

## FRAME-BASED GAUSSIAN BEAM SUMMATION AND SEISMIC HEAD WAVES

ROBERT L. NOWACK\*

**Abstract.** In this paper, a two-parameter discrete Gaussian beam summation is described and then applied to SH seismic waves in a layer over a half space. For this case the exact solution for the wide-angle reflection and head wave is known by the summation of plane waves or from the modified-Cagniard method. Using Gaussian beam summations over a single angle parameter, the wide-angle reflection and head wave can be generated by using wide beam widths. By using single-parameter beam summations with less wide beams, stable solutions of the wide-angle reflection can be obtained, but not the head wave solution which requires more spectral content. Here we show that both the wide-angle reflection and the head wave can be obtained when using finite beam widths by an over-determined, frame-based Gaussian beam summation over both position and wavenumber.

**1. Introduction.** Summations of Gaussian beams over either initial take-off angle or position along an initial surface have been applied for the computation of high frequency seismic wavefields in smoothly varying inhomogeneous media (see for example, Popov, 1982; Cerveny et al. 1982; Nowack and Aki, 1984). Reviews of Gaussian beam summation have been given by Cerveny (1985a,b), Babich and Popov (1989), and more recently by Popov (2002), Nowack (2003), Cerveny et al. (2007) and Bleistein (2007). An advantage of summations using Gaussian beams to construct more general wavefields is that the individual Gaussian beams have no singularities along their paths, no two-point ray tracing is required and triplicated arrivals are naturally incorporated into either forward or inverse modeling.

An early criticism of the single-parameter implementations of Gaussian beam summations was given by White et al. (1987) in terms of completeness and accuracy of the summations. However, more recently over-complete frame-based Gaussian beam summations have been developed based on window and wavelet transforms to address some of the issues related to completeness of beam summations (Lugara et al., 2003). In an over-complete frame based approach, the wavefield is decomposed into beam fields that are localized both in position and direction (Figure 1). In these decompositions, position plays the role of time and wavenumber plays the role of frequency in a time-frequency type of decomposition. Gabor originally suggested a decomposition using modulated and translated Gaussian windows. Although an orthonormal basis cannot be formed using a Gabor frame, an over-complete frame expansion can be constructed which has the added benefit of providing redundancy in the expansion (Feichtinger and Strohmer, 1998). Hill (1990, 2000) used an over-determined frame of Gaussian beams for the migration seismic data and gave criteria for the sampling of the beams in position and wavenumber based on physical reasoning (see also, Nowack et al., 2006; 2007).

For an SH line source in a layer over an halfspace, an exact solution for the wide-angle reflection and head wave can be obtained either by a summation of plane waves or by using the modified-Cagniard method (Aki and Richards, 1980; 2002). Nowack and Aki (1984) showed that seismic head waves can be generated by a single-parameter Gaussian beam summation by using wide beam widths. However, for more narrow beam widths, the spectral content is not sufficient to obtain the head wave, although the direct wave and wide-angle reflection can be obtained. An early attempt at generating the head wave using finite beam widths was given by Gao et al. (1990), however, they concluded that large beam widths were still required. Here I apply a two-parameter Gaussian beam summation using an overdetermined frame-based approach for the generation of wide-angle reflections and head waves with finite beam widths. Although the current study for the wide-angle response is for isotropic media, recent work by Landro and Tsvankin (2007) has investigated anisotropic parameter estimation using wide-angle amplitude variation with offset (AVO) response.

**2. Theory.** A 2D wave equation for SH waves in the frequency and horizontal wavenumber domain with for a line source can be written

---

\*Dept. of Earth and Atmos. Sci., Purdue University, West Lafayette, IN 47907, USA (nowack@purdue.edu).

