Evolution-equation approach to seismic image, and data, continuation [1]

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Abstract

In reflection seismology one places point sources and point receivers on the earth's surface, forming an acquisition geometry. Each source generates acoustic waves in the subsurface, that are reflected where the medium properties vary discontinuously. The reflections that can be observed at the receivers are used to image these discontinuities or reflectors assuming a background medium. We analyze methods to circumvent the repeated imaging of reflectors under varying background media, or the repeated modelling of reflections under varying acquisition geometries. These methods involve the introduction of the notion of seismic continuation. Here, we develop the foundation of, and a comprehensive framework for seismic continuation while extending earlier approaches to allow for the formation of caustics. Traditionally, seismic continuation has been viewed from a geometrical (ray asymptotic) point of view; here, we introduce the notion of wave-equation continuation through the appearance of evolution equations.

Key words: velocity continuation, Fourier integral operators, evolution systems, global Hamiltonians

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[1] This research was supported in part under NSF CMG grant EAR-0417891.
[2] Supported under NSF grant DMS-0500788.

1 Introduction

In reflection seismology one places point sources and point receivers on or near the earth's surface. Each source generates acoustic waves in the subsurface, that are reflected where the medium properties vary discontinuously. (In global earth applications the sources are earthquakes.) The recorded reflections that can be observed at the receivers are used to image these discontinuities or reflectors.

Seismic reflection data, in the single scattering or Born approximation, are commonly modelled by an integral operator mapping a medium contrast (containing reflectors), given a background medium (velocity model), to a wavefield (containing reflections). Imaging of seismic reflection data is then described by the adjoint of this integral operator with a given background medium. In exploration seismology, the process of imaging is also referred to as migration, while the process of modelling data from an image is referred to as demigration. In applications, however, the background medium may not be accurately known, and hence it becomes desirable to develop a family of modelling and imaging operators for a set of background media. Also, the data may have been acquired for one particular acquisition geometry, while it can become desirable to generate the data for different geometries, requiring the development of an associated family of imaging and modelling operators. The latter can be viewed as a method of data regularization.

In present day applications, the volume of data can be massive, whence it becomes advantageous to circumvent the repeated imaging or migration under varying background media or the repeated modelling or demigration under varying acquisition geometries. This leads to the introduction of the notion of seismic continuation: The continuation of an image following a path of background media without remigrating the data, or the continuation of data following a path of acquisition geometries without demigrating an image. The applications encompass the exploration of discontinuities in Earth's interior.

The notion of seismic continuation has been around for many years. Fomel [18] introduced the concept of data continuation in source-receiver offset [3]. In particular, under certain conditions, zero-offset data can be obtained from finite-offset data, as in the so-called data transformation to zero offset (TZO) obtained after dip moveout (DMO), through data continuation (see also [21,16]). The concept of image continuation and corresponding velocity rays may be dated back to the work by Fomel [19]; this continuation was based on time migration and assumed constant background media. An approach similar to image continuation is residual migration (see [15,43,35]). The concept of image continuation in varying background media was further developed by Goldin [20], Hubral, Tygel and Schleicher [47,24], Fomel [17], Iversen [25,27] and Adler [1]. In the process of continuation, one can also track the impulse response of the

[3] In [8] the authors erroneously attributed the equation generating this continuation, and the underlying geometrical construction, to S.V. Goldin.

imaging operator; in connection with this, Iversen [26] introduced the notion of isochron rays. Residual velocity DMO introduced by Alkhalifah and De Hoop [2,3] yields TZO in conjunction with continuation following a path of anisotropic velocity models and is reminiscent of both these concepts. Continuation in background velocity can be exploited in developing a method for determining it. For the case of image continuation, consistent with terminology from the seismic literature, we refer to such a method as 'continuation-based' migration velocity analysis. This idea was explored by Liu and Bleistein [28] and Meng and Bleistein [30,31].

The above mentioned, sometimes seemingly different concepts were developed either for constant background media or under the condition of absence of caustics. Here, we develop the foundation of, and a common, comprehensive, framework for seismic continuation while extending the earlier approaches to allow for the formation of caustics. Furthermore, we establish that the propagation of singularities by continuation can always be expressed in terms of a canonical transformation, while we address the question whether continuation can be described by an evolution system. We introduce necessary and sufficient conditions for the notion of continuation to be well-defined, and show the existence of an evolution equation that dynamically generates the continuation. The principal symbol of this equation defines a (global) Hamiltonian, the flow of which defines continuation bicharacteristics.

Our main analytical tools are taken from microlocal analysis, see e.g. [12,46,22,23]. Modelling and imaging of seismic reflection data can be mathematically described in terms of Fourier integral operators (FIOs) [4,34,45,7,38]. The basic properties of FIOs are summarized in Section 2.1. The class of FIOs the canonical relations of which are graphs, and which are invertible (Section 2.2), forms the key building block of seismic continuation theory. For the modelling (and imaging) operators, in the presence of caustics, to be contained in this class, these operators need to be extended (Stolk and De Hoop [38,39]). In Section 2.3 we introduce smooth one-parameter families of FIOs in the above mentioned class. In Appendix A, the principal fiber bundle (and Lie group) structure of the mentioned class of FIOs is discussed. The base space is formed by the canonical transformations, which reflects the central role that the geometry plays in seismic continuation. Continuation can then be formalized as a curve in a section of this principal fiber bundle. Canonical transformations are identifiable with contact transformations, which have been used in an alternative description of propagation of singularities through the notion of contact elements (Goldin [20]).

In Section 3.1 seismic continuation operators are defined satisfying a minimal set of requirements. All such operators can be identified as solution operators to pseudodifferential evolution equations. Thus, continuation operators have all the properties of propagators. The evolution equation leads to the introduction of continuation bicharacteristics (Section 3.2) which describe the propagation of singularities by continuation. The bicharacteristics solve a Hamilton system, in which the Hamiltonian is derived from the principal symbol of the evolution operator. (The bicharacteristics are also curves in phase space determined by the canonical transformation.) In applications, often, a vector field tangent to the Hamiltonian flow is directly constructed; using

Poincaré's lemma, a global Hamiltonian can then obtained.

In Section 4.1, we represent continuation as the composition of demigration and migration FIOs (which can be viewed as a factorization) selected from such families. The parameter becomes an evolution parameter – it imitates the time in wave propagation. For image continuation, for example, one can think of demigrating the image with one velocity model and remigrating the result with another velocity model. In Section 4.2 we discuss derivatives (with respect to the evolution parameter) of the above mentioned FIOs; we establish relationships between derivatives of modelling and imaging operators, and derivatives of continuation operators making use of the above mentioned factorization. Continuation operators are typically close to the identity and are connected to a Lie algebra of pseudodifferential operators. Composing the phases in the kernel representations of the FIOs making up the continuation operator, under infinitesimally differing parameter values, one obtains an alternative geometrical description, viz. through the evolution of fronts such as 'isochrons' in image continuation.

In Section 5 we establish the explicit relation between the phase function in the kernel representation of the FIO, the generating function corresponding with its canonical graph, and the pseudodifferential operator symbol appearing in the above mentioned evolution equation. This relation can be used as a construction in applications. The Hamiltonian that generates the continuation bicharacteristics is expressed in terms of this generating function. The continuation bicharacteristics can also be constructed directly from the evolution of fronts using the phase function. Essentially, the geometry associated with seismic continuation has been inferred from the geometry of migration operators.

In Sections 6 and 7 we conclude with showing examples of continuation in reflection seismology. The examples in Section 6 are derived from imaging with the generalized Radon transform (encompassing Kirchhoff migration). We give explicit expressions for phase functions and generating functions associated with the kernels of the relevant FIOs. In Section 6.2, we demonstrate how earlier concepts are contained in our theory. In Section 6.1 we discuss the original presentation (derived from the phase function) of image continuation under common-offset Kirchhoff migration in the absence of caustics, and specialize to constant background media. One of the motivations for developing the theory presented in this paper was indeed to establish the connection between continuation as a composition of migration with demigration and the construction of 'velocity rays' to describe the propagation of singularities under continuation (Section 6.2); a second motivation was to bring the system of ordinary differential equations for continuation bicharacteristics in Hamilton form (Sections 6.1 and 6.3, see also Appendix B). A third motivation was to establish the importance of canonical transformations (preserving the symplectic form) generating continuation operators.

In Section 7, we discuss the notion of image gathers and their velocity continuation in the presence of caustics. We show an example, revealing the potential of the comprehensive theory presented here (Section 3). Velocity continuation of image gathers can directly be exploited in reflection tomography, the problem of determining the background velocity, see [44,42,38,37,40].

2 Representations of 'migration' and 'demigration' operators

We formulate modelling and imaging of seismic reflection data within the general framework of linear integral operators. Let y denote a point in an acquisition manifold Y on which data u are defined. Let x denote a point in the subsurface manifold X on which a contrast v or an image w is defined. We let $n_X = \dim X$ and $n_Y = \dim Y$; naturally, $n_Y \geq n_X$. Typically, y consists of a combination of source and receiver points contained in ∂X, and time. We consider the operator pair F, F^*, where F^* is the adjoint of F, that is $\langle u, Fv \rangle_Y = \langle F^* u, v \rangle_X$. For any data u, there exists a $v \in \mathcal{E}'(X)$ such that

$$u = Fv. \tag{1}$$

In general, $u \in \mathcal{D}'(Y)$. Moreover, w in

$$w = F^* u, \tag{2}$$

is identified as the image.

2.1 Fourier integral operators

We assume that F is a Fourier integral operator (FIO) – this assumption is commonly satisfied in seismic data applications [4,34,45,7,38]. Then F^* is an FIO as well. The action of F, microlocally, can be written in the form

$$(Fw)(y) = \int A(y,x) w(x) \, \mathrm{d}x, \tag{3}$$

$$A(y,x) = \int_{\mathbb{R}^N} a(y,x,\theta) \exp[\mathrm{i}\phi(y,x,\theta)] \, \mathrm{d}\theta, \tag{4}$$

in which $\theta = (\theta_1, \ldots, \theta_N)$ are so-called phase variables. Here, ϕ is a phase function: ϕ is real-valued, $\phi \in C^\infty(Y \times X \times (\mathbb{R}^N \backslash 0))$, ϕ is positive-homogeneous of degree one in θ, and ϕ does not have critical points for $\theta \neq 0$, that is, $\partial_{(y,x)} \partial_\theta \phi(y,x,\theta) \neq 0$ for $(y,x) \in Y \times X$ and $\theta \in \mathbb{R}^N \backslash 0$.

Furthermore, a is an amplitude of order m, that is $a \in S^m((Y \times X, \mathbb{R}^N)$, which has the property: To every compact subset $\mathcal{K} \subset Y \times X$ and multi-indices α, β there is a constant $C_{\alpha,\beta}(\mathcal{K})$ such that

$$|\partial_\theta^\alpha \partial_{(y,x)}^\beta a(y,x,\theta)| \leq C_{\alpha,\beta}(\mathcal{K})\langle\theta\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \langle\theta\rangle = (1+\|\theta\|^2)^{1/2},$$
$$\text{for all } (y,x) \in \mathcal{K} \text{ and } \theta \in \mathbb{R}^N\backslash 0. \tag{5}$$

(With these estimates, and ϕ being a phase function, the integral representation for $A(y,x)$ in (4) can be regularized.) We restrict our analysis to amplitudes of the type $\rho = 1$, $\delta = 0$; amplitudes of order m of this type define the class of FIOs $\Psi^m(X)$. The operator F extends to a continuous linear map $F : \mathcal{E}'(X) \to \mathcal{D}'(Y)$. The operator F propagates singularities. Microlocally, this is determined by the phase function ϕ, and can be understood as follows. The stationary point set of the phase function is given by

$$S_\phi = \{(y,x,\theta) \mid \partial_\theta\phi(y,x,\theta) = 0\}. \tag{6}$$

The phase function will be assumed to be non-degenerate, that is, the rank of the Hessian matrix,

$$\left(\mathrm{d}_{(y,x,\theta)}\frac{\partial\phi}{\partial\theta}\right) \quad \text{is maximal (that is, } N\text{).}$$

Then S_ϕ is a $(n_Y + n_X)$- dimensional submanifold of $Y \times X \times (\mathbb{R}^N\backslash 0)$. Moreover, S_ϕ is a conic subset of $Y \times X \times (\mathbb{R}^N\backslash 0)$, i.e., if $(y_0, x_0, \theta_0) \in S_\phi$ then $(y_0, x_0, t\theta_0) \in S_\phi$ for any $t > 0$.

In view of the homogeneity of ϕ, we have $\phi = \theta \cdot \partial_\theta\phi$, so that $\phi(y,x,\theta) = 0$ if $(y,x,\theta) \in S_\phi$. Let $T^*Y\backslash 0$ denote the acquisition phase space and $T^*X\backslash 0$ denote the subsurface phase space. The stationary point set can be embedded in $T^*Y\backslash 0 \times T^*X\backslash 0$:

$$S_\phi \to \Lambda, \quad (y,x,\theta) \to (y, \partial_y\phi; x, -\partial_x\phi) \quad \text{is an immersion,} \tag{7}$$
$$\Lambda = \{(y, \partial_y\phi; x, -\partial_x\phi) \mid \partial_\theta\phi = 0\}\,. \tag{8}$$

Λ is (locally) a conic Lagrangian submanifold of $T^*(Y \times X)\backslash 0$, and is called the canonical relation of operator F; we sometimes write $\Lambda = \Lambda^F$ to indicate its association to F. It is immediate that $\Lambda^{F^*} = (\Lambda^F)^* = \{(x,\xi;y,\eta) \mid (y,\eta;x,\xi) \in \Lambda^F\}$. The canonical relation describes the propagation of singularities in (1): if WF denotes the wavefront set of a distribution,

$$\begin{aligned}\mathrm{WF}(u) &\subseteq \Lambda^F \circ \mathrm{WF}(w)\\ &= \{(y,\eta) \mid (y,\eta;x,\xi) \in \Lambda^F \text{ and } (x,\xi) \in \mathrm{WF}(w) \text{ for some } (x,\xi) \in T^*X\backslash 0\},\ u = Fw.\end{aligned}$$

Identifying reflections in $\mathrm{WF}(u)$ (η defines 'slopes') and reflectors in $\mathrm{WF}(w)$ or $\mathrm{WF}(v)$ (ξ defines 'dip'), following seismic terminology, we refer to F^* as 'migration'; if F acts on an image v, we speak of F as 'demigration' instead of modelling.

