W. Thus S has a unitary dilation, i.e. there exists a Hilbert space Z that contain l_W as a subspace and a unitary operator $U \in \mathcal{L}(Z)$ such that $S^n x = P_X U^n x$ for all $x \in l_W$ and $n \in \mathbb{Z}_+$, where P_{l_W} denotes the orthogonal projection onto l_W. (See [14] for more details.) Let $a \in l_W \cap c_{00}$ be arbitrary and recall that $e_0 = (1, 0, 0, \ldots) \in l_W$. Clearly $W = N_{e_0,S}$, $T = T_{e_0,S}$ and

$$\|a\|_W^2 = \|\sum a_n S^n e_0\|_{l_W}^2 = \|P_{l_W} \sum a_n U^n e_0\|_Z^2.$$

Moreover, it is easy to see that $T = T_{e_0,S} = T_{e_0,U} = N_{e_0,U}$, so

$$a'T_{e_0,S}a = a'N_{e_0,U}a = \|\sum a_n U^n e_0\|_Z^2.$$

Thus

$$a'(T-W)a = \|\sum a_n U^n e_0\|_Z^2 - \|P_{l_W} \sum a_n U^n e_0\|_Z^2 \geq 0, \tag{4.5}$$

as desired.

We turn to the general case. Note that $(Sa)'T(Sa) = a'Ta$ for any Toeplitz matrix T, and hence (4.3) is equivalent to

$$(Sa)'(T-W)(Sa) \geq a'(T-W)a$$

and (4.4) is equivalent to

$$(Sa)'(W-T)(Sa) \geq a'(W-T)a.$$

To conclude the proof, it is therefore enough to show that if $\tilde{W}$ is Hermitian symmetric with

$$\tilde{w}_{0,m} = \tilde{w}_{m,0} = 0 \tag{4.6}$$

and

$$(Sa)'\tilde{W}(Sa) \geq a'\tilde{W}a, \tag{4.7}$$

then

$$\tilde{W} \geq 0. \tag{4.8}$$

Henceforth, we assume that W is Hermitian symmetric and satisfies (4.6) and (4.7). For each $k \in \mathbb{N}$ we define a new matrix $W_k = (w^k_{m,n})$ given by

$$\begin{cases} w^k_{m,n} = w_{m,n} & m, n \leq k \\ w^k_{m+l,k+l} = w_{m,k} & m \leq k, l > 0 \\ w^k_{k+l,m+l} = w_{k,m} & m \leq k, l > 0 \\ w^k_{m,n} = 0 & \text{elsewhere} \end{cases}$$

This cumbersome definition is easily visualized, here is W_3 :

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{1,1} & w_{1,2} & w_{1,3} & 0 & 0 \\ 0 & w_{2,1} & w_{2,2} & w_{2,3} & w_{1,3} & \iddots \\ 0 & w_{3,1} & w_{3,2} & w_{3,3} & w_{2,3} & \iddots \\ 0 & 0 & w_{3,1} & w_{3,2} & w_{3,3} & \iddots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

For each fixed $a \in c_{00}$ we clearly have

$$\lim_{k\to\infty} a'W_k a = a'Wa, \tag{4.9}$$

and hence it is enough to show that $W_k \geq 0$ for all $k \in \mathbb{Z}_+$. Decompose an arbitrary a as $a = a_b + a_e$, where $a_b = (a_0, a_1, \ldots, a_{k-1}, 0, 0, \ldots)$. Note that

$$\begin{aligned} &(Sa_b)'W_k(Sa_e) = a_b'W_k a_e \\ &(Sa_e)'W_k(Sa_e) = a_e'W_k a_e \\ &a_b'W_k a_b = a_b'Wa_b \leq (Sa_b)'W(Sa_b) = (Sa_b)'W_k(Sa_b) \end{aligned}$$

from which it follows that

$$a'W_k a \leq (Sa)'W_k(Sa) \tag{4.10}$$

Let I denote the identity matrix, set $c_k = \max\{w_{m,n} : 0 \leq m, n \leq k\}$ and note that

$$\left| \, a'W_k a \right| < 2kc_k \|a\|_I^2,$$

which follows from the calculation

$$\begin{aligned} \left| \, a'W_k a \right| &= \left| \sum_n \sum_{|m-n|\leq k-1} w_{m,n}^k \overline{a_m} a_n \right| \leq \\ &\leq \left| \sum_n \sum_{|m-n|\leq k-1} c_k \frac{(|a_m|^2 + |a_n|^2)}{2} \right| \leq \frac{4k-2}{2} c_k \|a\|_I^2. \end{aligned}$$