## Frame-Based Gaussian Beam Decomposition

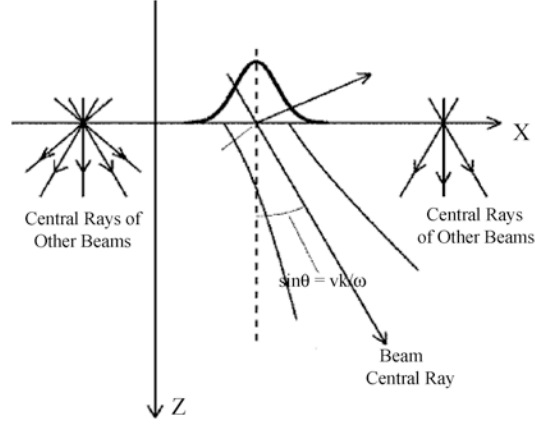


FIG. 1. An illustration of an over-determined frame-based decomposition in terms of position and wavenumber of an initial wavefield and propagation using paraxial Gaussian beams.

$$(2.1) \quad \left[ \frac{\partial^2}{\partial x_3^2} + k_3^2 \right] u(k, x_3, \omega) = \delta(x_3)$$

where  $k_3^2 = (\omega/v)^2 - k_1^2$  is the squared vertical wavenumber. Matching singularities of this equation at the source depth results in

$$u(k, x_3, \omega) = \frac{-i}{2k_3} e^{ik_3|x_3|}.$$

An integral expression over wavenumber for a source at  $x_1 = x_s$  can then be written as

$$(2.2) \quad u(x_1, x_3, \omega) = \frac{-i}{4\pi} \int \frac{dk_1}{k_3} e^{ik_3|x_3|} e^{ik_1(x_1 - x_s)}$$

(Chew, 1990; Eqn. 2.2.10).

The division of unity by Gaussians can be written as,  $1 \sim \frac{\Delta L}{\sqrt{2\pi}\gamma} \sum_L e^{-(x_1 - x_L)^2/2\gamma^2}$  (Kaiser, 1994), where  $\Delta L \ll 2\gamma$  and  $x_L = L\Delta L$ . Inserting this into Eqn. (2.1) gives

$$(2.3) \quad u(x_1, x_3, \omega) \sim \frac{-i}{4\pi} \frac{\Delta L}{\sqrt{2\pi}\gamma} \int \frac{dp_1}{p_3} \sum_L e^{-(x_1 - x_L)^2/2\gamma^2} e^{i\omega p_3|x_3|} e^{i\omega p_1(x_1 - x_s)}$$

where the wavenumber integral is written in terms of horizontal slowness  $p_1$  with  $k_1 = \omega p_1$ . Using a discretization of the  $p_1$  integral results in a two-parameter summation evaluated over Gaussian tapered wave components at the receiver datum level as

$$(2.4) \quad u(x_1, x_3, \omega) = \frac{-i}{4\pi} \frac{\Delta L \Delta p}{\sqrt{2\pi}\gamma} \sum_M \sum_L \frac{1}{p_3} e^{-(x_1 - x_L)^2/2\gamma^2} e^{i\omega p_3|x_3|} e^{i\omega p_1(x_1 - x_s)}$$

where  $p_1 = M\Delta p$  and  $p_3 = \sqrt{v^{-2} - p_1^2}$  is the vertical slowness. For  $x_3 = 0$ , this gives the expansion in the aperture plane as

$$(2.5) \quad u(x_1, x_3 = 0, \omega) = \frac{-i}{4\pi} \frac{\Delta L \Delta p}{\sqrt{2\pi}\gamma} \sum_M \sum_L \frac{1}{p_3} \{e^{-(x_1-x_L)^2/2\gamma^2} e^{+i\omega p_1(x_1-x_L)}\} e^{i\omega p_1(x_L-x_s)}$$