The kernel A in (4) is a Lagrangian distribution. Its singular support is also determined by the phase function ϕ: Let $\pi : Y \times X \times (\mathbb{R}^N\backslash 0) \to Y \times X$ denote the natural projection, then sing supp $A \subset \pi S_\phi$. Viewing sing supp A at a fixed x, x_0 say, yields the physical notion of a front: $\mathcal{W}(x_0) = \{y \in Y \mid (y, x_0) \in \pi S_\phi\}$. In the reflection seismology literature, one refers to such a front as the (geometrical) 'impulse response' and 'special surfaces', see Goldin [20]. In case of modelling or demigration, the fronts are also called 'diffraction surfaces', while in the case of imaging these fronts are also called 'isochrons'.

In general, Λ admits local coordinates $(y_{I'}, \eta_{J'}, x_I, \xi_J)$ with $(I' \cup I) \cup (J' \cup J) = \{1, 2, \dots, n_Y + n_X\}$ together with the existence of a generating function $S = S(y_{I'}, \eta_{J'}, x_I, \xi_J)$ such that

$$x_J = \frac{\partial S}{\partial \xi_J} \ , \ \xi_I = -\frac{\partial S}{\partial x_I},$$
$$y_{J'} = \frac{\partial S}{\partial \eta_{J'}} \ , \eta_{I'} = \frac{\partial S}{\partial y_{I'}}$$

[22, Thm. 21.2.18]. Then the phase variables in (4) can be locally chosen to be $\theta = (\eta_{J'}, \xi_J)$, whence the phase function attains the form

$$\phi(y, x, \eta_{J'}, \xi_J) = S(y_{I'}, \eta_{J'}, x_I, \xi_J) - \langle \eta_{J'}, y_{J'} \rangle - \langle \xi_J, x_J \rangle. \tag{9}$$

Let F_1 and F_2 both be FIOs. F_2 maps functions on X to functions on Y, and F_1 maps functions on Y to functions on Z. The composition $F_1 F_2$ is well defined if the intersection of $\Lambda^{F_1} \times \Lambda^{F_2}$ with $T^*Z\backslash 0 \times \mathrm{diag}(T^*Y\backslash 0) \times T^*X\backslash 0$ is transversal [46, Ch. VIII, p.464]. The canonical relation of the composition is given by $\Lambda^{F_1} \circ \Lambda^{F_2}$, following

$$\begin{gathered} \Lambda^{F_1} \times \Lambda^{F_2} \cap T^*Z\backslash 0 \times \mathrm{diag}(T^*Y\backslash 0) \times T^*X\backslash 0 \\ \downarrow \text{ projection} \\ \Lambda^{F_1} \circ \Lambda^{F_2} \subset T^*Z\backslash 0 \times T^*X\backslash 0 \end{gathered} \tag{10}$$

If $Y = X$ and $\Lambda^F \subset \mathrm{diag}\, T^*X\backslash 0$, then F becomes a pseudodifferential operator. Pseudodifferential operators admit representations of the type (4) with $\theta = \xi$ (i.e., $|I'| = n_Y$ and $|J| = n_X$), while $S(y, \xi) = \langle \xi, y \rangle$, so that $\phi(y, x, \xi) = \langle \xi, y - x \rangle$.

2.2 *Fourier integral operators associated with canonical graphs*

Here, we develop the necessary preparation of continuation theory, which leads to a certain class of allowable FIOs. To begin with, we need to assume that $n_Y = n_X$.

Graph assumption. The canonical relation (cf. (8)) is a graph, that is, there exists a transformation $\Sigma: T^*X \to T^*Y$ such that

$$\Lambda = \{(\Sigma(x,\xi); x, \xi)\}. \tag{11}$$

The transformation Σ will be a canonical transformation, that is, it preserves the symplectic form. (If Σ is the identity, the associated operator will simply be pseudodifferential.)

Without restriction, we can assume that the FIOs are of order $m = 0$ (cf. (5)). Indeed, if F were of order m, that is, $F: H^s \to H^{s-m}$, then $F': H^s \to H^s$ in $F = (I - \Delta_y)^{m/2} F'$ is of order 0 and will be the operator under consideration here; Δ_y denotes the Laplacian in the y coordinates. The FIOs of order 0 satisfying the graph assumption form a semi-group. If the canonical relations of F_1 and F_2 are generated by canonical transformations then their canonical relations will compose transversally, and $F_1 F_2$ is an FIO of order 0 the canonical relation of which is generated by a canonical transformation, again.

Subject to the graph assumption, the kernel of an FIO F admits a representation (cf. (4), (9) with $|I'| = n_Y$, $|J| = n_X$)

$$A(y,x) = \int a(y,\xi) \exp[\mathrm{i}\phi(y,x,\xi)] \, \mathrm{d}\xi, \tag{12}$$

$$\phi(y,x,\xi) = S(y,\xi) - \langle \xi, x \rangle, \tag{13}$$

where S is homogeneous of degree 1 in ξ; this representation is close to the one for a pseudodifferential operator kernel. In (12) we have reduced the amplitude $a(y,x,\xi)$ to $a(y,\xi)$ by standard methods. Up to principal parts, $a_0(y,\xi) = a(y, \partial_\xi S, \xi)$. By an iteration argument [23, p.27] the amplitude $a(y,\xi)$ is obtained, leading to a kernel equivalent to the original operator kernel modulo C^∞.

The principal symbol of F with an integral kernel (12) is defined to be

$$\sigma_0(F)(y,\xi) = a(y, \partial_\xi S(y,\xi), \xi) \left|\det \partial_y \partial_\xi S(y,\xi)\right|^{-\frac{1}{2}}. \tag{14}$$

The canonical relation attains the form (cf. (8)-(9))

$$\Lambda = \{(y, \partial_y S; \partial_\xi S, \xi)\}. \tag{15}$$

(Indeed, a canonical transformation (cf. (11)) provides S, which generates a phase function as in (13)). In conjunction with this, the matrix $\partial_\xi \partial_y S$ is non-singular. Naturally, (x,ξ) form coordinates on Λ as well. How to change between representations of the type (4) and (12)-(13), with different phase variables, is discussed in Appendix B.

DEFINITION 1. *We reserve the notation $\mathcal{C}$ for the class of invertible FIOs of order 0 that satisfy the graph assumption (then the Σ are diffeomorphisms).*

We will show that seismic continuation needs to be formulated within this class. The class $\mathcal{C}$ is an infinite-dimensional manifold with the structure of a principal fibre bundle: The base manifold consists of all canonical transformations, the fibres are isomorphic to the algebra of pseudodifferential operators of order 0, while the structure group is that same algebra of pseudodifferential operators. In fact, the class $\mathcal{C}$ admits an infinite-dimensional Lie group structure [14,33,36]; see also Appendix A for a precise description. This structure is implicit in the original treatment and characterization of seismic data processing by Goldin.

With F being invertible, the graph assumption also holds for F^*; thus, Λ^*, and Λ, admit coordinates (x, η). To suppress the detailed account of amplitudes, without loss of generality, we can assume that the FIOs are unitary, that is $F^* = F^{-1}$. Indeed, the normal operator $N = F^*F$ is pseudodifferential, and, by standard arguments, the polar decomposition $F = \widetilde{F}\, N^{1/2}$ provides $\widetilde{F}$ with $\widetilde{F}^* = \widetilde{F}^{-1}$; thus our further analysis applies to $\widetilde{F}$.

Moreover, the notion of ellipticity is well defined for FIOs which are associated to the graph of a canonical transformation, see [23, p.27]. An FIO F associated to the graph of a canonical transformation Σ is non-characteristic at a point $(y_0, \eta_0, x_0, \xi_0)$ if its principal symbol does not vanish at this point. F is elliptic if it is non-characteristic at every point on the graph of Σ. The procedure for constructing parametrices of elliptic pseudodifferential operators can be used to prove the following: If $F : C^\infty(X) \longrightarrow C^\infty(Y)$ is a properly supported elliptic FIO of order m associated to the graph of a canonical transformation Σ, then there exists $\widehat{F}^{-1} : C^\infty(Y) \longrightarrow C^\infty(X)$, an elliptic FIO of order $-m$ associated to the graph of Σ^{-1}, such that

$$F\widehat{F}^{-1} - I \in \Psi^{-\infty}(Y), \quad \widehat{F}^{-1}F - I \in \Psi^{-\infty}(X). \tag{16}$$

In the developement of continuation theory we assume invertibility of the relevant FIOs (cf. Definition 1); however, we will also indicate how to weaken the assumption of invertibility to ellipticity.

2.3 Smooth families of Fourier Integral Operators

Here we consider particular smooth one-parameter families of Fourier integral operators

DEFINITION 2. *Let $I \subset \mathbb{R}$ be an interval, and for each $\alpha \in I$, let $F(\alpha) : \mathcal{E}'(X) \longrightarrow \mathcal{D}'(Y)$ be a properly supported Fourier integral operator of order m associated to the graph of a canonical transformation Σ_α. We say that $F(\alpha)$ is a smooth, or C^∞, family of FIOs if the following condition holds:*

$$\textit{for all } f \in C_0^\infty(X), \quad F(\alpha)f \in C^\infty(I \times Y). \tag{17}$$

We define the operator $\partial_\alpha F(\alpha)$ as

$$(\partial_\alpha F(\alpha))f = \partial_\alpha (F(\alpha) f). \tag{18}$$

We assume that

$$\partial_\alpha F(\alpha) \textit{ is an FIO of order } m+1 \textit{ associated to the graph of } \Sigma_\alpha. \tag{19}$$

If $F(\alpha)$, $\alpha \in I$ form a smooth family, and for each α, $F(\alpha)$ is either elliptic or invertible the family can be characterized as follows:

THEOREM 3. *Let $F(\alpha) : \mathcal{E}'(X) \longrightarrow \mathcal{D}'(Y)$, $\alpha \in I$ be a C^∞ one-parameter family of FIOs. If for each $\alpha \in I$, $F(\alpha)$ is properly supported invertible then there exists a C^∞ one-parameter family of ΨDOs, $P(\alpha) = P(\alpha; y, D_y) \in \Psi^1(Y)$ such that*

$$(\partial_\alpha - \mathrm{i}P(\alpha))\, F(\alpha) = 0. \tag{20}$$

If for each $\alpha \in I$, $F(\alpha)$ is properly supported elliptic, then there exists a C^∞ one-parameter family of ΨDOs, $P(\alpha; y, D_y) \in \Psi^1(Y)$ such that

$$(\partial_\alpha - \mathrm{i}P(\alpha))\, F(\alpha) \in \Psi^{-\infty}(Y). \tag{21}$$

Proof: If $\partial_\alpha F(\alpha)$ is a FIO of order $m+1$ associated to the graph of Σ_α and $F(\alpha)$ is invertible then by the calculus of FIOs, the operator

$$\mathrm{i}P(\alpha) = \partial_\alpha F(\alpha) F(\alpha)^{-1} \in \Psi^1(Y).$$

Then it is immediate that (20) is satisfied. The same argument can be used in the elliptic case, where instead of using $F(\alpha)^{-1}$ we use the parametrix $\widehat{F}(\alpha)^{-1}$ given by (16).

Conversely, the solution of evolution equation (20) generates a C^∞ α-family of FIOs in $\mathcal{C}$. Let $F(\alpha)$, $\alpha \in I$, be such a C^∞ one-parameter family of FIOs associated to the graphs of canonical transformations Σ_α as above. For each $\alpha_0 \in I$ and $(x_0, \xi_0, y_0, \eta_0)$ on the graph of Σ_{α_0} there exists a neighborhood $E \subset I$ of α_0 and conic neighborhoods, Γ, of $(x_0, \xi_0, y_0, \eta_0)$ such that the Schwartz kernel, $A = A_{F(\alpha)}$, of $F(\alpha)$ microlocally is given by (12)-(13) with $a = a(\alpha, y, \xi)$ and $\phi = \phi(\alpha, y, x, \xi)$. Indeed, because

$$\partial_\alpha A_{F(\alpha)}(y, x) = \int_{\mathbb{R}^N} \exp[\mathrm{i}\phi(\alpha, y, x, \theta)]\, (\mathrm{i}(\partial_\alpha \phi)\, a(\alpha, y, x, \theta) + \partial_\alpha a(\alpha, y, x, \theta))\ \mathrm{d}\theta, \tag{22}$$

it follows that $\partial_\alpha F(\alpha)$ is an FIO of order one associated to the graph of Σ_α.