Putting $V_k = I - 2kc_k W_k$, we thus have $V_k > 0$ and (4.10) implies that $(Sa)'V_k(Sa) \leq a'V_k a$ for all $a \in c_{00}$. But then the first part of the proof applies, so (4.5) implies that

$$0 \leq (I - V_k) = 2kc_k W_k,$$

and so $W \geq 0$ follows by (4.9).
□

Given an operator $\Gamma \in \mathcal{L}(X, \tilde{X})$ and $x \in X$, recall that $\|\Gamma|_{[x]_S}\|$ denotes the operator norm of Γ restricted to the invariant subspace $[x]_S$ generated by x. We are now in a position to prove Theorem 1.6.

THEOREM 4.3. *Let $X, \tilde{X}$ be such that S is expansive and $\tilde{B}$ is a contraction. Let $\Gamma : X \to \tilde{X}$ be a Hankel operator with singular values $\sigma_0 \geq \sigma_1 \geq \ldots$. Let $u \in X$ be a singular vector with singular value σ_N. Then*

$$\|\Gamma|_{[u]_S}\| = \sigma_N.$$

Proof. Put $v = \Gamma u$. By the polar decomposition of Γ it follows that $\|v\| = \sigma_N$. Obviously then $\|\Gamma|_{[u]_S}\| \geq \|\Gamma u\| = \|v\| = \sigma_N$, so we focus on proving the reverse inequality. It suffices to show that

$$\|\Gamma(\sum a_n S^n u)\| \leq \sigma_N \|\sum a_n S^n u\|$$

for all $a \in c_{00}$. Note that $\|\sum a_n S^n u\|^2 = a' N_{u,S}\ a$ and similarly

$$\|\Gamma(\sum a_n S^n u)\|^2 = \|\sum a_n \tilde{B}^n \Gamma u\|^2 = a' N_{v,\tilde{B}}\ a.$$

Also note that by (1.9) we have

$$\langle \tilde{B}^n v, v\rangle = \langle \tilde{B}^n \Gamma u, \Gamma u\rangle = \langle \Gamma S^n u, \Gamma u\rangle = \langle S^n u, \Gamma^* \Gamma u\rangle = \sigma_N^2 \langle S^n u, u\rangle$$

which implies that $T_{v,\tilde{B}} = \sigma_N^2 T_{u,S}$. The desired inequality follows via Proposition 4.1 and the calculation

$$\begin{aligned}\sigma_N^2 \|\sum a_n S^n u\|^2 = \sigma_N^2 (a' N_{u,S}\ a) \geq \sigma_N^2 (a' T_{u,S}\ a) = \\ = a' T_{v,\tilde{B}}\ a \geq a' N_{v,\tilde{B}}\ a = \|\Gamma(\sum a_n S^n u)\|^2.\end{aligned}$$

□

COROLLARY 4.4. *Let $X, \tilde{X}$ and Γ be as in Theorem 4.3. Let $u_N \in X$ be a singular vector with singular value σ_N and assume that $\sigma_{N-1} > \sigma_N$. Then* $\mathsf{codim}\ [u_N]_S \geq N$.