The term in the parentheses represents a sum of initial planar Gaussian tapered wave components at an angle given by  $p_1 = \sin\vartheta/v$  at locations  $x_1 = x_L$  at the source depth  $x_3 = 0$ . The initial Gaussian tapered wavefronts components can then be propagated to the receiver using paraxial Gaussian beams as

$$(2.6) \quad u(x_1, x_3, \omega) = \frac{-i}{4\pi} \frac{\Delta L \Delta p}{\sqrt{2\pi}\gamma} \sum_M \sum_L \frac{1}{p_3} u^{\text{gb}}(x_1, x_3; x_L, 0, \omega) e^{i\omega p_1(x_L-x_s)}$$

where in ray centered coordinates, an individual SH Gaussian beam can be written as

$$(2.7) \quad g^{\text{gb}}(x_1, x_3, \omega) = \sqrt{\frac{v_0 \rho_0 q_0}{v(s) \rho(s) q(s)}} e^{i\omega(\tau(s) + \frac{1}{2}M(s)n^2)}$$

where  $(s, n)$  are the ray centered coordinates along and transverse to the ray,  $\rho$  and  $v$  are the density and velocity,  $\tau(s)$  is the travel-time along the ray and  $M(s) = M_R(s) + iM_I(s)$  is the complex second derivative of the time field with respect to the transverse coordinate  $n$ . To form a beam,  $M_I(s) > 0$ . Along the horizontal coordinate at the source, the beam half-width at a reference lower frequency  $\omega_{\text{ref}}$  can be written as  $\gamma_{\text{ref}}^2 = (\omega_{\text{ref}} \cos^2(\vartheta) M_I(s_0))^{-1}$  where  $\vartheta$  is the angle of the ray from the vertical at the source. The horizontal beam widths at other frequencies are then  $\gamma^2 = (\omega_{\text{ref}}/\omega) \gamma_{\text{ref}}^2$ . A choice of sampling for Eqn. (2.6) can be made as  $\Delta L = \sqrt{2} \left(\frac{\omega_{\text{ref}}}{\omega_H}\right)^{1/2} \gamma_{\text{ref}}$  and  $\Delta p_1 = \left(\frac{1}{\omega_{\text{ref}} \omega_H}\right)^{1/2} \gamma_{\text{ref}}^{-1}$  where  $\omega_H$  is the highest frequency of interest (Hale, 1992; Hill, 1990). This ensures 4 times over-sampling at the highest frequency.

In order to incorporate a reflection coefficient into the above formulas in Eqns. (2.6 and 2.7), the term  $\frac{1}{p_3}$  is replaced by  $\frac{R}{p_3}$  where  $R = (I_1 - I_2)/(I_1 + I_2)$  with  $I = \rho v \cos\vartheta$  for an SH wave. The above sampling in ray parameter could need to be increased when sampling near the critical distance and for the examples below I sample at 1/3 the above sampling in  $\Delta p_1$ .

As an alternative approach, a Kirchoff integral of boundary value data could be used. Assuming that the field values are specified at  $x_3 = 0$ , the Kirchoff integral can be written as (Hill, 2001; Goodman, 1996)

$$u(x_1, x_3, \omega) = \frac{-1}{2\pi} \int dx'_1 u(x'_1, x'_3 = 0, \omega) \frac{\partial g(x_1, x_3; x'_1, x'_3 = 0, \omega)}{\partial x'_3}$$

Expanding the Green's function  $g$  into plane waves results in

$$g(x_1, x_3; x'_1, 0, \omega) = \frac{-i}{4\pi} \int \frac{dk_1}{k_3} e^{ik_3|x_3|} e^{ik_1(x_1-x'_1)}$$

and

$$\frac{\partial g(x_1, x_3; x'_1, 0, \omega)}{\partial x'_3} = \frac{1}{4\pi} \int dk_1 e^{ik_3|x_3|} e^{ik_1(x_1-x'_1)}$$