3 Continuation theory

In seismic applications, to which we return in Sections 6 and 7, one distinguishes two types of operators F connecting spaces X and Y. We note that for invertible F we have $\dim X = \dim Y$. For the first type of operators, Y and X are different spaces, with different physical roles. Examples are the scattering or modelling operators and the imaging operator (after extension, as discussed in [38]) where the subsurface is connected to the data space. For the second type of operators, we have $Y = X$. Examples are image continuation following a path of background media, and data continuation with source-receiver offset.

Suppose that (i) given data $u(y)$, an image, $w(x) = w(\alpha_0, x)$, has been obtained in a model parameterized by α_0, or (ii) given an image $w(x)$, data $u(y) = u(\alpha_0, y)$ have been obtained in a model parameterized by α_0, or (iii) data $u(y) = u(\alpha_0, y)$ have been acquired in an acquisition geometry parameterized by α_0. Suppose that models or acquisition geometries of interest can be connected along a path parameterized by α taking values in an interval, $I = [\alpha_1, \alpha_2] \subset \mathbb{R}$, containing α_0. Here, we develop a common framework for directly 'continuing' $w(\alpha_0, x)$ or $u(\alpha_0, y)$ along such a path; that is, we introduce continuation operators, C^Y, C^X, such that

$$w(\alpha, x) = (C^X_{(\alpha,\alpha_0)} w(\alpha_0, .))(x), \quad u(\alpha, y) = (C^Y_{(\alpha,\alpha_0)} u(\alpha_0, .))(y), \tag{23}$$

with $\alpha_1 \leq \alpha_0 \leq \alpha \leq \alpha_2$. We discuss (i) sufficient and necessary conditions to be able to develop a continuation theory, (ii) the existence of evolution equations that describe the process of continuation, and (iii) the existence of continuation bicharacteristics and associated global Hamiltonians.

3.1 Operator definition, evolution equation

In this subsection, we consider $C^Y_{(\alpha,\alpha_0)}$, but omit the superscript Y for convenience of notation. An operator $C_{(\alpha,\alpha_0)}$ is called a continuation operator if $C_{(\alpha,\alpha_0)}$ can be viewed as a smooth family of FIOs (cf. Definition 2) depending on the parameter α, satisfying the assumptions that for all $\alpha_0, \alpha \in I$, $\alpha \geq \alpha_0$,

$$\begin{aligned} &- C_{(\alpha,\alpha_0)} \in \mathcal{C}, \\ &- C_{(\alpha,\alpha)} = \mathsf{Id}. \end{aligned} \tag{24}$$

Thus, for fixed α_0, $C_{(\alpha,\alpha_0)}$ defines a one-parameter family of operators in $\mathcal{C}$ (cf. Definition 1). The canonical relation of $C_{(\alpha,\alpha_0)}$ is denoted by $\Lambda_{(\alpha,\alpha_0)}$; we denote the canonical transformation that generates $\Lambda_{(\alpha,\alpha_0)} \subset T^*Y\backslash 0 \times T^*Y\backslash 0$ by $\Sigma_{(\alpha,\alpha_0)}$, while $\Sigma_{(\alpha_0,\alpha_0)} = \mathsf{Id}$. Here, we simplify the notation by setting $\alpha_0 = 0$.

With $C_{(0,0)} = \mathsf{Id}$, equation (2.3) signifies that

$$\partial_\alpha C_{(\alpha,0)}|_0 = \mathrm{i}P(0). \tag{25}$$

With $C_{(\alpha,\alpha)} = \mathsf{Id}$ holding on the interval, the one-parameter family of operators $C_{(\alpha,0)}$ forms a curve containing the identity in the earlier mentioned infinite-dimensional Lie group; the property that

$$C_{(\alpha,0)} = C_{(\alpha,\alpha')}C_{(\alpha',0)}, \quad 0 \le \alpha' \le \alpha$$

reflects this. Thus $C_{(\alpha,0)}$ can be called a propagator. Clearly, the propagation of singularities by $C_{(\alpha,0)}$ is described by

$$\Lambda_{(\alpha,0)} \circ \mathrm{WF}(u(0,.)) = \{\Sigma_{(\alpha,0)}(y_0,\eta_0) \mid (y_0,\eta_0) \in \mathrm{WF}(u(0,.))\},$$

through curves on $T^*Y\backslash 0$. Theorem 3 guarantees the existence of an evolution equation the solution to which is described by the continuation operator. When no confusion is possible, we will omit $\alpha_0 = 0$ in our notation.

Remark. The invertibility condition in (24) can be slightly weakened to the condition that $C_{(\alpha,\alpha_0)}$ is elliptic, see the remark below (15) and Theorem 3.

3.2 Geometry

Continuation bicharacteristics. Evolution equation (20) propagates singularities in accordance with the Hamilton flow with Hamiltonian

$$\mathcal{H}(\alpha, y, \eta_\alpha, \eta) = \eta_\alpha - p_1(\alpha, y, \eta), \tag{26}$$

where p_1 denotes the principal symbol of P; p_1 is homogeneous of degree 1 in η. The Hamilton system is

$$\frac{\mathrm{d}y}{\mathrm{d}\alpha} = \partial_\eta \mathcal{H} = -\partial_\eta p_1, \;\; \frac{\mathrm{d}\eta}{\mathrm{d}\alpha} = -\partial_y \mathcal{H} = \partial_y p_1, \tag{27}$$

$$\frac{\mathrm{d}\eta_\alpha}{\mathrm{d}\alpha} = -\partial_\alpha \mathcal{H} = \partial_\alpha p_1, \tag{28}$$

supplemented with initial conditions $y(0) = y_0$, $\eta(0) = \eta_0$, and $\eta_\alpha(0) = \eta_{\alpha 0}$. In general, the Hamiltonian is anisotropic even when one restricts to isotropic background media. In view of the homogeneity in η we have the usual relation between (anisotropic) group velocity and slowness vectors, $\eta \cdot \frac{\mathrm{d}y}{\mathrm{d}\alpha} = -\eta_\alpha$.

Equation (27) does not depend on η_α and thus may be solved independently. Solutions to (27) describe the canonical transformation generating $C_{(\alpha,0)}$ for each $\alpha \in [0,1]$ fixed (cf. [13, section 5.3], see also [23, Theorem 21.3.4]). We refer to the solutions $y(\alpha, y_0, \eta_0), \eta(\alpha, y_0, \eta_0)$, determining $\Sigma_{(\alpha,0)}$, that is, $\Sigma_{(\alpha,0)}(y_0, \eta_0) = (y(\alpha, y_0, \eta_0), \eta(\alpha, y_0, \eta_0))$, as the continuation bicharacteristics.

Vector field and global Hamiltonian. If ω is the canonical symplectic form on $T^*Y\backslash 0$, then, by definition,

$$\Sigma^*_{(\alpha,0)}\omega = \omega. \tag{29}$$

The curves defined by the canonical transformation determine a vector field, $V_\alpha(\tilde{y}, \tilde{\eta}) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Sigma_{(\alpha,0)}(\tilde{y}, \tilde{\eta})$, $(\tilde{y}, \tilde{\eta}) = \Sigma_{(\alpha,0)}(y_0, \eta_0)$, on $T(T^*Y\backslash 0)$; that is, $V_\alpha := (V_1, V_2)$ is the tangent vector to the curve $(y(\alpha, y_0, \eta_0), \eta(\alpha, y_0, \eta_0))$ in T^*Y,

$$\frac{\mathrm{d}y}{\mathrm{d}\alpha} = V_1(\alpha, y, \eta), \quad \frac{\mathrm{d}\eta}{\mathrm{d}\alpha} = V_2(\alpha, y, \eta). \tag{30}$$

Differentiating (29), we get [49]

$$0 = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Sigma^*_{(\alpha,0)}\omega(\cdot) = \Sigma^*_{(\alpha,0)}\left(\mathrm{d}\omega(\cdot, V_\alpha)\right).$$

But then

$$\mathrm{d}\omega(\cdot, V_\alpha) = 0. \tag{31}$$

In applications, commonly, a construction (based on perturbation arguments) leads directly to equations of the type (30), that is, a vector field V_α. Then one may question the applicability of the theory presented here, in particular, the existence of a global Hamiltonian (cf. (26)). To guarantee the validity of (31), one checks whether the Lie derivative, $\mathcal{L}_{V_\alpha}\,\omega = 0$.

In local coordinates, (y, η), and $V_\alpha = (V_1(\alpha, y, \eta), V_2(\alpha, y, \eta))$, with

$V_1(\alpha, y, \eta) = (V_{11}(\alpha, y, \eta), \ldots, V_{1n}(\alpha, y, \eta))$ and $V_2(\alpha, y, \eta) = (V_{21}(\alpha, y, \eta), \ldots, V_{2n}(\alpha, y, \eta))$

being the ∂_y and ∂_η components of V_α respectively, equation (31) is equivalent to

$$\begin{gathered} \frac{\partial V_{2j}}{\partial y_k} = \frac{\partial V_{2k}}{\partial y_j}, \ \ j \neq k, \quad \frac{\partial V_{1j}}{\partial \eta_k} = \frac{\partial V_{1k}}{\partial \eta_j}, \ \ j \neq k, \\ \frac{\partial V_{2j}}{\partial \eta_k} = -\frac{\partial V_{1k}}{\partial \eta_j}, \quad \text{for all } j, k. \end{gathered} \tag{32}$$

If $\mathrm{d}\omega(\cdot, V_\alpha) = 0$, that is if equations (32) hold, there exists a C^∞ function $q = q(\alpha, y, \eta)$ such that

$$\omega(\cdot, V_\alpha) = \mathrm{d}_{(y,\eta)} q,$$

and q can be computed by the formula

$$q(\alpha, y, \eta) = \sum_{j=1}^{n_Y} \int_0^1 t^{-1} \left[V_{1j}(\alpha, ty, t\eta)\eta_j - V_{2j}(\alpha, ty, t\eta) y_j \right] \mathrm{d}t, \tag{33}$$

see [48, Theorem 4.18]. We have assumed that the manifold T^*Y is essentially an open set in $\mathbb{R}^{2n_Y}$, so that we can apply Poincaré's lemma (whence the form $\omega(\cdot, V)$ is exact). It is clear that if V_α is smooth in α, so is q. It follows that q coincides with p_1 up to an additive constant.

4 Factorization in 'migration' and 'demigration'

Here, we discuss how the continuation operators introduced in the previous section are possibly (but not necessarily) developed from demigration (modelling) and migration (imaging) operators. In general, the amplitude and phase of F are determined by a model $m \in M$, where M stands for a model (such as background velocity) or configuration (such as source-receiver acquisition) space. A curve, $m[\alpha]$, in M thus defines a a one-parameter family of demigration FIOs $F(\alpha) \in \mathcal{C}$ (cf. Definition 1) with amplitudes $a = a(\alpha; y, \xi)$ and phase functions $\phi = \phi(\alpha; y, x, \xi)$, cf. (12)-(13). We denote the canonical relation of $F(\alpha)$ by Λ_α and that of the migration operators $F(\alpha)^*$ by Λ_α^*.

4.1 Continuation operators revisited

We construct continuation operators by the composition,

$$C^X_{(\alpha,\alpha_0)} = F(\alpha)^* F(\alpha_0), \quad C^Y_{(\alpha,\alpha_0)} = F(\alpha)\, F(\alpha_0)^*. \tag{34}$$

These operators satisfy assumptions (24): Indeed, $F(\alpha)^* F(\alpha_0)$ and $F(\alpha)\, F(\alpha_0)^*$ satisfy the graph condition. The canonical relations follow the compositions (cf. (10))

$$\begin{aligned} \Lambda^{C^X}_{(\alpha,\alpha_0)} = \Lambda^*_\alpha \circ \Lambda_{\alpha_0} = \{(x, \xi; x_0, \xi_0) \mid (x, \xi; y, \eta) \in \Lambda^*_\alpha \\ \text{and } (y, \eta; x_0, \xi_0) \in \Lambda_{\alpha_0} \text{ for some } (y, \eta) \in T^*Y \backslash 0\}, \\ \Lambda^{C^Y}_{(\alpha,\alpha_0)} = \Lambda_\alpha \circ \Lambda^*_{\alpha_0} = \{(y, \eta; y_0, \eta_0) \mid (x, \xi; y_0, \eta_0) \in \Lambda^*_{\alpha_0} \\ \text{and } (y, \eta; x, \xi) \in \Lambda_\alpha \text{ for some } (x, \xi) \in T^*X \backslash 0\}; \end{aligned} \tag{35}$$

both are generated by the respective composition of canonical transformations.

In terms of phase functions, the composition of canonical relations (cf. (10)) in $\Lambda^{C^X}_{(\alpha,\alpha_0)}$ follows the construction of the stationary point set (cf. (7)-(8)) for $-\phi(\alpha; y, x, \xi) + \phi(\alpha_0; y, x_0, \xi')$:

$$\partial_\xi \phi(\alpha; y, x, \xi) = 0, \quad \partial_{\xi'} \phi(\alpha_0; y, x_0, \xi') = 0, \\ \partial_y[-\phi(\alpha; y, x, \xi) + \phi(\alpha_0; y, x_0, \xi')] = 0, \quad (36)$$

on which $(x, -\partial_x \phi(\alpha; y, x, \xi); x_0, -\partial_{x_0} \phi(\alpha_0; y, x_0, \xi'))$ determines the points in $\Lambda^*_\alpha \circ \Lambda_{\alpha_0} = \Lambda^{C^X}_{(\alpha,\alpha_0)}$. A similar construction holds for $\Lambda_\alpha \circ \Lambda^*_{\alpha_0}$ and the continuation operator $C^Y_{(\alpha,\alpha_0)}$. We will exploit this observation in the later applications.