Proof. This follows immediately by the definition of singular numbers;

$$\inf\{\|\Gamma|_{\mathcal{M}}\| : \mathcal{M} \leq X \text{ and } \mathsf{codim}\ \mathcal{M} = N-1\} = \sigma_{N-1} > \sigma_N = \|\Gamma|_{[u]_S}\|.$$

□

5. Proof of Theorem 1.9 and 1.11. We begin with some definitions and lemmas. We let $\{e_m\}_{m=0}^n$ denote the usual basis in $\mathbb{C}^{n+1}$, where the context will determine the value of n, and for $0 \leq i, j \leq n$ we let $e_i \otimes e_j$ denote the matrix with 1 in the i, j'th position and zeroes elsewhere. We will sometimes treat $\mathbb{C}^n$ as a subset of c_{00} in the obvious way. Let $S_n : \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ denote the restriction of the shift S on c_{00} and let $B_n : \mathbb{C}^{n+1} \to \mathbb{C}^n$ denote the restriction of the backward shift B on c_{00}. Given $W \in \mathbb{M}_\infty$, (or $W \in \cup_{m>n} \mathbb{M}_m$), let $\mathcal{R}_n(W) \in \mathbb{M}_n$ denote the "upper left corner of W" and note that

$$\mathcal{R}_n(S'WS) = S_n' \mathcal{R}_{n+1}(W) S_n = \begin{pmatrix} w_{1,1} & w_{1,2} & \dots & w_{1,n+1} \\ w_{2,1} & w_{2,2} & \dots & w_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n+1,1} & w_{n+1,2} & \dots & w_{n+1,n+1} \end{pmatrix}. \tag{5.1}$$

Similarly,

$$\mathcal{R}_n(B'WB) = B_n' \mathcal{R}_{n-1}(W) B_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & w_{0,0} & \dots & w_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & w_{n-1,0} & \dots & w_{n-1,n-1} \end{pmatrix}. \tag{5.2}$$

It is easy to see that

$$S \text{ is expansive on } l_W \iff \mathcal{R}_n(S'WS) \geq \mathcal{R}_n(W),\ \forall n \in \mathbb{N}, \tag{5.3}$$

$$S \text{ is contractive on } l_W \iff \mathcal{R}_n(S'WS) \leq \mathcal{R}_n(W),\ \forall n \in \mathbb{N} \tag{5.4}$$

and

$$B \text{ is contractive on } l_W \Longleftrightarrow \mathcal{R}_n(B'WB) \le \mathcal{R}_n(W),\ \forall n \in \mathbb{N}. \tag{5.5}$$

LEMMA 5.1. *Let $W \in \mathbb{M}_n$ be given and assume that $\mathcal{R}_{n-1}(W)$ has no eigenvalues on $[-\infty, 0)$. Then we can chose a $c > 0$ such that $W + c\ e_n \otimes e_n$ has no eigenvalues on $[-\infty, 0)$.*

Proof. By evaluating $p(\lambda) = \det(W + c\ e_n \otimes e_n - \lambda I)$ by expansion along the bottom row, we obtain that

$$p(\lambda) = (c + w_{n,n} - \lambda)f(\lambda) - g(\lambda) = \left((c + w_{n,n} - \lambda) - \frac{g(\lambda)}{f(\lambda)}\right) f(\lambda).$$

It is easy to see that $\deg(f) = n$ and $\deg(g) = n - 1$, that $f(\lambda) = \det(\mathcal{R}_{n-1}(W) - \lambda I)$ so $f(\lambda) \neq 0$ for all $\lambda \in [-\infty, 0)$. If $f(0) \neq 0$ we thus get

$$C = \sup\{|g(\lambda)/f(\lambda)| : \lambda \in [-\infty, 0)\} < \infty, \tag{5.6}$$

and hence the lemma follows by choosing c such that $\inf\{|c + w_{n,n} - \lambda| : \lambda \in [-\infty, 0]\} > C$.