This can be written in terms of an integral of horizontal slowness as

$$\frac{\partial g(x_1, x_3; x'_1, 0, \omega)}{\partial x'_3} = \frac{|\omega|}{4\pi} \int dp_1 e^{i\omega p_3 |x_3|} e^{i\omega p_1 (x_1 - x_L)} e^{i\omega p_1 (x_L - x'_1)}$$

where  $k_1 = \omega p_1$  and similarly for  $k_3$ . Inserting this into the Kirchhoff integral and using the division of unity by Gaussians,  $1 \sim \frac{\Delta L}{\sqrt{2\pi}\gamma} \sum_L e^{-(x_1 - x_L)^2/2\gamma^2}$ , with  $\Delta L \ll 2\gamma$  and  $x_L = L\Delta L$  gives at  $x_3 = 0$

$$(2.8) \quad u(x_1, x_3 = 0, \omega) = \frac{|\omega|}{4\pi} \frac{\Delta L}{\sqrt{2\pi}\gamma} \int dx'_1 \int dp_1 \sum_L u(x'_1, 0, \omega) \{e^{-(x_1 - x_L)^2/2\gamma^2} e^{i\omega p_1 (x_1 - x_L)}\} e^{i\omega p_1 (x_L - x'_1)}$$

The term in the parentheses is an initial Gaussian tapered wave components which can be propagated in depth using paraxial Gaussian beams as

$$(2.9) \quad u(x_1, x_3, \omega) = \frac{|\omega|}{4\pi} \frac{\Delta L \Delta p}{\sqrt{2\pi}\gamma} \sum_M \sum_L u^{\text{gb}}(x_1, x_3; x_L, 0, \omega) \int dx'_1 u(x'_1, 0, \omega) e^{-i\omega p_1 (x'_1 - x_L)}$$

where  $p_1 = M\Delta p$  and  $p_3 = \sqrt{v^{-2} - p_1^2}$ . Thus,

$$(2.10) \quad u(x_1, x_3, \omega) = \frac{|\omega|}{4\pi} \frac{\Delta L \Delta p}{\sqrt{2\pi}\gamma} \sum_M \sum_L u^{\text{gb}}(x_1, x_3; x_L, 0, \omega) D(x_L, p_1, \omega)$$

where  $D(x_L, p_1, \omega)$  is a slant stack of the wavefield on the initial surface,

$$(2.11) \quad D(x_L, p_1, \omega) = \int dx'_1 u(x'_1, 0, \omega) e^{-i\omega p_1 (x'_1 - x_L)}$$

If we assume a single point source as an initial field displacement then

$$u(x'_1, 0, \omega) = \delta(x'_1 - x_s)$$

and

$$D(x_L, p_1, \omega) = e^{i\omega p_1 (x_L - x_s)}$$

and for the above expansion then

$$(2.12) \quad u(x_1, x_3, \omega) = \frac{|\omega|}{4\pi} \frac{\Delta L \Delta p}{\sqrt{2\pi}\gamma} \sum_M \sum_L u^{\text{gb}}(x_1, x_3; x_L, 0, \omega) e^{i\omega p_1 (x_L - x_s)}$$

which is an approximate expansion of the line source into plane beams in the aperture plane and then propagated to the receiver. Aside from the frequency weighting and lack of the  $1/p_3$  resulting from the Kirchhoff formulation, this is similar to Eqn. (2.6).

In the approach of Hill (2001), a local slant stack is used of the form

$$D(x_L, p_1, \omega, \gamma) = \int dx'_1 u(x'_1, 0, \omega) e^{-(x'_1 - x_L)^2/2\gamma^2} e^{-i\omega p_1 (x'_1 - x_L)},$$

instead of Eqn. (2.11). This results from an initial expansion of the Green's functions into Gaussian beams. It causes an additional decay term which possibly could diminish the spectral content of the results when using finite beam widths, in contrast to the full slant stack.