4.2 Derivatives

The derivative $\partial_\alpha F(\alpha)$ is determined by $\partial_\alpha C^Y_{(\alpha,\alpha_0)}$: We have (cf. (25))

$$\partial_\alpha F(\alpha)\, F(\alpha_0)^* = \partial_\alpha C^Y_{(\alpha,\alpha_0)} = \mathrm{i}P(\alpha) C^Y_{(\alpha,\alpha_0)} = \mathrm{i}P(\alpha) F(\alpha) F(\alpha_0)^*; \quad (37)$$

because $F(\alpha_0)^*$ is invertible it follows that

$$\partial_\alpha F(\alpha) = \mathrm{i}P(\alpha) F(\alpha).$$

Conversely, $\partial_\alpha F(\alpha)$ determines $\partial_\alpha C^Y_{(\alpha,\alpha_0)}$. Likewise, $\partial_\alpha F(\alpha)^*$ is determined by $\partial_\alpha C^X_{(\alpha,\alpha_0)}$ and vice versa, with

$$\partial_\alpha F(\alpha)^* = \mathrm{i}\widetilde{P}(\alpha) F(\alpha)^*; \quad (38)$$

we have

$$F(\alpha) \widetilde{P}(\alpha) = -P(\alpha) F(\alpha). \quad (39)$$

Derivatives at α_0 can be found by means of perturbation theory, yielding $F(\alpha_0 + \mathit{\Delta})^*\, F(\alpha_0)$ and $F(\alpha_0 + \mathit{\Delta})\, F(\alpha_0)^*$ for small $\mathit{\Delta}$. This can be carried over to (36) to determine, infinitesimally, the propagation of singularities under continuation.

Remark. From equation (37) it follows that perturbation of the image continuation operator $C^Y_{(\alpha,\alpha_0)}$ is completely determined by perturbation of migration operators $F(\alpha)^*$; we use this observation in the next section.

5 Phase functions, generating functions, and continuation Hamiltonians revisited

Here, we connect the phase functions of $F(\alpha)^*$ and the generating functions for Λ^*_α to the pseudodifferential operator in the evolution equation (cf. (20)) generating the image continuation operator and the Hamiltonian (cf. (26)) generating the continuation bicharacteristics. Earlier, in Section 3.2, we discussed how to obtain the principal part of the pseudodifferential operator symbol from geometrical considerations. We will also discuss how the points on an evolving front (Section 2.1; here, an isochron) follow the continuation characteristics.

Since the canonical relation of $F(\alpha)^*$ is a graph, it admits coordinates (x, η) and a generating function $\widetilde{S} = \widetilde{S}(\alpha; x, \eta)$ (cf. (12)-(13)). The kernel of $F(\alpha)^*$ then admits the representation

$$A_{F(\alpha)^*}(x, y) = \int \widetilde{a}(\alpha; x, \eta) \exp[\mathrm{i}\widetilde{\phi}(\alpha; x, y, \eta)] \,\mathrm{d}\eta, \quad \widetilde{\phi}(\alpha; x, y, \eta) = \widetilde{S}(\alpha; x, \eta) - \langle \eta, y \rangle. \tag{40}$$

With this kernel representation, we can replace (36) by:

$$\begin{aligned} \partial_\eta \, \widetilde{\phi}(\alpha; x, y, \eta) = 0, \quad \partial_{\eta'} \widetilde{\phi}(\alpha_0; x_0, y, \eta') = 0, \\ \partial_y[\widetilde{\phi}(\alpha; x, y, \eta) - \widetilde{\phi}(\alpha_0; x_0, y, \eta')] = 0. \end{aligned} \tag{41}$$

We eliminate the bottom equation, and substitute its solution, $\eta' = \eta$, in the top equations, that is, $\partial_\eta[\widetilde{S}(\alpha; x, \eta)] = y = \partial_\eta[\widetilde{S}(\alpha; x_0, \eta)]$, whence

$$\partial_\eta[\widetilde{S}(\alpha; x, \eta) - \widetilde{S}(\alpha_0; x_0, \eta)] = 0. \tag{42}$$

With $x(\alpha)$ denoting a continuation characteristic as before (cf. Section 3.2), while perturbing α about α_0 and $x = x(\alpha)$ about $x_0 = x(\alpha_0)$, it follows that

$$\partial_\alpha \partial_\eta \widetilde{S}(\alpha; x, \eta) + \frac{\mathrm{d}x}{\mathrm{d}\alpha} \cdot [\partial_x \partial_\eta \widetilde{S}(\alpha; x, \eta)] = 0, \quad x = x(\alpha). \tag{43}$$

Because $\partial_x \partial_\eta \widetilde{S}$ is non-singular, this is a system of n_X equations that provides a solution for $\frac{\mathrm{d}x}{\mathrm{d}\alpha}$ for each η $(= -\partial_y \widetilde{\phi})$. With initial condition, $x(\alpha_0, y_0, \eta_0) = x_0$, it holds true that $\widetilde{S}(\alpha_0; x_0, \eta_0) - \langle \eta_0, y_0 \rangle = 0$.

To leading order, the kernel of $\partial_\alpha F(\alpha)^*$ has the representation (cf. (22))

$$\mathrm{i} \int \partial_\alpha \widetilde{S}(\alpha; x, \eta) \, \widetilde{a}(\alpha; x, \eta) \exp[\mathrm{i}\widetilde{\phi}(\alpha; x, y, \eta)] \,\mathrm{d}\eta. \tag{44}$$

We introduce the change of coordinates, $(x, \eta) \to (x, \xi)$, by solving the equation

$$\xi = \partial_x \widetilde{S}(\alpha; x, \eta), \tag{45}$$

for $\eta = \eta(\alpha; x, \xi)$. The principal symbol of pseudodifferential $\widetilde{P}(\alpha)$ then follows to be

$$\widetilde{p}_1(\alpha, x, \xi) = \partial_\alpha \widetilde{S}(\alpha; x, \eta(\alpha; x, \xi)). \tag{46}$$

Since $\widetilde{S}$ is homogeneous of degree 1 in η, $\widetilde{p}_1$ is a symbol of order 1. Indeed, applying the composition rule for a pseudodifferential operator with an FIO [46, Ch. VIII, p.465],

$$\widetilde{P}(\alpha, x, D_x) A_{F(\alpha)^*}(x, y) = \int \widetilde{p}(\alpha, x, \partial_x \widetilde{S}(\alpha; x, \eta))\, \widetilde{a}(\alpha; x, \eta) \exp[\mathrm{i}\widetilde{\phi}(\alpha; x, y, \eta)]\, \mathrm{d}\eta, \tag{47}$$

and, with the property

$$\partial_\alpha \widetilde{S}(\alpha; x, \eta(\alpha; x, \partial_x \widetilde{S}(\alpha; x, \eta))) = \partial_\alpha \widetilde{S}(\alpha; x, \eta),$$

we recover (44), to leading order.

The continuation bicharacteristics are the solution to Hamilton system (27), which, with (46), attains the form

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -[\partial_\xi \eta(\alpha; x, \xi)] \cdot (\partial_\eta \partial_\alpha \widetilde{S})(\alpha; x, \eta(\alpha; x, \xi)), \tag{48}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}\alpha} = \partial_x \partial_\alpha \widetilde{S}(\alpha; x, \eta(\alpha; x, \xi)) + [\partial_x \eta(\alpha; x, \xi)] \cdot (\partial_\eta \partial_\alpha \widetilde{S})(\alpha; x, \eta(\alpha; x, \xi)). \tag{49}$$

It is straightforward to verify that the solution to (43) coincides with (48):

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -[\partial_\eta \partial_x \widetilde{S}]^{-1} \cdot (\partial_\alpha \partial_\eta \widetilde{S}) = -[\partial_\eta \xi]^{-1} \cdot (\partial_\eta \partial_\alpha \widetilde{S}), \quad \text{using that } \xi = \partial_\xi \widetilde{S}. \tag{50}$$

6 Examples

Here, we connect some known procedures for continuation to the general framework developed in this paper. In particular, we discuss the velocity continuation of images and isochrons in common-offset Kirchhoff migration [1,25], and the continuation of offset image gathers [28,30] in the absence of caustics. We arrive at a general formulation leading to a new Hamiltonian for the latter type of continuation. By introducing the amplitude in the kernel representation for common-offset Kirchhoff migration, one could also obtain the full evolution operator. A family of migration-demigration operators $F(\alpha)^*$, $F(\alpha)$ is defined by a smooth family of background velocities $v[\alpha]$. In the case of constant velocity, $v[\alpha] \equiv v = \text{const}$ and v itself plays a role of α.

In Kirchhoff migration, we consider $F(\alpha)^*$ transforming data $u(y)$ into an image $w(x)$. We assume the absence of caustics. On the acquisition manifold we introduce the following coordinates: $y = (t, y')$, where t is the time, $y' = (r + s)/2$ is the source-receiver mid-point, and $h = (r - s)/2$ is half-offset; s indicates a source position and r indicates a receiver position, see Fig. 1. In common offset [4] migration we consider h to be a set of (constant) parameters so that $F(\alpha)^* : \ u(t, y') \to w(x)$.

A phase function $\widetilde{\phi}$ for the oscillatory integral representation of the kernel of operator $F(\alpha)^*$ can be chosen of the form

$$\begin{aligned} &\widetilde{\phi}(\alpha; x, t, y', \tau) = -\phi(\alpha; t, y', x, \tau) = \phi(\alpha; t, y', x, -\tau), \\ &\text{with} \quad \phi(\alpha; t, y', x, \tau) = \tau\,(T(\alpha; y', x) - t), \end{aligned} \tag{51}$$

in which $T(\alpha; y', x)$ denotes the 'two-way' travel time along a broken ray connecting a receiver at r to a source at s (defining y') via the scattering point x, that is,

$$T(\alpha; y', x) = t_s(\alpha; y' - h, x) + t_r(\alpha; y' + h, x), \tag{52}$$

if t_s denotes the travel time along a source ray (connecting $s = y' - h$ with x) and t_r denotes the travel time along a receiver ray (connecting $r = y' + h$ with x), see Fig. 1. In the above, ϕ is the phase function for demigration operator $F(\alpha)$; τ is the only phase variable (cf. (9)).

The propagation of singularities by $F(\alpha)^* F(\alpha_0)$, illustrated in Fig. 1, is then derived from the equations determining the stationary point set of $\widetilde{\phi}(\alpha; x, t, y', \tau) + \phi(\alpha_0; t, y', x_0, \tau')$ in the composition (in analogy with (36) or (41)):

$$\partial_\tau[\widetilde{\phi}(\alpha; x, t, y', \tau) + \phi(\alpha_0; t, y', x_0, \tau)] = T(\alpha_0; y', x_0) - T(\alpha; y', x) = 0, \tag{53}$$

$$\tau^{-1}\partial_{y'}[\widetilde{\phi}(\alpha; x, t, y', \tau) + \phi(\alpha_0; t, y', x_0, \tau)] = \partial_{y'} T(\alpha_0; y', x_0) - \partial_{y'} T(\alpha; y', x) = 0. \tag{54}$$

(We substituted the solution to the omitted equation, $\partial_t[\widetilde{\phi}(\alpha; x, t, y', \tau) + \phi(\alpha_0; t, y', x_0, \tau')] = 0$, that is, $\tau' = \tau$.) Equation (54) arises naturally in continuation theory through the composition calculus of FIOs, but does not always appear in application specific approaches; often, (53) alone is used as the point of departure, while an additional equation is introduced on the basis of ad hoc assumptions. We will review these assumptions in the context of the framework presented here, in the next subsection.

[4] In three-dimensional configurations, offset is a two-dimensional vector; representing this vector in polar coordinates, one refers to the angular coordinate as azimuth.

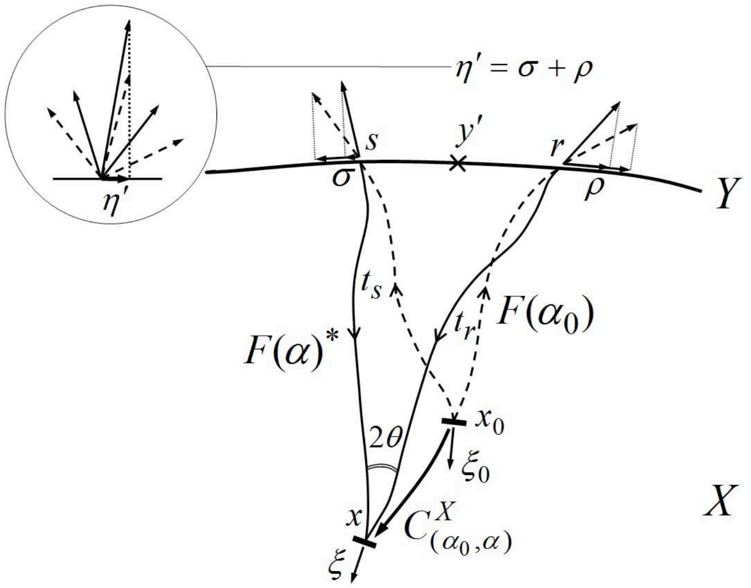

Fig. 1. Continuation principle and the composition $C^X_{(\alpha,\alpha_0)} = F(\alpha)^* F(\alpha_0)$ in the case of common-offset migration. The source position is indicated by s, the receiver position is indicated by r, and the scattering point by x; t_s represents travel time along a source ray, and t_r represents travel time along a receiver ray.