Now suppose that $f(0) = 0$, i.e. that 0 is an eigenvalue of $\mathcal{R}_{n-1}(W)$. The multiplicity of the root of f at $\lambda = 0$ is equal then to the dimension of the kernel of $\mathcal{R}_{n-1}(W)$. Similarly, $g(0) = \det(W - w_{n,n}e_n \otimes e_n) = 0$ and the multiplicity of the root $\lambda = 0$ is equal to $\dim \mathsf{Ker}\ (W - w_{n,n}e_n \otimes e_n)$. As obviously

$$\dim \mathsf{Ker}\ (W - w_{n,n}e_n \otimes e_n) \ge \dim \mathsf{Ker}\ (\mathcal{R}_{n-1}(W)),$$

we conclude that (5.6) holds even in this case, and the Lemma follows as above. □

Given $n > 1$, we will say that a matrix is W is n-diagonal if $w_{i,j} = 0$ whenever $i \neq j$ and $\max(i, j) > n$.

PROPOSITION 5.2. *Let $W \in \mathbb{M}_n$ be Hermitian symmetric and strictly positive. If*

$$S'_{n-1}WS_{n-1} \ge \mathcal{R}_{n-1}(W) \tag{5.7}$$

and/or

$$B'_n\mathcal{R}_{n-1}(W)B_n \le W \tag{5.8}$$

then there exists a bounded Hermitian symmetric strictly positive n-diagonal matrix W_{nd} satisfying

$$\mathcal{R}_n(W_{nd}) = W$$

such that the shift S is expansive on $l_{W_{nd}}$ and/or the backward shift B is a contraction on $l_{W_{nd}}$.

Proof. We begin with the part concerning S. Let $W_n \in \mathbb{M}_\infty$ denote the matrix that equals W in the upper left corner and has zeroes elsewhere. Put

$$W_{nd} = W_n + \sum_{m>n} c\ e_m \otimes e_m,$$

where $c \in \mathbb{R}^+$ is to be determined. By (5.1) and (5.7) we have

$$\mathcal{R}_{n-1}(S'W_{nd}S - W_{nd}) = S'_{n-1}WS_{n-1} - \mathcal{R}_{n-1}(W) \ge 0,$$

and hence by Lemma 5.1 we can choose c such that $\mathcal{R}_n(W_{nd}) \le \mathcal{R}_n(S'W_{nd}S)$. It follows that S is expansive on $l_{W_{nd}}$ by (5.3) and the fact that $S'W_{nd}S - W_{nd}$ has zeroes "outside the range of $\mathcal{R}_n$", (i.e. all elements with index (i, j) satisfying $\max(i, j) \ge n$ are zero). The part concerning B is similar. □

COROLLARY 5.3. *Let $W \in \mathbb{M}_\infty$ be Hermitian symmetric and strictly positive. If S is expansive and/or B is a contraction on l_W, then there exists a bounded Hermitian symmetric strictly positive n-diagonal matrix W_{nd} satisfying*

$$\mathcal{R}_n(W_{nd}) = \mathcal{R}_n(W)$$

such that S is expansive on $l_{W_{nd}}$ and/or B is a contraction.

Proof. This follows by combining Proposition 5.2 with the equivalences (5.1)-(5.5). □

Recall that $F_n \in Hank(n)$ is given by

$$F_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

LEMMA 5.4. *Let $W \in \mathbb{M}_n$ be Hermitian symmetric and strictly positive. Then*

$$S'_{n-1} W S_{n-1} \geq \mathcal{R}_{n-1}(W) \Longleftrightarrow S'_{n-1} F_n W F_n S_{n-1} \leq \mathcal{R}_{n-1}(F_n W F_n)$$

Proof. We use the notation $W = (w_{i,j}) = (w_{i,j})_{0 \leq i,j \leq n}$. Note that

$$F_n W F_n = (w_{n-i,n-j}) \tag{5.9}$$

so $S'_{n-1} F_n W F_n S_{n-1} = (w_{n-1-i,n-j-1})_{0 \leq i,j < n}$ and $\mathcal{R}_{n-1}(F_n W F_n) = (w_{n-i,n-j})_{0 \leq i,j < n}$. Thus

$$\begin{aligned} & a'\Big(\mathcal{R}_{n-1}(F_n W F_n)) - S'_{n-1}(F_n W F_n) S_{n-1}\Big) a = \\ = & \sum_{0 \leq i,j < n} \overline{a_i}(w_{n-i,n-j} - w_{n-1-i,n-j-1}) a_j = \sum_{0 < i,j \leq n} \overline{a_{n-i}}(w_{i,j} - w_{i-1,j-1}) a_{n-j} = \\ = & a'\Big(S'_{n-1} W S_{n-1} - \mathcal{R}_{n-1}(W)\Big) a. \end{aligned}$$