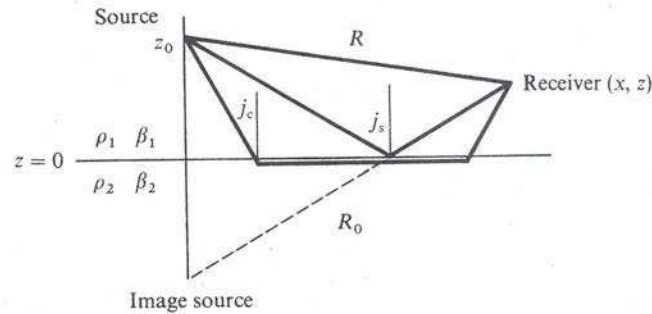


FIG. 2. *The layer over a half space geometry (from Aki and Richards, 1980).*

**3. Application to a Line Source in a Layer Over a Halfspace.** For this example, a homogeneous layer over a half space is used where the geometry is shown in Figure 2. The reflected arrival for an SH line source in the post-critical range gives rise to a wide angle reflection, as well as a head wave (Aki and Richards, 1980; 2002). This problem can be solved by direct integration of the wave-number integral in Eqn. (2.2). Alternatively, the modified-Cagniard method can be used to solve the problem (Aki and Richards, 1990; 2002). By using the Gaussian beam method, the head wave can be generated by using wide beams to model for the wide-angle reflection (Nowack and Aki, 1984, Nowack, 2003). For a layer over a gradient, Nowack and Stacy (2002) and Stacy and Nowack (2002), investigated using Gaussian beams for the wide-angle interference head wave and could simulate the amplitude and frequency effects with Gaussian beams. More narrow beams were used for the diving rays and broader beams were used for the wide-angle reflection to obtain the interference waves and head wave contributions. Recently, Thomson (2004) using the coherent state approach of Thomson (2001) to show asymptotically that the head wave could be generated using beam tapered solutions.

A 10 km layer over a halfspace is used with the upper velocity of 3.5 km/sec and the half-space velocity of 4.6 km/sec and only the SH case is considered. Both the source and receiver are located along the surface. The critical angle for the wide-angle reflection is 49.5 degrees and the critical distance is at 23.4 km. A Gabor pulse with a center frequency of 2 Hz is used and low reference frequency of .25 Hz and the highest frequency considered is 5 Hz. The initial beam widths are selected so that the reflected beams at the receiver have optimally narrow beams at the receiver (Cerveny et al., 1982). This was shown by Nowack and Aki (1984) to result in stable beam contributions for the wide-angle reflection but a more limited head wave contribution when using a single-parameter beam summation over take-off angle.

Figure 3 shows a waveform comparison at a distance of 50 km between a two-parameter, frame-based Gaussian beam summation using the optimally narrow beams at the receiver and the results from the modified-Cagniard method. The initial beams are chosen to be planar at the source. For this case the frame-based Gaussian beam summation generates both the wide-angle reflection and the head wave with the finite beam widths used. The overall waveforms are similar between the two methods.

Figure 4 shows a waveform comparison at a distance of 40 km between finite beam summations using frame-based summations with planar wavefronts at the source and at the receiver. Cerveny (1985a,b) recommended using planar wavefronts at the receiver locations since they can result in more stable summations. These are compared with the summation of plane waves (Fig. 4, center). Overall the waveforms are similar between the frame-based beam summations and the summation of plane waves.

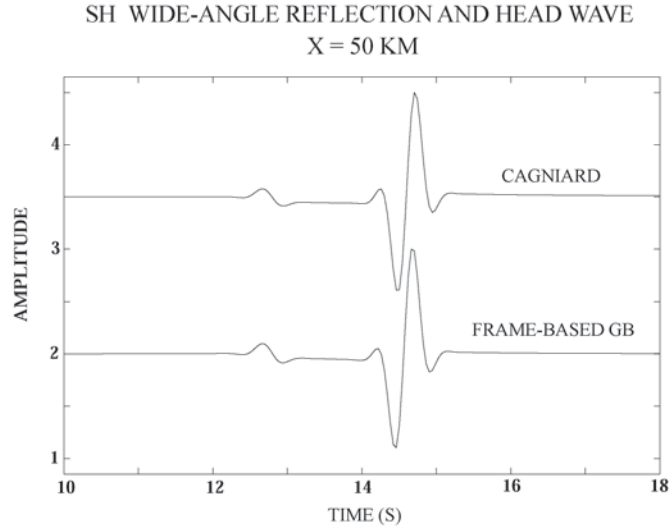


FIG. 3. Comparison of waveforms for a receiver at 50 km from the source. The bottom trace is from a two-parameter Gaussian beam summation with planar beams at the source. The upper trace is from the modified-Cagniard method.