Repeating the reasoning from (41) to (43) but with different phase variables (namely τ) leads to

$$\partial_{(\tau,y')} \left(\tau \partial_\alpha T(\alpha; y', x)\right) + \frac{\mathrm{d}x}{\mathrm{d}\alpha} \cdot \left[\partial_x \partial_{(\tau,y')} \left(\tau T(\alpha; y', x)\right)\right] = 0, \tag{55}$$

see [1, (C-4)]. The quantities $\partial_{(\tau,y')}\partial_\alpha(\tau T)$ and $[\partial_x \partial_{(\tau,y')}(\tau T)]$ can be obtained by methods of ray tracing, dynamic ray tracing, and ray perturbation. The matrix, $[\partial_{(\tau,y')}\partial_x(\tau T)]$, using that $\xi = \partial_x(\tau T)$, has a determinant which appears in 'true-amplitude' common-offset imaging based on the generalized Radon transform and has been attributed to Beylkin [5, p.223]. It is non-singular, whence (55) can be solved, providing:

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = \left[\partial_{(\tau,y')}\partial_x \left(\tau T(\alpha; y', x)\right)\right]^{-1} \cdot \partial_{(\tau,y')} \left(\tau \partial_\alpha T(\alpha; y', x)\right). \tag{56}$$

This equation is the analogue of (50). However, unlike (50), it does not aid in constructing a global Hamiltonian or an explicit evolution equation.

Remark. We can obtain $\widetilde{S}$ explicitly using the formalism developed in Section 5, leading to the principal part of the evolution operator. To this end, we apply Appendix B to the generating function $S'(\alpha; x, y', \tau) = \tau\, T(\alpha; y', x)$, to obtain $\widetilde{S}(\alpha; x, \eta) = (\tau\, T(\alpha; y', x) + \langle \eta', y' \rangle)|_{y'=y'(\alpha;x,\eta)}$ (cf. (40)) with $\eta = (\tau', \eta')$; here, the stationary points $y'(\alpha; x, \eta)$ are found from the equations

$$\tau\, \partial_{y'} T(\alpha; y', x) = -\eta. \tag{57}$$

After substituting the solution of (57) into (45) we find that $\eta' = \eta'(\alpha; x, \xi)$ and $\tau = \tau(\alpha; x, \xi)$ solve

$$\begin{aligned}\xi &= \partial_x \widetilde{S}(\alpha; x, \eta) \\ &= \tau\, \partial_x T(\alpha; y'(\alpha; x, \eta), x) + \tau\, \partial_x y'(\alpha; x, \eta) \cdot \partial_{y'} T(\alpha; y'(\alpha; x, \eta), x) + \partial_x y'(\alpha; x, \eta) \cdot \eta' \\ &= \tau\, \partial_x T(\alpha; y'(\alpha; x, \eta), x). \end{aligned} \tag{58}$$

One can combine equations (57) and (58) and solve directly

$$\tau\, \partial_x T(\alpha; y', x) = \xi \tag{59}$$

for $y'(\alpha; x, \xi)$ and $\tau(\alpha; x, \xi)$. The Hamiltonian for continuation bicharacteristics, using (46), the follows to be

$$\mathcal{H}(\alpha, x, \xi_\alpha, \xi) = \xi_\alpha - \underbrace{\tau(\alpha; x, \xi)\, \partial_\alpha T(\alpha; y'(\alpha; x, \xi), x)}_{=\widetilde{p}_1(\alpha, x, \xi)}. \tag{60}$$

We will compute this Hamiltonian in the case of constant background velocities in the next subsection. We note that along continuation bicharacteristics, $\xi_\alpha = \tau(\alpha; x, \xi)\, \partial_\alpha T(\alpha; y'(\alpha; x, \xi), x)$.

A similar procedure of changing phase variables was followed in the development of map migration using curvelets [10].

6.2 'Velocity rays' as curves connecting evolving isochrons

Isochrons, generated by $F(\alpha)^*$, are given by

$$\mathcal{W}(\alpha; t, y') = \{x \in X \mid T(\alpha; y', x) = t\},$$

see Section 2.1. 'Velocity rays' were introduced in the literature as curves connecting isochrons evolving with α. Such curves are written as $x(\alpha)$, and must then satisfy

$$\partial_\alpha T = -\partial_x T \cdot \frac{\mathrm{d}x}{\mathrm{d}\alpha} \tag{61}$$

(keeping (t, y') fixed); this is also the ∂_τ component of equation (55).

Since, for now, we leave out the $\partial_{y'}$ component of (55), we will have to supplement this equation with another equation for $\frac{\mathrm{d}x}{\mathrm{d}\alpha}$ to be determined. This implies that the continuation will no longer be a composition of the type introduced in Section 4. In this subsection, we discuss four different supplementary equations from the literature, each leading to a notion of 'velocity rays'.

To illustrate the different notions of 'velocity rays', we consider the special case of constant background media. We remind that with constant background velocities, v plays the role of α.

We introduce the vertical coordinate z; y' and h lie in a horizontal plane ($z = 0$). The points x now have coordinates (x, z). We get (cf. (52))

$$t_s(v; y' - h, x, z) = \frac{\rho_s(y', x, z)}{v}, \;\; t_r(v; y' + h, x, z) = \frac{\rho_r(y', x, z)}{v},$$
$$\rho_s(y', x, z) = \sqrt{(x - y' + h)^2 + z^2}, \;\; \rho_r(y', x, z) = \sqrt{(x - y' - h)^2 + z^2}. \quad (62)$$

The coordinates, and three, evolving, isochrons (half ellipses, in this case) are shown in Fig. 2. By differentiating (52) with (62) with respect to v, we obtain (61) for the constant background media case:

$$-\frac{\rho_s + \rho_r}{v^2} + \left(\frac{\partial_x \rho_s + \partial_x \rho_r}{v}\right) \frac{\mathrm{d}x}{\mathrm{d}v} + \left(\frac{\partial_z \rho_s + \partial_z \rho_r}{v}\right) \frac{\mathrm{d}z}{\mathrm{d}v} = 0, \quad (63)$$

where simply

$$\partial_x \rho_s = \frac{x - y' + h}{\rho_s}, \quad \partial_z \rho_s = \frac{z}{\rho_s} \quad (64)$$

and similarly for $\partial_x \rho_r$ and $\partial_z \rho_r$.

1. Liu and Bleistein [28]: vertical 'ray'. The authors assume that the curves that connect an initial with a perturbed isochron are vertical: $\frac{\mathrm{d}x}{\mathrm{d}\alpha} = 0$; in constant background media,

$$\frac{\mathrm{d}x}{\mathrm{d}v} = 0. \quad (65)$$

Solving equations (63) and (65), one obtains [28, (13)]

$$\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{\rho_s \rho_r}{v z}. \quad (66)$$

The corresponding curves are illustrated, and indexed by 1, in Fig. 2.

2. Iversen [25]: source-'ray' parametrization. Iversen defines a 'velocity ray' as the curve connecting an initial with a perturbed isochron, subject to the condition

$$[\partial_\alpha (v[\alpha](s, 0)\, \partial_s t_s(\alpha; s, x, z)) + v[\alpha](s, 0)\, \frac{\mathrm{d}(x, z)}{\mathrm{d}\alpha} \cdot \partial_{(x,z)} \partial_s t_s(\alpha; s, x, z)]_{s=y'-h} = 0,$$
$$(x, z) = (x, z)(\alpha). \quad (67)$$

(This equation arises from the composition-like relation

$$v[\alpha](s, 0)\, \partial_s t_s(\alpha; s, x, z) - v[\alpha_0](s, 0)\, \partial_s t_s(\alpha_0; s, x_0, z_0) = 0, \quad s = y' - h.)$$

For the constant velocity models, with t_s as in (62), we thus obtain the supplementary equation

$$\frac{\mathrm{d}x}{\mathrm{d}v} - \tan\beta \frac{\mathrm{d}z}{\mathrm{d}v} = 0, \quad \tan\beta = \frac{x - y' + h}{z}. \quad (68)$$

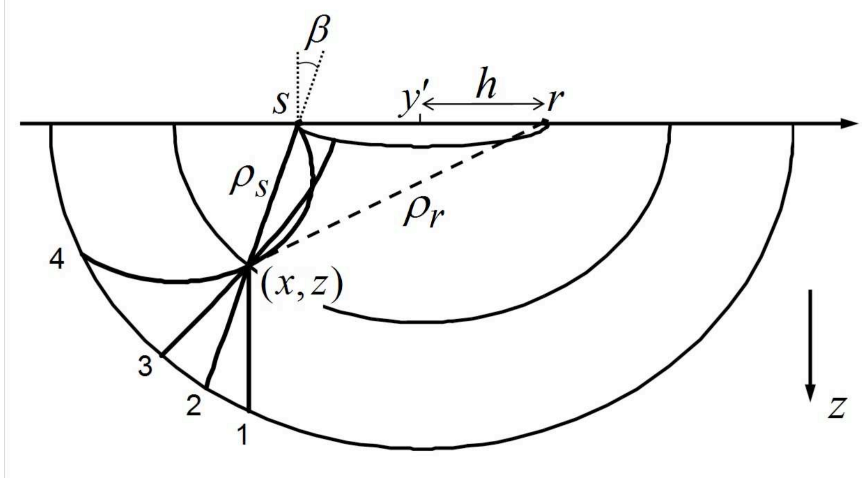

Fig. 2. Three isochrons for fixed $(t, y', h) = (2, 0, 0.5)$ and different velocities, $v = 0.51, 1.0$ and 1.5. The different 'velocity rays' are indexed: 1 – vertical ray, 2 – source ray, 3 – isochron-normal ray, and 4 – canonical ray (continuation characteristic).

Equations (63) and (68) can be solved for $\frac{\mathrm{d}(x,z)}{\mathrm{d}v}$; the corresponding curves are illustrated, and indexed by 2, in Fig. 2.

3. Meng and Bleistein [31]: isochron-normal 'ray'. The authors define a 'velocity ray' as the curve connecting an initial with a perturbed isochron, with the provision that the curve is normal to the (initial) isochron:

$$\partial_{(x,z)} T(\alpha; y', x, z) \wedge \frac{\mathrm{d}(x,z)}{\mathrm{d}\alpha} = 0. \tag{69}$$

We introduce the isochron-normal vector, $n = \partial_{(x,z)} T / \|\partial_{(x,z)} T\|$, and half of opening angle between incident and reflected rays, θ (Fig. 1), so that $\|\partial_{(x,z)} T\| = \frac{2\cos\theta}{v[\alpha](x,z)}$; $v[\alpha] \in C^\infty$ describes a curve of velocity models. The isochrone-normal 'rays' have the property that

$$\left(n \cdot \frac{\mathrm{d}(x,z)}{\mathrm{d}\alpha} \right) n = \frac{\mathrm{d}(x,z)}{\mathrm{d}\alpha}.$$

From equation (61) it then follows that

$$n \cdot \frac{\mathrm{d}(x,z)}{\mathrm{d}\alpha} = -\partial_\alpha T \, \frac{v[\alpha]}{2\cos\theta} \tag{70}$$

[30, (4.4.24), (4.4.25)] is the velocity of an isochron-normal 'ray'. We note that the quantity $n \cdot \frac{\mathrm{d}(x,z)}{\mathrm{d}\alpha}$ in the above is the same for all notions of 'velocity rays' (and continuation characteristics).

For the constant velocity models, with T as in (52), (62), we thus obtain the supplementary equation

$$\partial_z T(v; y', x, z) \frac{\mathrm{d}x}{\mathrm{d}v} - \partial_x T(v; y', x, z) \frac{\mathrm{d}z}{\mathrm{d}v} = 0. \tag{71}$$

Equations (63) and (71) can be solved for $\frac{\mathrm{d}(x,z)}{\mathrm{d}v}$; the corresponding curves are illustrated, and indexed by 3, in Fig. 2.

4. Adler [1]: canonical 'ray' (continuation characteristic). Adler honors all components of equation (55); the $\partial_{y'}$ component of this equation in the constant background media case yields the equation,

$$-\frac{\partial_{y'}\rho_s + \partial_{y'}\rho_r}{v^2} + \frac{\partial_x\partial_{y'}\rho_s + \partial_x\partial_{y'}\rho_r}{v} \frac{\mathrm{d}x}{\mathrm{d}v} + \frac{\partial_z\partial_{y'}\rho_s + \partial_z\partial_{y'}\rho_r}{v} \frac{\mathrm{d}z}{\mathrm{d}v} = 0, \tag{72}$$

supplementary to (63). Equations (63) and (72) can be solved for $\frac{\mathrm{d}(x,z)}{\mathrm{d}v}$; the corresponding curves are illustrated, and indexed by 4, in Fig. 2. They can be related to the so-called combined-ray parametrization of velocity rays in [25,27]; see also the Appendices in [1].

Remark. We verify which of the velocity 'rays' are actually rays, by computing their respective Lie derivatives $\mathcal{L}_{V_\alpha}\,\omega = 0$, thus checking integrability conditions (32). The construction above led to expressions for $\frac{\mathrm{d}x}{\mathrm{d}\alpha}$ in terms of coordinates (x, τ, y'). Here, we introduce a procedure to, consistently, construct $\frac{\mathrm{d}\xi}{\mathrm{d}\alpha}$. We consider heterogeneous media, but assume the absence of caustics as in Section 6.1.