□

LEMMA 5.5. *Let $W \in \mathbb{M}_\infty$ be Hermitian symmetric, strictly positive and such that B is a contraction on l_W. Then there exists a Hermitian symmetric, strictly positive n-diagonal matrix $V \in \mathbb{M}_\infty$ such that B is a contraction on l_V and*

$$\mathcal{R}_n(V) = F_n(\mathcal{R}_n(W))^{-1} F_n.$$

Proof. First note that by Corollary 5.3 we may assume that W is n-diagonal and bounded, and in this case we have

$$(\mathcal{R}_n(W))^{-1} = \mathcal{R}_n(W^{-1}).$$

By Lemma 3.8 and equality (3.6), we conclude that S is a contraction on $l_{W^{-1}}$, so by (5.1) and (5.4) we obtain

$$S_{n+1} \mathcal{R}_{n+1}(W^{-1}) S_{n+1} \leq \mathcal{R}_n(W^{-1}). \tag{5.10}$$

Explicitly, if $\mathcal{R}_{n+1}(W^{-1}) = (a_{i,j})_{0 \leq i,j \leq n+1}$ then $a_{i,j} = 0$ for $i \neq j$ and $\max(i,j) = n+1$, so

$$S_{n+1} \mathcal{R}_{n+1}(W^{-1}) S_{n+1} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 \\ 0 & \dots & 0 & a_{n+1,n+1} \end{pmatrix}$$

By (5.9) and the fact that $a_{n+1,n+1} > 0$, (5.10) implies that

$$\text{(5.11)} \qquad \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & a_{n,n} & \dots & a_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{1,n} & \dots & a_{1,1} \end{pmatrix} \leq \begin{pmatrix} a_{n,n} & a_{n,n-1} & \dots & a_{n,0} \\ a_{n-1,n} & a_{n-1,n-1} & \dots & a_{n-1,0} \\ \vdots & \vdots & \ddots & \vdots \\ a_{0,n} & a_{0,n-1} & \dots & a_{0,0} \end{pmatrix}$$

Let V be any n-diagonal bounded extension of the matrix to the right, i.e.

$$\mathcal{R}_n(V) = F_n \mathcal{R}_n(W^{-1}) F_n.$$

(5.11) can then be rewritten as $B_n \mathcal{R}_{n-1}(V) B_n \leq \mathcal{R}_n(V)$ by (5.2). Proposition 5.2 implies that V can be chosen such that B on l_V is a contraction. ☐

Given $R \geq 1$, recall that I_R denotes the diagonal matrix with $1, R, R^2$ etc. on the diagonal, (see (3.4)). By Section 3.1 we have $H^2(R\mathbb{D}) \cong l_{I_{R^2}}$. Note that I_{R^2} acts as a bounded operator from $H^2(R\mathbb{D})$ onto $H^2(R^{-1}\mathbb{D})$.

LEMMA 5.6. *Let $R \geq 1$ and let $\tilde{X}$ be such that $\tilde{B}$ is a contraction. Let $\Gamma : H^2(R\mathbb{D}) \to \tilde{X}$ be a Hankel operator and assume that $u \in H^2(R^{-1}\mathbb{D})$ is a singular vector to $\Gamma I_{R^2}^{-1} : H^2(R^{-1}\mathbb{D}) \to \tilde{X}$ with singular value σ. Then*

$$\|\Gamma I_{R^2}^{-1}|_{[u]_{M_z}}\| = \sigma.$$

Proof. The proof is similar to that of Theorem 4.3. Put $v = \Gamma I_{R^2}^{-1} u$ and note that $\|v\| = \sigma$. A short calculation yields that $\Gamma I_{R^2}^{-1} S^n = R^{-2n} \tilde{B}^n \Gamma I_{R^2}^{-1}$ for all $n \in \mathbb{Z}_+$, so we need to show that