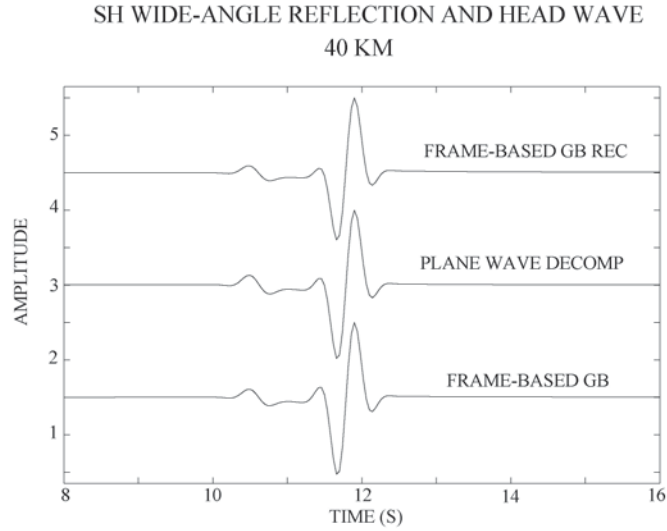


FIG. 4. Comparison of waveforms for receiver at 40 km from the source. The bottom trace is from a two-parameter Gaussian beam summation with planar beams at the source. The middle trace for a plane-wave summation and the lower trace is from a two-parameter Gaussian beam summation with planar beams at the receiver.

**4. Conclusions.** In this paper, an over-determined frame-based decomposition is developed based on a two parameter summation over position and wavenumber. This is used to compute the wide-angle reflection and head wave for an SH line source in a layer over a half space for a finite beam-widths. The use of the two-parameter, frame-based beam summation approach restores some of the spectral content that was lost in the single-parameter beam summations when using finite beam widths.

**Acknowledgements.** This work was funded in part by the National Science Foundation and the members of the GeoMathematical Imaging Group (GMIG) at Purdue University.