(i) Equation (59) must hold, providing a change of coordinates, $\tau = \tau(\alpha; x, \xi)$, $y' = y'(\alpha; x, \xi)$, for each value of α. (The constant background media case is treated explicitly in the next subsection.)

(ii) We consider any of the velocity 'rays' written in the form,

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = f(\alpha; x, y'). \tag{73}$$

By substituting the transformation $y' = y'(\alpha; x, \xi)$ from (i), we obtain $V_1(\alpha; x, \xi) = \frac{\mathrm{d}x}{\mathrm{d}\alpha} = f(\alpha; x, y'(\alpha; x, \xi))$ which can be used for checking integrability.

(iii) Equation (59) must hold along $x = x(\alpha)$, providing $\xi(\alpha) = \tau\partial_x T(\alpha; y', x(\alpha))$. The latter expression can be differentiated with respect to α (keeping τ and y' fixed):

$$\frac{\mathrm{d}\xi}{\mathrm{d}\alpha} = g(\alpha; x, \tau, y') = \tau\partial_\alpha\partial_x T(\alpha; y', x) + \tau\frac{\mathrm{d}x}{\mathrm{d}\alpha} \cdot \partial_x\partial_x T(\alpha; y', x). \tag{74}$$

After substituting $\tau = \tau(\alpha; x, \xi)$ and $y' = y'(\alpha; x, \xi)$ from (i) into (74), we obtain $V_2(\alpha; x, \xi) = \frac{\mathrm{d}\xi}{\mathrm{d}\alpha} = g(\alpha; x, \tau(\alpha; x, \xi), y'(\alpha; x, \xi))$, which can be used for checking integrability.

With expressions for $(V_1(\alpha; x, \xi), V_2(\alpha; x, \xi))$ obtained under (ii) and (iii), we can test conditions (32): For example, the 'source-ray' parameterized velocity ray fails this test and hence no global Hamiltonian can be found; of course, the canonical ray passes this test. We argue that, hence, 'velocity rays' other than the canonical rays (continuation characteristics) should perhaps not be called rays.

6.3 Global Hamiltonian for constant velocity continuation

Only 'velocity rays' corresponding with canonical transformations yield the appropriate geometry underlying an evolution-equation based approach to image continuation. Following the Remark in Section 6.1, we, here, construct the global Hamiltonian for velocity continuation (constant velocity models). Equations (57) and (58) take the following form (see (62) for the definitions of ρ_r, ρ_s)

$$-\frac{(x-y'-h)}{v\rho_r} - \frac{(x-y'+h)}{v\rho_s} = -\frac{\eta'}{\tau}, \tag{75}$$

$$\frac{(x-y'-h)}{v\rho_r} + \frac{(x-y'+h)}{v\rho_s} = \frac{\xi}{\tau}, \tag{76}$$

$$\frac{z}{v\rho_r} + \frac{z}{v\rho_s} = \frac{\zeta}{\tau}, \tag{77}$$

were (ξ, ζ) are variables dual to (x, z). Equations (75)-(76) imply that $\eta' = \xi$. Eliminating τ from equations (76)-(77) results in

$$y'(x, z, \xi, \zeta) = x + \frac{z(\zeta^2 - \xi^2) - \sqrt{(2h\xi\zeta)^2 + z^2(\xi^2 + \zeta^2)^2}}{2\xi\zeta}, \tag{78}$$

being independent of v. Following (46), we have $\widetilde{p} = \tau\partial_v T = -\frac{\tau}{v^2}(\rho_r + \rho_s)$, so that, with (77),

$$\begin{aligned}\widetilde{p}(v, x, z, \xi, \zeta) &= -\left.\frac{\zeta}{zv}\rho_r\rho_s\right|_{y'=y'(x,z,\xi,\zeta)} \\ &= -\frac{\xi^2+\zeta^2}{2v\xi^2\zeta}\sqrt{(2h\xi\zeta)^2 + 2z^2(\xi^4+\zeta^4) + 2z(\xi^2-\zeta^2)\sqrt{(2h\xi\zeta)^2 + z^2(\xi^2+\zeta^2)^2}},\end{aligned} \tag{79}$$

yielding the symbol of the evolution operator. We can further simplify (79) and obtain the global Hamiltonian for velocity continuation (cf. (60); an alternative form of this Hamiltonian was derived by Fomel [11])

$$\mathcal{H}(v, x, z, \xi_v, \xi, \zeta) = \xi_v + \frac{\xi^2+\zeta^2}{2v\xi^2\zeta}\left[z(\xi^2-\zeta^2) + \sqrt{(2h\xi\zeta)^2 + z^2(\xi^2+\zeta^2)^2}\right]. \tag{80}$$

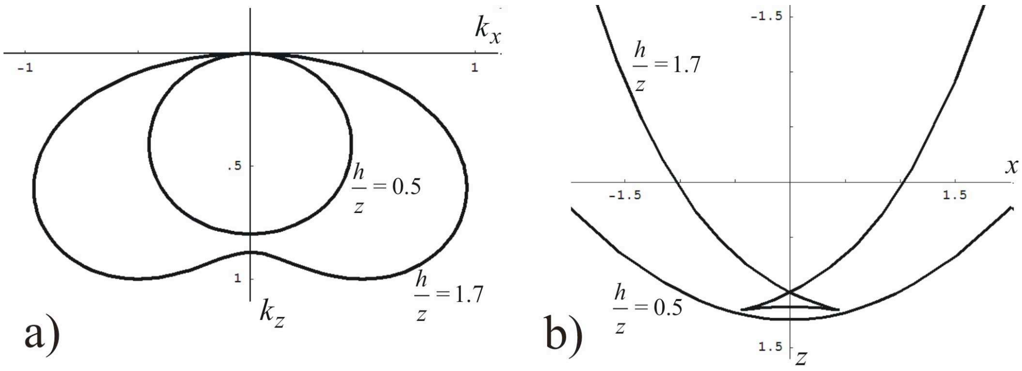

Fig. 3. (a) – Slowness surface for small offset ($h/z = 0.5$) and large offset ($h/z = 1.7$); (b) – group velocity surface for small offset ($h/z = 0.5$) and large offset ($h/z = 1.7$).

Remark. On the slowness surface associated with the Hamiltonian in (80), we have that if $\xi_v > 0$ then $\zeta < 0$ (we note that the expression in square brackets in (80) is always non-negative, so that the sign of the second term is controlled by the sign of ζ), and, hence, if $\xi_v < 0$ then $\zeta > 0$. The slowness surface ($\mathcal{H} = 0$ for given (x, z)) is depicted in Fig. 3 a) (where we introduce the normalized vector components, $k_x = \xi/\xi_v$ and $k_z = \zeta/\xi_v$), while the group velocities (cf. (27) or (48)) are shown in Fig. 3 b). We consider two cases: Small offset ($h/z = 0.5$) and large offset ($h/z = 1.7$), while setting $z = 1$ and $v = 1$ in the computation. For small offsets, the slowness surface approaches a circle and the group velocity surface approaches a parabola. For large offsets, the slowness surface develops inflection points, leading to cusps in the group velocity surface. Note that the group velocity surface corresponds to an 'instantaneous front' generated at a point in the initial image; see [15,17].

Making use of the global Hamiltonian (80), we illustrate common-offset image continuation and the notion of continuation characteristics. In Fig. 4 (left) we show continuation characteristics calculated for a segment (in bold) of a planar (line) reflector. The initial (correct) common-offset image corresponds to a background velocity $v = 1$ km/s; common-offset migrations for different values of h will produce the same image. Continuation characteristics (thin lines) take off from the original image and terminate at an image for $v = 1.3$ km/s (straight line segment to the left) and an image for $v = 0.5$ km/s (straight line segment to the right). Thin solid lines represent continuation characteristics corresponding to offset $h = 0.1$ km, and dashed thin lines represent continuation characteristics corresponding to $h = 0.7$ km. In Fig. 4 (right) the reflector, and the initial (correct) common-offset image, are parabolic. The correct (and initial) background velocity is $v = 1$ km/s; the image is continued to $v = 1.06$ km/s. Even in this simple model, we observe the formation of caustics.

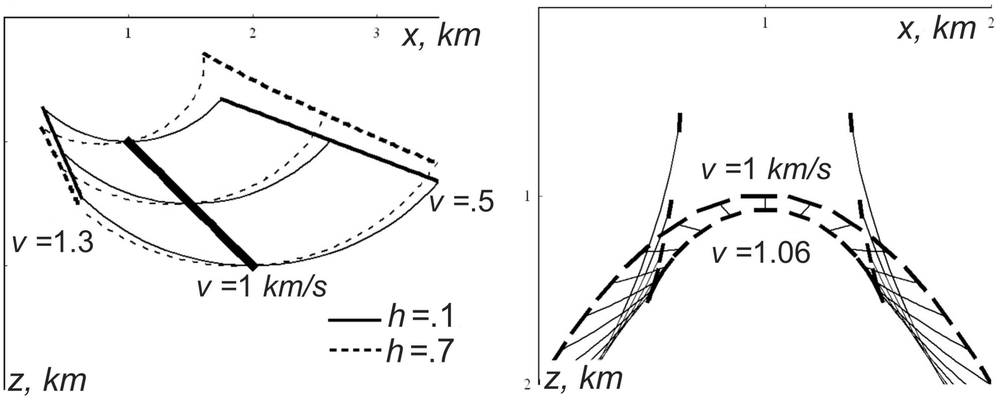

Fig. 4. Common-offset image continuation and continuation characteristics. Left: plane reflector (solid line segment); thin solid lines indicate continuation characteristics for $h = 0.1$ km, while thin dashed lines indicate continuation characteristics for $h = 0.7$ km. Right: parabolic reflector; thin solid lines indicate continuation characteristics.

7 Velocity continuation of common-image point gathers in the presence of caustics

Here, we apply the presented theory to the problem of velocity continuation of so-called common-image point gathers in the presence of caustics. The formation of such gathers is explained below. In the presence of caustics, the framework of common-offset migration no longer applies, and we resort to an alternative invertible transformation. With data $u = u(s, r, t)$ (identifying (s, r, t) as coordinates for y), common-image point gathers are now formed as follows. We will have $n_X = n_Y = 3$ for two-dimensional configurations. Let G denote the causal Green's function of the scalar wave equation, that is,

$$\begin{aligned} [v^{-2}(x)\partial_t^2 - \partial_x^2 - \partial_z^2]\, G(x, z, t, x', z') = \delta(x - x')\delta(z - z')\delta(t), \\ G(x, z, t, x', z') = 0,\ t \ll 0. \end{aligned} \tag{81}$$

We then introduce [37,40,9]

$$\begin{aligned} &\mathsf{D}(x - h_x, x + h_x, z - h_z, z + h_z, t') = \\ &\iiiint G(x+h_x, z+h_z, -(\widetilde{t}-t), r, 0) G(x-h_x, z-h_z, \widetilde{t}-t', s, 0)\, \mathrm{d}\widetilde{t}\ \partial_t^2 u(s, r, t)\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}t. \end{aligned} \tag{82}$$

We have the freedom of choosing the direction of (h_x, h_z) [41]. For the case of non-horizontal wave propagation and non-vertical reflectors, a natural choice is $h_z = 0$ (leading to the downward continuation approach to imaging [40]). In the case of near vertical reflectors, we choose

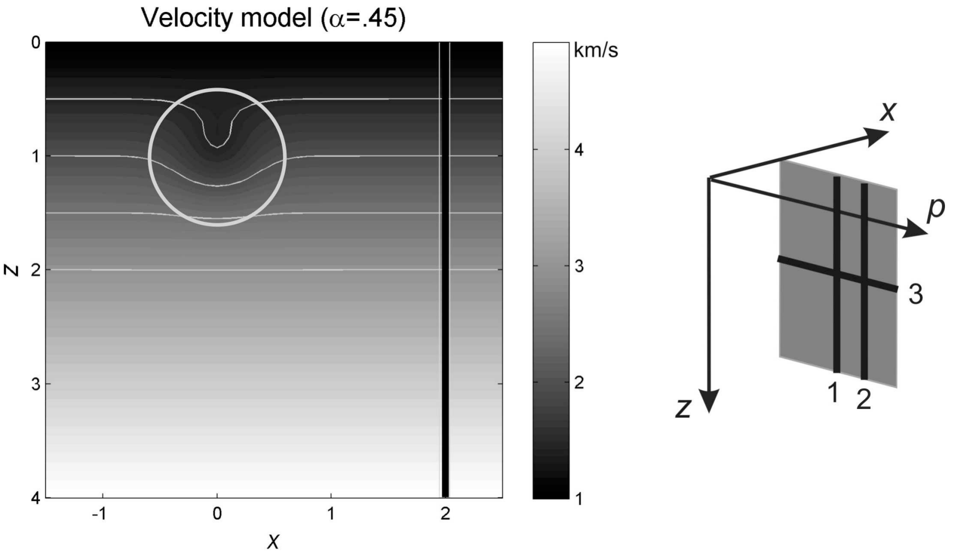

Fig. 5. Left: Background velocity model including a low velocity lens, and vertical reflector (which can be thought of as a toy model for the flank of a salt dome). Right: A schematic view of the (x, z, p) box on which $w_g(x, z, p)$ is defined; the singular support of the correct image – corresponding with the vertical reflector – in the (x, z, p) box, is indicated by a gray plane. Lines 1 ($x = 2$, $p = 0$) and 2 ($x = 2$, $p = 0.1$) in the gray plane indicate the restrictions to which $w_g(x, z, p)$ is subjected in Fig. 9. Line 3 is to be identified with a line or 'string' in a 'vertical' common-image point gather (for x fixed), used to illustrate continuation in Fig. 10.