$$\|\sum a_n R^{-2n} \tilde{B}^n v\| \leq \sigma \|\sum a_n S^n u\|,$$

for all $a \in c_{00}$. This is clearly equivalent to $\|\sum a_n \tilde{B}^n v\| \leq \sigma \|\sum a_n (R^2 S)^n u\|$for all $a \in c_{00}$. Note that

$$\langle \tilde{B}^n v, v\rangle_{\tilde{X}} = \langle \tilde{B}^n \Gamma I_{R^2}^{-1} u, \Gamma I_{R^2}^{-1} u\rangle = \langle \Gamma I_{R^2}^{-1} (R^2 S)^n u, \Gamma I_{R^2}^{-1} u\rangle = \sigma^2 \langle (R^2 S)^n u, u\rangle,$$

which implies that $T_{v,\tilde{B}} = \sigma^2 T_{u,R^2 S}$, and also note that $R^2 S$ is expansive on $H^2(R^{-1}\mathbb{D})$. The desired inequality follows by applying the same argument as in Theorem 4.3. ☐

Recall that a Hankel operator Γ is defined to be in $Hank(n)$ if $\Gamma(z^{n+1}) = 0$. We are finally in a position to prove Theorems 1.9 and 1.11.

THEOREM 5.7. *Let $R \geq 1$, let $\tilde{X}$ be a space that satisfies $(i)-(iii)$ such that $\tilde{B}$ is a contraction. Let $\Gamma : H^2(R\mathbb{D}) \to \tilde{X}$ be a Hankel operator in $Hank(n)$. Let $\sigma_N > 0$ be a fixed singular value of Γ with multiplicity 1. Then there exists a σ_N-singular vector u such that*

$$\#(\mathcal{Z}_R(u)) = N,$$

and $\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u))}\| = \sigma_N$.

Proof. Let $\tilde{W}$ be such that $\tilde{X} \cong l_{\tilde{W}}$, as in Section 3.1, and recall that $H^2(R\mathbb{D}) \cong l_{I_{R^2}}$. It will be convenient to consider Γ as an operator from $l_{I_{R^2}}$ to $l_{\tilde{W}}$, and thus u will denote an element in $l_{I_{R^2}}$, whereas $\breve{u}(z) = \sum_{j=0}^{\infty} u_j z^j$ will denote the corresponding element in $H^2(R\mathbb{D})$. To shorten notation, we will write $I_{R^2,n}, \tilde{W}_n$ and Γ_n for $\mathcal{R}_n(I_{R^2}), \mathcal{R}_n(\tilde{W})$ and $\mathcal{R}_n(\Gamma)$. Note that $(\Gamma^*)_n$ is typically not the same as $(\Gamma_n)'$, in fact, by a calculation very similar to (3.1) we obtain

$$(\Gamma^*)_n = I_{R^2,n}^{-1} {\Gamma'}_n \tilde{W}_n.$$

Also note that there is no restriction to assume that $\Gamma(z^n) \neq 0$, so that Γ_n is invertible.

Clearly $\mathsf{Span}\ \{e_m : m > n\} = \mathsf{Ker}\ \Gamma$, so as $\sigma_N > \sigma_\infty = 0$, any singular vector u to σ_N lies in $\mathsf{Span}\ \{e_m : m \leq n\}$. It therefore suffices to prove that $\check{u} = \sum_{m=0}^{n} u_m z^m$ satisfies $\#\mathcal{Z}_R(\check{u}) = N$. By Beurling's theorem and the fact that $\check{u}$ is a polynomial, it follows that

$$[\check{u}]_{M_z} = \mathcal{M}(\mathcal{Z}_R(\check{u})),$$

so $\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u))}\| = \sigma_N$ follows directly from Theorem 4.3. Moreover

$$\#\mathcal{Z}_R(\check{u}) \geq N \tag{5.12}$$

by Corollary 4.4. Hence we are done if we show the reverse inequality. This part of the proof is inspired by Butz's proof of Theorem 1.1, [7]. We will from now on consider u as an element of $\mathbb{C}^{K+1}$, as the remaining entries are zero anyway. Then u satisfies

$$\sigma_N^2 u = I_{R^2,n}^{-1} \Gamma_n' \tilde{W}_n \Gamma_n u.$$

Thus

$$\sigma_N^{-2} u = \Gamma_n^{-1} (\tilde{W}_n)^{-1} (\Gamma_n^{-1})' I_{R^2,n} u.$$