## REFERENCES

- [1] AKI K. AND P. G. RICHARDS, (1980) *Quantitative Seismology*, Freeman, San Francisco.
- [2] AKI K. AND P. G. RICHARDS, (2002) *Quantitative Seismology*, 2<sup>nd</sup> edition, Krueger.
- [3] BABICH, V.M AND M.M. POPOV, (1989) *Gaussian summation method (Review)*, Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika 32, 1447-1466 (Translated in Radiophysics and Quantum Electronics 32, 1063-1081, 1990).
- [4] BLEISTEIN, N., (2007) *Mathematics of modeling, migration and inversion with Gaussian beams*, Lecture Notes (online at [www.cwp.mines.edu/~norm/](http://www.cwp.mines.edu/~norm/)).
- [5] CERVENY, V., (1985a) *Gaussian beam synthetic seismograms*, J. Geophys., 58, 44-72.
- [6] CERVENY, V., (1985b) *The applications of ray tracing to the numerical modeling of seismic wave fields in complex structures*, in Seismic Shear Waves (ed. G. Dohr) Geophysical Press, London, pp. 1-124.
- [7] CERVENY, V., L. KLIMES AND I. PSENCIK, (2007) *Seismic ray method: some recent developments*, Advances in Geophysics, 48, 1-128.
- [8] CERVENY, V., M.M. POPOV AND I. PSENCIK, (1982) *Computation of wavefields in inhomogeneous media – Gaussian beam approach*, Geophys. J.R. Astr. Soc., 70, 109-128.
- [9] CHEW, W.C., (1990) *Waves and Fields in Inhomogeneous Media*, Van Nostrand Reinhold, New York.
- [10] FEICHTINGER, H.G. AND T. STROHMER (EDS.), (1998) *Gabor Analysis and Algorithms: Theory and Applications*, Birkhauser.
- [11] GAO, X.J., L.B. FELSEN AND I.T. LU, (1990) *Spectral options to improve the paraxial narrow Gaussian beam algorithm for critical reflection and head waves*, in Computational Acoustics (eds. D. Lee, A Cakmak and R. Vlchnevetsky) Elsevier, 149-166.
- [12] GOODMAN, J. W., (1996) *Introduction to Fourier Optics*, McGraw-Hill.
- [13] HALE, D., (1992) *Migration by the Kirchhoff, slant stack and Gaussian beam methods*, CWD-121, Center for Wave Phenomena, Colorado School of Mines, Golden, CO.
- [14] HILL, N. R., (1990) *Gaussian beam migration*, Geophysics, 55, 1416-1428.
- [15] HILL, N. R., (2001) *Prestack Gaussian beam depth migration*, Geophysics, 66, 1240-1250.
- [16] KAISER, G., (1994) *A Friendly Guide to Wavelets*, Birkhauser Boston.
- [17] LANDRO, M. AND I. TSVANKIN, (2007) *Seismic critical-angle reflectometry: A method to characterize azimuthal anisotropy?*, Geophysics, 72, D41-D50.
- [18] LUGARA, D., C. LETROU A, SHLIVINSKI, E. HEYMAN AND A. BOAG, (2003) *Frame-based Gaussian beam summation method: Theory and applications*, Radio Science, Vol. 38, No. 2, 8026.
- [19] NOWACK, R.L., (2003) *Calculation of synthetic seismograms with Gaussian Beams*, Pure and Applied Geophys., 160, 487-507.
- [20] NOWACK, R.L. AND K. AKI, (1984) *The two-dimensional Gaussian beam synthetic method: testing and application*, J. Geophys. Res., 89, 7797-7819.
- [21] NOWACK R.L. AND S. M. STACY, (2003) *Synthetic seismograms and wide-angle attributes from the Gaussian beam and reflectivity methods for models with interfaces and velocity gradients*, Pure and Applied Geophys., 159, 1447-1464.
- [22] NOWACK, R. L., W.P. CHEN, U. KRUSE AND S. DASGUPTA, (2007) *Imaging offsets in the Moho: synthetic tests using Gaussian beams with teleseismic waves*, Pure and Applied Geophys., 164, 1921- 1936.
- [23] NOWACK, R.L., S. DASGUPTA, G. T. SCHUSTER AND J. M. SHENG, (2006) *Correlation migration using Gaussian beams scattered teleseismic body waves*, Bull. Seism. Soc. Am., 96, 1-10.
- [24] POPOV, M. M., (1982) *A new method of computation of wave fields using Gaussian beams*, Wave Motion, 4, 85-97.
- [25] POPOV, M. M., (2002) *Ray Theory and Gaussian beam method for geophysicists*, Lecture notes University of Bahia, Salvador Brazil, 172 pp.
- [26] STACY, S.M. AND R. L. NOWACK, (2002) *Modeling of wide-angle seismic attributes using the Gaussian beam method*, Studia Geophys. Geod., 46, 667-690.
- [27] THOMSON, C.J., (2001) *Seismic coherent states and ray geometrical spreading*, Geophys. J. Int., 144, 320-342.
- [28] THOMSON, C.J., (2004) *Coherent-state analysis of the seismic head-wave problem: an overcomplete representation and its relationship to rays and beams*, Geophys. J. Int., 157, 1189-1205.
- [29] WHITE, B.S., A. NORRIS, A. BAYLISS, AND R. BURRIDGE, (1987) *Some remarks on the Gaussian beam summation method*, Geophys. J.R. astr. Soc., 89, 579-636.