$h_x = 0$. We then form an image gather according to

$$w_g(x, z, p) = \int \mathsf{D}(x, x, z - h_z, z + h_z, 2ph_z)\chi(x, z, h_z)\, \mathrm{d}h_z, \tag{83}$$

where p is a variable related to the scattering (opening) angle at point (x, z), and $\chi(x, z, h_z)$ is a cutoff in h_z. (This type of transform was introduced in [6]; here, (x, z, p) are coordinates defining x in Sections 2 and 3.) We arrive at the so-called angle transform [37,40], $\mathcal{A}_{we} : u(s, r, t) \to w_g(x, z, p)$. It can be shown that the operator $\mathcal{A}_{we}$ is microlocally invertible, given a proper choice of χ. Thus we can use $\mathcal{A}_{we}$ and its inverse as the basis for velocity continuation of common-image point gathers.

We present an example, making use of a background velocity model containing a vertical gradient and a low velocity lens:

$$v[\alpha] = 1 + z - \alpha \exp[-7.5(x^2 + (1 - z)^2)], \tag{84}$$

where α defines the 'strength' of the lens. We take $\alpha = 0.45$ as the true model (see Fig. 5 left) and use it to construct rays and calculate travel times (and the wavefront set of the data, u). In Fig. 6 we show incident (a)) and reflected (b)) rays for a single point source. As expected, we

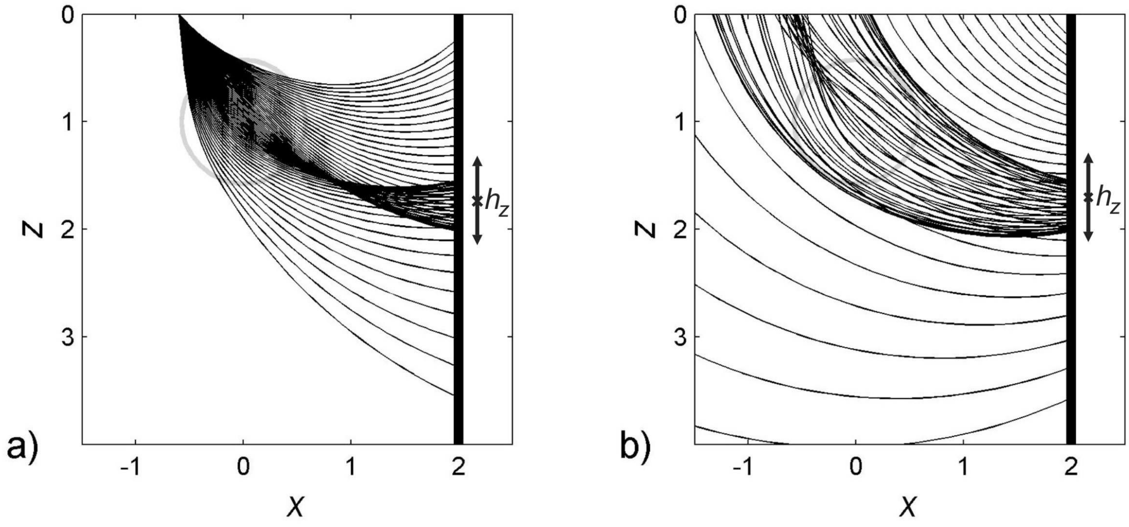

Fig. 6. Incident (a) and reflected (b) rays for a single point source, computed in the model shown in Fig. 5 left. The gray circle indicates the location of the lens.

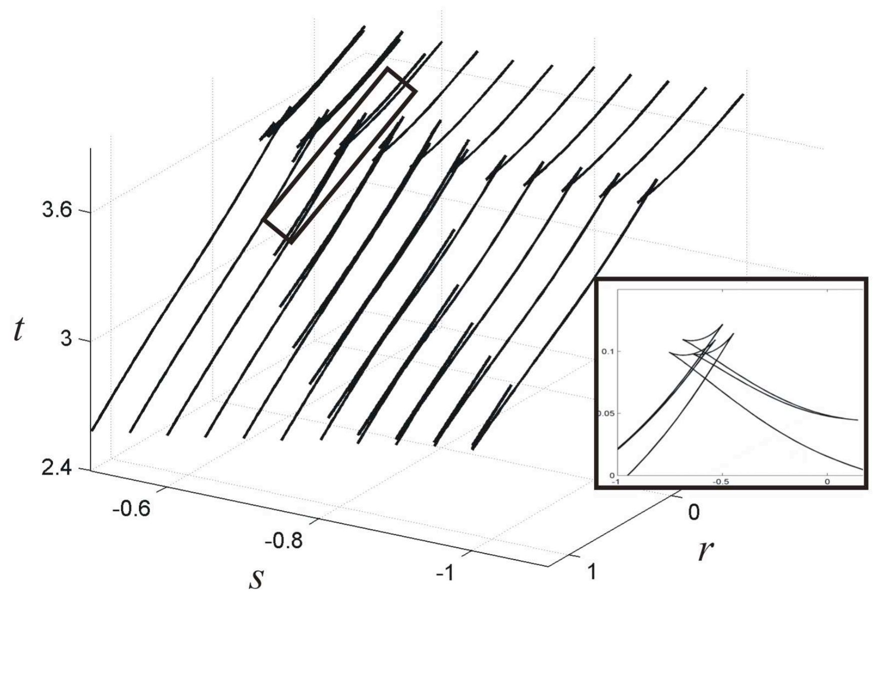

Fig. 7. Two-way travel time curves for several sources, and a blow up (insert) for the source at -0.6.

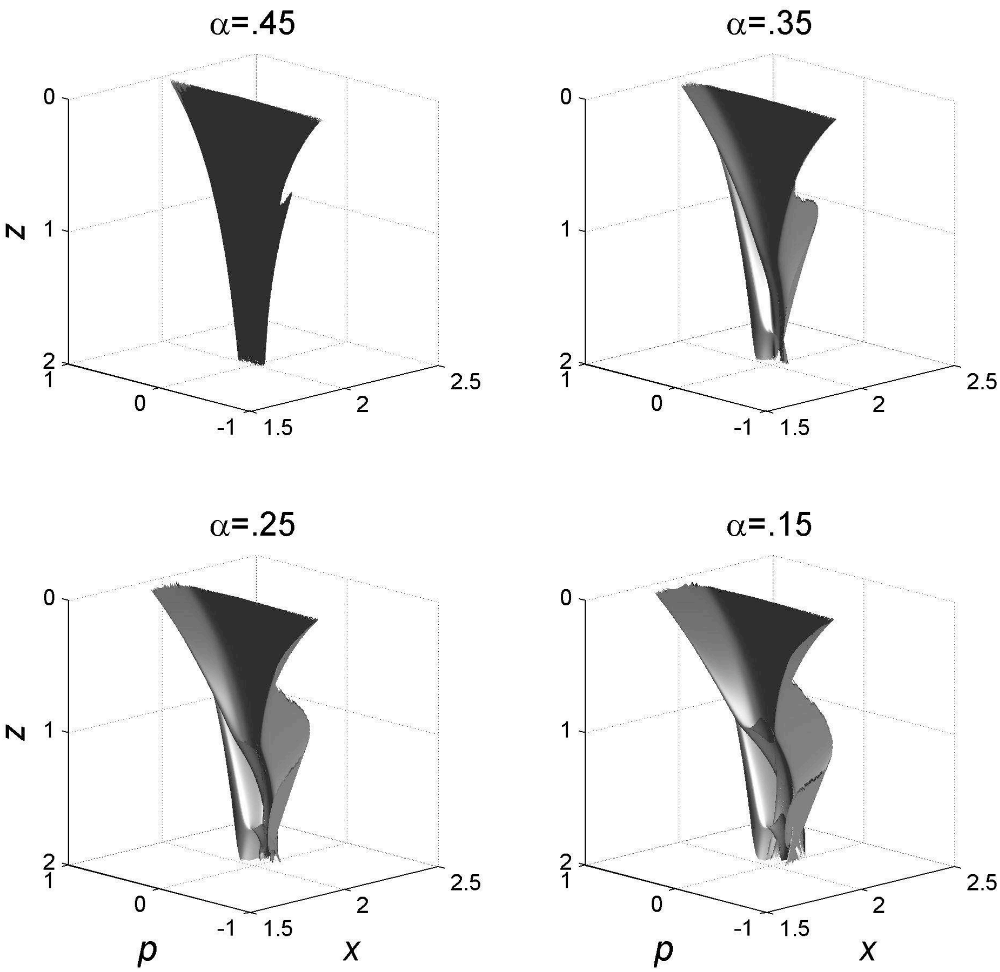

Fig. 8. Evolution of the gray plane in Fig. 5 right, with α: $\alpha = \alpha_0 = 0.45$ (true model), $\alpha = 0.35$, $\alpha = 0.25$ and $\alpha = 0.15$. Only part of the plane is illuminated due to the limited acquisition aperture used in the computation: The illuminated part shrinks in the p direction with increasing depth z.

observe the presence of caustics and turning rays. In Fig. 7, we show two-way travel time curves for the reflected wave (u) for different point sources evenly distributed along the acquisition surface (at the top of Fig. 6).

The continuation of the singular support of $w_g(x, z, p)$ (Fig. 5 left) is illustrated in Figs. 8, 9 and 10. Fig. 8 shows the evolution of the gray (image) plane in Fig. 5 right; Fig. 9 shows the evolution of lines 1 and 2, both representative of the vertical reflector, in Fig. 5 right; and Fig. 10 shows the evolution of line 3, a ‘string’ in a common-image point gather, in Fig. 5 right.

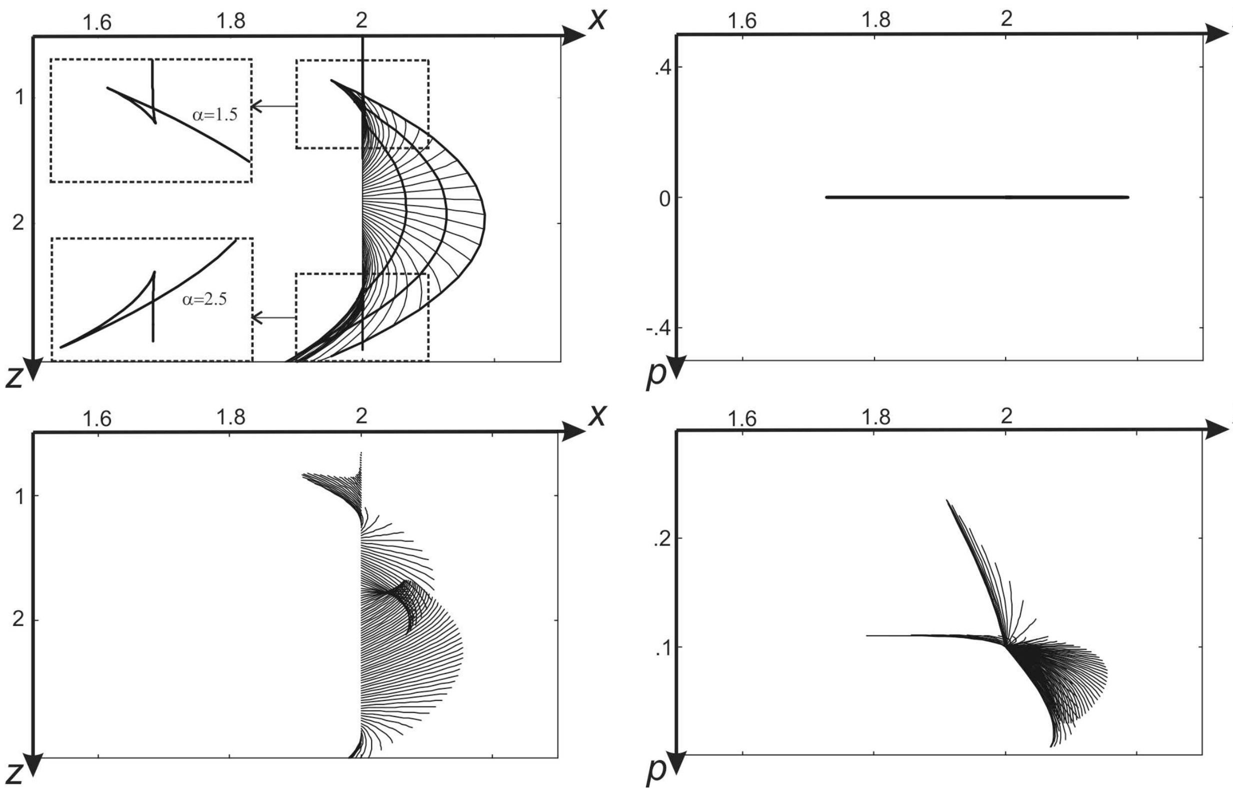

Fig. 9. Image continuation, $w_g(x = 2, z, p = 0)$ (top) and $w_g(x = 2, z, p = 0.1)$ (bottom). The continuation characteristics are projected onto the (x, z) plane (left) and the (x, p) plane (right). In the top left figure we also plotted four fronts (thick solid lines), at $\alpha = \alpha_0 = 0.45$ (true model), $\alpha = 0.35$, $\alpha = 0.25$ and $\alpha = 0.15$. The inserts show the cusps at the top and the bottom in more detail; these cusps are formed in a transitional region, where the influence of the lens vanishes. Note that, for $p = 0$, the continuation characteristics stay in a plane, unlike for $p = 0.1$ (bottom, right).