Moreover, by Lemma 3.1 it follows that

$$\sigma_N^{-2} u = F_n \Pi_n F_n (\tilde{W}_n)^{-1} F_n \Pi_n' F_n I_{R^2,n} u,$$

where $\Pi_n \in hank(n)$. Setting $V_n = F_n(\tilde{W}_n)^{-1} F_n$ and noting that $F_n I_{R^2,n} F_n = R^{2n} I_{R^{-2},n}$, this yields

$$\sigma_N^{-2} F_n u = R^{2n} \Pi_n V_n \Pi_n' I_{R^{-2},n} F_n u.$$

Finally, with $v = F_n u$ we obtain

$$(R^n \sigma_N)^{-2} v = \Pi_n V_n \Pi_n' I_{R^{-2},n} v. \tag{5.13}$$

The proof will be complete if we show that

$$\#\mathcal{Z}_{R^{-1}}(\check{v}) \geq n - N, \tag{5.14}$$

because the zeroes of $\check{v}$ are the inverses of the zeroes of $\check{u}$, and therefore $\#\mathcal{Z}_{R^{-1}}(\check{v}) + \#\mathcal{Z}_R(\check{u}) \leq n$ so that (5.14) would imply that

$$\#\mathcal{Z}_R(\check{u}) \leq n - (n - N) = N, \tag{5.15}$$

which, combined with (5.12), yields the desired result.

By Lemma 5.5, let V be an $n-$diagonal matrix V such that B on l_V is a contraction and $V_n = \mathcal{R}_n(V)$. Let $\Pi \in Hank(n)$ be the Hankel matrix obtained from Π_n by "adding zeroes". Clearly $I_{R^{-2}}$ can be considered as a bounded operator from $l_{I_{R^{-2}}}$ onto $l_{I_{R^2}}$, and Π' as an operator from $l_{I_{R^2}}$ into l_V. The adjoint of $\Pi' I_{R^{-2}}$ is then given by ΠV, (cf. (3.1)). Thus v, considered as an element of $l_{I_{R^{-2}}}$ by "adding zeroes", is a singular vector to $\Pi' I_{R^{-2}}$ with singular value $(R^n \sigma_N)^{-2}$.

Recall that the non-zero singular values $\sigma_0 \geq \ldots \geq \sigma_n$ of Γ are assumed to be distinct. Denote the corresponding singular vectors by u_m ($m = 0, \ldots, n$) and set $v_m = F_n(u_m)$. (So $u = u_N$ and $v = v_N$ in the previous notation). In the above calculations, there was nothing special about the value N, so it follows that each v_m is a singular vector to $\Pi' I_{R^{-2}}$ with corresponding singular value $(R^n \sigma_m)^{-2}$, $m = 0, \ldots, n$. Moreover, $\Pi' I_{R^{-2}}$ has rank $n+1$, and hence these are all non-zero singular values. Hence $(R^n \sigma_N)^{-2}$ is the $n - N$:th singular value, ($(R^n \sigma_n)^{-2}$ is the 0:th). By Lemma 5.6 we have that

$$\|\Pi' I_R^{-2}|_{[\check{v_N}]_{M_z}}\| = (R^n \sigma_N)^{-2}$$

which, by the same calculation as in Corollary 4.4 yields that

$$\mathsf{codim}\ [\check{v_N}]_{M_z} \geq n - N.$$

(5.14) now follows by Beurling's theorem and the fact that $\check{v}_N$ is a polynomial.
□

Recall the operators $\widetilde{P_n}$ from Lemma 3.4.

THEOREM 5.8. *Let $R \geq 1$, let $l_{\tilde{W}}$ be such that $\tilde{B}$ is a contraction. Moreover assume that $\|\tilde{P}_n I_r\|_{\mathcal{L}(l_{\tilde{W}})} \to 0$ as $n \to \infty$ for each $r < 1$. Let $\Gamma \in \mathcal{L}(H^2(R\mathbb{D}), l_{\tilde{W}})$ be a Hankel operator and let N be such that*

$$\sigma_{N-1} > \sigma_N = \ldots = \sigma_{N+\mu} > \sigma_{N+\mu+1} \geq \sigma_\infty.$$

Then there exists mutually orthogonal σ_N-singular vectors $u_N, \ldots, u_{N+\mu}$ such that

$$N \leq \#(\mathcal{Z}_R(u_N)) \leq N + r,$$

and $\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u_N))}\| = \sigma_N$.

Proof. Once the first statement has been established, the latter follows as before from Theorem 4.3 and Beurling's theorem. By Lemmas 3.3 and 3.4 it follows that there is a sequence (Γ_n) of Hankel operators such that $\Gamma_n \in Hank(n)$ and $\lim_n \Gamma_n = \Gamma$ in the strong operator topology. Let $\mathcal{E}_\Gamma$ be the spectral projection measure associated with $\sqrt{\Gamma^*\Gamma}$. Clearly $\mathsf{Ran}\ \mathcal{E}_\Gamma(\sigma_N)$ has dimension $\mu+1$. We need to show that $\mathcal{E}_\Gamma(\sigma_N)$ has an orthonormal basis $\{u_k\}_{k=N}^{N+\mu}$ with the desired amount of zeroes. Let P denote the orthogonal projection onto $\mathcal{E}_\Gamma(\sigma_N)$ and let $u_k(\Gamma_n)$ be the singular vector of Γ_n corresponding to $\sigma_k(\Gamma_n)$ for $k = 0, \ldots, n$. By Lemma 3.3 and Proposition 3.5,

$$\lim_n \|P(u_k(\Gamma_n)) - u_k(\Gamma_n)\| = 0 \tag{5.16}$$

for all $N \leq k \leq N + r$. Consider $\big(P(u_k(\Gamma_n))\big)_{k=N}^{N+\mu}$ in $\times_{k=N}^{N+\mu}\mathcal{E}_\Gamma(\sigma_N)$ as a sequence in n. By compactness, this sequence has a convergent subsequence $\big(P(u_k(\Gamma_{n_j}))\big)_{k=N}^{N+\mu}$, $j = 1, \ldots$. Let $(u_k)_{k=N}^{N+\mu}$ be the limit and note that by (5.16),

$$\big(u_k(\Gamma_{n_j})\big)_{k=N}^{N+\mu} \to (u_k)_{k=N}^{N+\mu}$$

as $j \to \infty$. This implies that $u_k(\Gamma_{n_j}) \to u_k$ uniformly on compacts in $R\mathbb{D}$, and hence u_k satisfies $\mathcal{Z}_R(u_k) \leq N + r$ by Theorem 5.7. On the other hand $\mathcal{Z}_R(u_k) \geq N$ by Theorem 4.3 and the assumption that $\sigma_{N-1} > \sigma_N$. The proof is complete. □

COROLLARY 5.9. *Let $R \geq 1$ and let $w = (w_j)$ be an increasing strictly positive sequence. Let $\Gamma : H^2(R\mathbb{D}) \to D^2(w)$ be a Hankel operator and let N be such that*

$$\sigma_{N-1} > \sigma_N = \ldots = \sigma_{N+\mu} > \sigma_{N+\mu+1} \geq \sigma_\infty.$$

Then there exists mutually orthogonal σ_N-singular vectors $u_N, \ldots, u_{N+\mu}$ such that

$$N \leq \#(\mathcal{Z}_R(u_N)) \leq N + r,$$

and $\|\Gamma|_{\mathcal{M}(\mathcal{Z}_R(u_N))}\| = \sigma_N$.

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