We conclude by mentioning that (i) the continuation of common-image point gathers aids in the understanding of 'coherent noise' in such gathers due to background velocity errors [29], and (ii) the continuation of common-image point gathers is directly applicable to the reflection tomography problem [9]. The measure whether a background model (the value of α) is acceptable for imaging, essentially, depends on the vanishing of $\frac{\partial}{\partial p} w_g(x, z, p)$; this quantity can be evaluated during image gather continuation – without remigrating the data.

8 Discussion

We developed the foundation of, and a comprehensive framework for seismic continuation, while extending the earlier approaches to this type of continuation to allow for the formation of

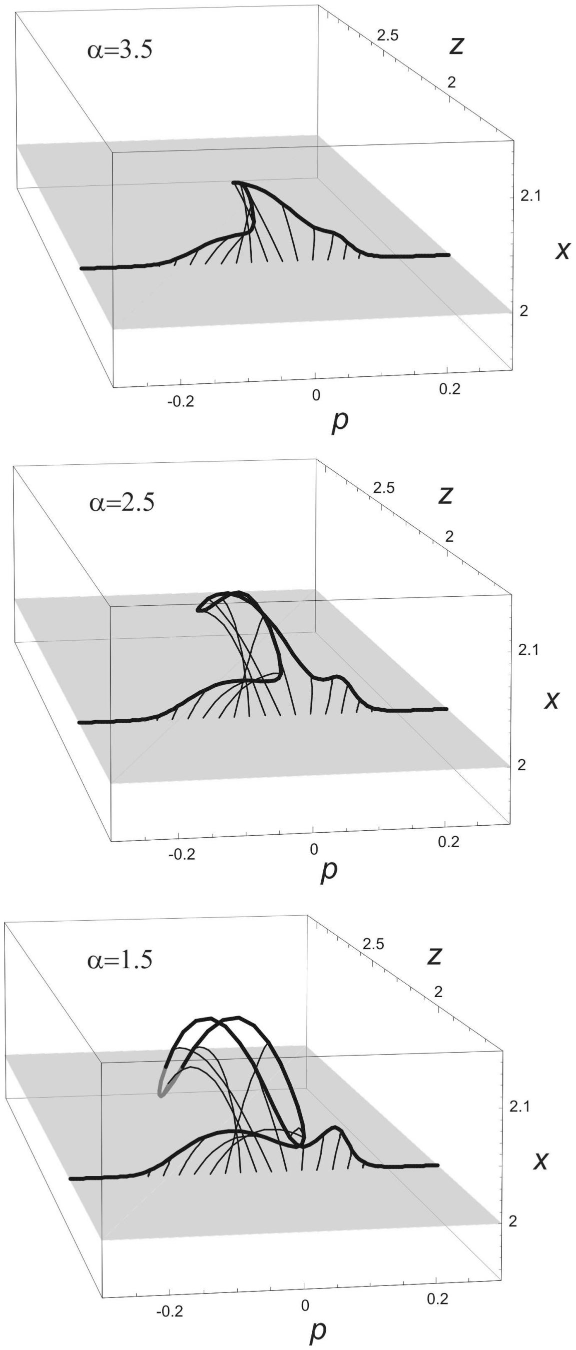

Fig. 10. Continuation of a line or 'string', initially at $z = 2$, in a 'vertical' common-image point gather initially at $x = 2$ (line 3 in Fig. 5). Top: $\alpha = 0.35$, middle: $\alpha = 0.25$, bottom: $\alpha = 0.15$. The thin lines indicate continuation characteristics.

caustics. We illustrated how the concepts introduced and developed by Fomel (partial differential equations for data and image continuation, corresponding Hamiltonians), Goldin (continuation by composing remigration with demigration, underlying contact transformations), Hubral *et al*. (image waves), Iversen (system of ordinary differential equations for continuation characteristics, connection with ray perturbation theory), Adler (velocity rays, connection with a migration Jacobian) and Liu, Meng and Bleistein (common-offset image continuation and residual moveout) are contained in our theory. Traditionally, seismic continuation has been viewed from a geometrical (ray) point of view; here, we introduced the notion of wave-equation continuation through the appearance of evolution equations. This notion has applications, for example, in wave-equation reflection tomography and in imaging Earth's deep interior.

Acknowledgements

MdH has fond memories of the many discussions with Sergey Goldin that led to the research presented in this paper. This research was supported in part by the members, BP, ExxonMobil and Total, of the Geo-Mathematical Imaging Group.

A Canonical transformations and contact transformations

Let X (or Y) be a n-dimensional manifold as in the main text, and let T^*X denote its $2n$-dimensional cotangent bundle (phase space). The $(2n-1)$-dimensional cosphere bundle of X is given by $S(T^*X) = T^*X\backslash 0/\mathbb{R}_+$. In fact, $\pi :\ T^*X\backslash 0 \rightarrow S(T^*X)$ defines a principal fiber bundle with structure group $\mathbb{R}_+$.

Let θ denote the canonical 1-form on T^*X, whence $\mathrm{d}\theta$ is the fundamental symplectic form. To obtain a contact structure on $S(T^*X)$ one introduces a global section $\sigma :\ S(T^*X) \rightarrow T^*X\backslash 0$ with the property that $\pi \circ \sigma = \mathsf{id}$. Such a section is determined by a function $f_\sigma :\ T^*X\backslash 0 \rightarrow \mathbb{R}$:

$$\sigma(\pi(\xi_x)) = f_\sigma(\xi_x)\,\xi_x.$$

Then $\theta_\sigma = \sigma^*\theta$ defines a contact 1-form on $S(T^*X)$, with $\theta_\sigma \wedge (\mathrm{d}\theta_\sigma)^{n-1}$ defining a volume form on $S(T^*X)$. We have

$$\pi^*\theta_\sigma = f_\sigma\,\theta.$$

Let $\Sigma :\ T^*X \rightarrow T^*X$ denote a canonical transformation, which has the property that $\Sigma^*\theta = \theta$.

There is a unique diffeomorphism $\varphi : \ S(T^*X) \to S(T^*X)$ such that $\varphi \circ \pi = \pi \circ \Sigma$, namely

$$\varphi = \pi \circ \frac{1}{f_\sigma} \Sigma \circ \sigma.$$

Indeed,

$$\varphi(\pi(\xi_x)) = \pi(\frac{1}{f_\sigma}\Sigma(\sigma(\pi(\xi_x)))) = \pi(\frac{1}{f_\sigma}\Sigma(f_\sigma(\xi_x)\xi_x)) = \pi(\Sigma(\xi_x)).$$

If one defines

$$h_\sigma := f_\sigma \circ \frac{1}{f_\sigma}\Sigma \circ \sigma : \ S(T^*X) \to \mathbb{R}_+,$$

it follows that $\varphi^*\theta_\sigma = h_\sigma \, \theta_\sigma$, that is, (φ, h_σ) is a contact transformation.

Conversely, for a pair (φ, h_σ) defining a contact transformation, one defines

$$\Sigma = \frac{1}{(h \circ \pi) \, f_\sigma} \, \sigma \circ \varphi \circ \pi,$$

with the property that $\Sigma^*\theta = \theta$ so that Σ is a canonical transformation. Thus the base space of the Lie group of invertible FIOs of order 0, the canonical relations of which are graphs, can be identified with these contact transformations. This geometrical point of view has been preferred in the original work of Goldin [20].

The contact 1-form is defined on $TS(T^*X)$, that is, $\theta_\sigma : \ TS(T^*X) \to \mathbb{R}$. For each point $s \in S(T^*X)$, we have the decomposition

$$T_sS(T^*X) = \ker{(\theta_\sigma)_s} \oplus \ker{(\mathrm{d}\theta_\sigma)_s}; \tag{A.1}$$

$\ker \mathrm{d}\theta_\sigma$ is 1-dimensional and determines the so-called characteristic direction of the contact form θ_σ. Moreover, $R_s = \ker{(\theta_\sigma)_s}$ defines a tangent hyperplane at s. A contact element on $S(T^*X)$ is a point $s \in S(T^*X)$, called a contact point, paired with a tangent hyperplane at s; $R : \ S(T^*X) \ni s \to R_s$ defines a smooth field of such contact elements. Thus, given a contact 1-form, a contact element (s, R_s) is assigned to each point $s \in S(T^*X)$. (Moreover, given σ, a point $(x, \xi_x) \in T^*_xX \backslash 0$ determines a contact element (s, R_s); sometimes, if s projects to $x \in X$, one relates to (x, R_s) instead of (s, R_s)). The pair $(S(T^*X), R)$ is called a contact manifold.

Darboux's theorem essentially states that all contact structures (of the same dimension) look the same near a point: There exists a coordinate system $(x_1, \ldots, x_{n-1}, \bar{\xi}_1, \ldots, \bar{\xi}_{n-1}, x_n)$ on a neighborhood of a point, s say, in $S(T^*X)$ such that

$$\theta_\sigma = \mathrm{d}x_n - \sum_{i=1}^{n-1} \bar{\xi}_i \mathrm{d}x_i. \tag{A.2}$$

This contact form also defines the standard contact structure on $\mathbb{R}^{2n-1}$. For example, for $n=3$, at the point $(x_1, x_2, \bar{\xi}_1, \bar{\xi}_2, x_3)$, the contact hyperplane $R_{(x_1,x_2,\bar{\xi}_1,\bar{\xi}_2,x_3)}$ is spanned by

$$\{(\bar{\xi}_1 + \bar{\xi}_2)\,\partial_{x_3} + \partial_{x_1} + \partial_{x_2}, \bar{\xi}_2\partial_{x_1} - \bar{\xi}_1\partial_{x_2}, \partial_{\bar{\xi}_1}, \partial_{\bar{\xi}_2}\}.$$

In this case, the symplectic form on T^*X providing the contact 1-form (A.2) is simply given by $\mathrm{d}\theta = \sum_{i=1}^{n} \mathrm{d}\xi_i \wedge \mathrm{d}x_i$, while $f_\sigma(\xi_x) = \frac{1}{\xi_n}$ (without loss of generality, we can assume that $\xi_n \neq 0$; then $\bar{\xi}_i = -\frac{\xi_i}{\xi_n}$, $i = 1, \ldots, n-1$).

B Oscillatory integral representations – change of phase variables

In applications, the oscillatory integral representation of the relevant operator kernel often appears naturally in a form different from the canonical form (12)-(13). We set $n_Y = n_X = n$. If for the kernel of an FIO in $\mathcal{C}$ we have an oscillatory integral representation – making use of coordinates (y, x_I, ξ_J) with $I \cup J = \{1, \ldots, n\}$ on Λ – with amplitude $a = a'(y, x_I, \xi_J)$, we can obtain $a = a(y, \xi)$ by the relation [32, 4.1.2] [5]

$$a(y,\xi)\,\exp[\mathrm{i}\,S(y,\xi)] = \int a'(y, x'_I, \xi_J)\,\exp[\mathrm{i}\,(S'(y, x'_I, \xi_J) + \langle \xi_I, x'_I\rangle)]\,\mathrm{d}x'_I,$$

which follows from writing the action of the associated FIO as

$$(Fu)(y) = \int (2\pi)^{-n} \iiint a'(y, x'_I, \xi_J)\,\exp[\mathrm{i}\,(S'(y, x'_I, \xi_J) + \langle \xi_I, x'_I\rangle)]\,\mathrm{d}x'_I \\ \exp[\mathrm{i}\,(-\langle \xi_J, x_J\rangle - \langle \xi_I, x_I\rangle)]\,\mathrm{d}\xi_J \mathrm{d}\xi_I\,u(x)\,\mathrm{d}x.$$

Invoking the method of stationary phase in x_I yields

$$\begin{aligned} \int a'(y, x_I, \xi_J)\,\exp[\mathrm{i}\,(S'(y, x_I, \xi_J) + \langle \xi_I, x_I\rangle)]\,\mathrm{d}x_I \\ = (2\pi)^{|I|/2} \exp[\mathrm{i}(\pi/4)\ \mathrm{sgn}\,\Delta(y, x_I, \xi_J)] a'(y, x_I, \xi_J) \\ \exp[\mathrm{i}\,(S'(y, x_I, \xi_J)]\,\exp[\mathrm{i}\,\langle \xi_I, x_I\rangle]\,[\det \Delta(y, x_I, \xi_J)]^{-1/2}\,|_{x_I = x_I(y,\xi)}, \end{aligned} \tag{B.1}$$

where Δ is the $|I| \times |I|$ Hessian

$$\Delta(y, x_I, \xi_J) = \frac{\partial^2 S'(y, x_I, \xi_J)}{\partial x_I^2}.$$

[5] This follows by inserting the Fourier transforms $\mathcal{F}^{-1}_{\xi_I \to x_I} \mathcal{F}_{x_I \to \xi_I}$ in front of $u(x)$ in the action of F on u.

We identify $S(y,\xi) = (S'(y,x_I,\xi_J) + \langle \xi_I, x_I\rangle)|_{x_I=x_I(y,\xi)}$. The stationary points are the x_I, satisfying the system of equations

$$-\frac{\partial S'}{\partial x_I}(y,x_I,\xi_J) = \xi_I,$$

with solution $x_I = x_I(y,\xi_I,\xi_J) = x_I(y,\xi)$ revealing the coordinate transformation (y,x_I,ξ_J) $\to (y,\xi_I,\xi_J)$ on Λ. We note that

$$(-)^{|I|} \det \frac{\partial^2 S'(y,x_I,\xi_J)}{\partial x_I^2} = \det \frac{\partial(\xi_I)}{\partial(x_I)}.$$

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